Generalized spinwave theory

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S=1 Heisenberg model with single-ion anisotropy

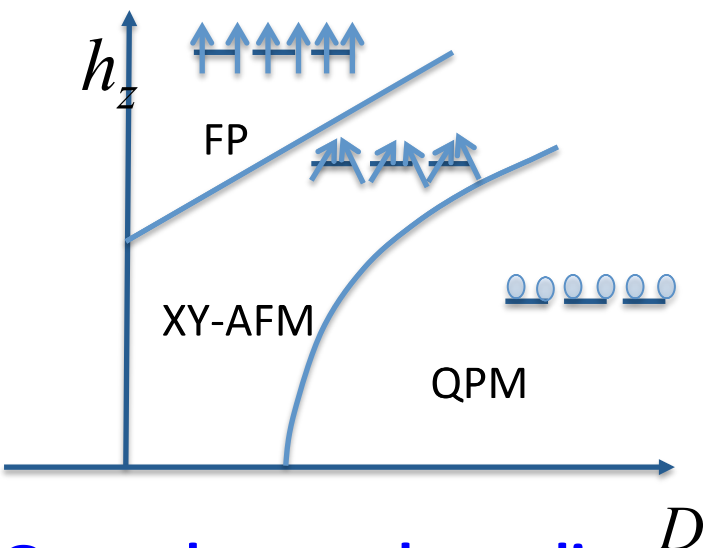
Apply the generalized spin-wave theory to analyze the ground sate properties of the $S=1$ Heisenberg model with single-ion anisotropy in an external magnetic field on a bipartite lattice.

Assume isotropic lattice – generalization to anisotropic lattice is straightforward

$$\mathcal{H} = \sum_{\mathbf{i},\nu} \frac{J_\nu}{2} \mathbf{S_i} \cdot \mathbf{S_{i+e_\nu}} + D \sum_{\mathbf{i}} (S^z_{\mathbf{i}})^2 - h_z \sum_{\mathbf{i}} S^z_{\mathbf{i}}$$

Rich ground state phase diagram – known from SW, QMC studies.

Realized in many quantum magnets, e.g., DTN



Ground state phase diagram

Schwinger bosons

Local Hilbert space has dimension D_l=3 – introduce 3 Schwinger bosons

$$b_{i,m}^\dagger, m \in \{0,1,2\}$$

The Schwinger bosons create eigenstates of S_i^z

$$b_0^\dagger|vac\rangle = |0\rangle, \;\; b_1^\dagger|vac\rangle = |1\rangle, \;\; b_2^\dagger|vac\rangle = |-1\rangle$$

And satisfy the local constraint at each site $\sum_m b_{i,m}^\dagger b_{i,m} = 1$

The spin operators assume bilinear forms in the bosonic operators

$$\begin{aligned}
S_i^z &= \mathbf{b}_i^\dagger \mathcal{S}^z \mathbf{b}_i = b_{i,1}^\dagger b_{i,1} - b_{i,2}^\dagger b_{i,2} \\
S_i^+ &= \mathbf{b}_i^\dagger \mathcal{S}^+ \mathbf{b}_i = \sqrt{2}(b_{i,1}^\dagger b_{i,0} + b_{i,0}^\dagger b_{i,2}) \\
S_i^- &= \mathbf{b}_i^\dagger \mathcal{S}^- \mathbf{b}_i = \sqrt{2}(b_{i,0}^\dagger b_{i,1} + b_{i,2}^\dagger b_{i,0})
\end{aligned}$$

Schwinger bosons

We have introduced the vectors

$$\mathbf{b}^\dagger = \begin{pmatrix} b_0^\dagger & b_1^\dagger & b_2^\dagger \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

The spin matrices take the following forms

$$\mathcal{S}^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \mathcal{S}^+ = \sqrt{2} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathcal{S}^- = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In the bosonic representation, the Hamiltonian is

$$\begin{aligned} \mathcal{H} &= J \sum_{\langle i,j \rangle} \left[\frac{1}{2} (\mathbf{b}_i^\dagger \mathcal{S}^+ \mathbf{b}_i \mathbf{b}_i^\dagger \mathcal{S}^- \mathbf{b}_j + \mathbf{b}_j^\dagger \mathcal{S}^+ \mathbf{b}_j \mathbf{b}_i^\dagger \mathcal{S}^- \mathbf{b}_i) + \mathbf{b}_i^\dagger \mathcal{S}^z \mathbf{b}_i \mathbf{b}_i^\dagger \mathcal{S}^z \mathbf{b}_j \right] \\ &+ D \sum_i \left(1 - \mathbf{b}_i^\dagger \mathcal{A} \mathbf{b}_i \right) - h \sum_i \mathbf{b}_i^\dagger \mathcal{S}^z \mathbf{b}_i, \qquad A_{\alpha,\beta} = \delta_{\alpha,0} \delta_{\beta,0} \end{aligned}$$

Mean field ground state

The “classical” ground state is represented by the condensation of a bosonic operator

$$|\psi_{cl}\rangle = \prod_i \tilde{b}^\dagger_{i,0}|0\rangle$$

The relevant operator is written in the working basis $\tilde{b}^\dagger_{i,0} = \sum_{m=0}^{2} c_m b^\dagger_{i,m}$

$\tilde{b}^\dagger_{i,0}$ is obtained by minimizing the classical ground state energy

$$e_0 = \langle\psi_{cl}|\mathcal{H}|\psi_{cl}\rangle/N$$

The parameter space for minimization is spanned by the group of $\mathrm{SU}(D_l)$ of unitary operators that act on the local Hilbert space

Change of basis $\tilde{\mathbf{b}}^\dagger_i = \mathcal{U}\mathbf{b}^\dagger_i$

In terms of the usual spin wave approach, this corresponds to choosing the quantization axis along the direction of the classical order parameter

Sublattice rotation

In general, to describe any ordered state that breaks discrete translational symmetry making the 2 sublattices inequivalent, one needs a different transformation matrix for each sublattice, e.g., AFM state.

$$\mathcal{U}_\alpha , \;\; \alpha = A, B$$

However, if the sublattice order parameters are simply related by a phase transformation, then for a bipartite lattice, one can apply a sublattice rotation making the two sublattices equivalent and work with a homogeneous ground state and a single transformation matrix. This maps the exchange interaction from AFM to FM.

This works only if the magnitude of the order parameter is the same on both sublattices.

The Hamiltonian in the transformed basis (with sublattice rotation applied)

$$\begin{aligned}\mathcal{H} &= J\sum_{\langle i,j\rangle}\left[-\frac{1}{2}(\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^+\tilde{\mathbf{b}}_i\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^-\tilde{\mathbf{b}}_j + \tilde{\mathbf{b}}_j^\dagger\tilde{\mathcal{S}}^+\tilde{\mathbf{b}}_j\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^-\tilde{\mathbf{b}}_i) + \tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^z\tilde{\mathbf{b}}_i\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^z\tilde{\mathbf{b}}_j\right] \\ &+ D\sum_i\left(1-\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{A}}\tilde{\mathbf{b}}_i\right) - h\sum_i\tilde{\mathbf{b}}_i^\dagger\tilde{\mathcal{S}}^z\tilde{\mathbf{b}}_i, \quad \tilde{\mathcal{S}}^\mu = \mathcal{U}\mathcal{S}^\mu\mathcal{U}^\dagger \text{ and } \tilde{\mathcal{A}} = \mathcal{U}\mathcal{A}\mathcal{U}^\dagger\end{aligned}$$

The local constraint remains invariant under a unitary transformation

$$\sum_m \tilde{b}_{i,m}^\dagger\tilde{b}_{i,m} = 1$$

Condensation of the new operator corresponds to the condition

$$\langle\tilde{b}_{i,0}^\dagger\rangle = \langle\tilde{b}_{i,0}\rangle > 0$$

In conjunction with the local constraint, this implies

$$\tilde{b}_{i,0}^\dagger = \tilde{b}_{i,0} = \sqrt{1-\tilde{b}_{i,1}^\dagger\tilde{b}_{i,1}-\tilde{b}_{i,2}^\dagger\tilde{b}_{i,2}}$$

Note: Accuracy of the method is improved by treating the expectation value as a parameter and minimizing the total energy.

Applying the condensation condition and keeping terms up to quadratic order in bosonic operators, the spin operators become

$$\begin{aligned}
\tilde{\mathcal{S}}_i^{\mu} &= \tilde{\mathbf{b}}_i^{\dagger}\tilde{\mathcal{S}}^{\mu}\tilde{\mathbf{b}}_i \\
&\approx \tilde{\mathcal{S}}_{00}^{\mu} + \sum_{\alpha=1,2}(\tilde{\mathcal{S}}_{\alpha 0}^{\mu} b_{i\alpha}^{\dagger} + \tilde{\mathcal{S}}_{0\alpha}^{\mu} b_{i\alpha}) + \sum_{\alpha,\beta}(\tilde{\mathcal{S}}_{\alpha\beta}^{\mu} - \tilde{\mathcal{S}}_{00}^{\mu}\delta_{\alpha\beta}) b_{i\alpha}^{\dagger} b_{i\beta}, \quad \mu = +, -, z
\end{aligned}$$

The bilinear spin operators in the Hamiltonian take the forms

$$\begin{aligned}
\sum_{\langle i,j\rangle}\tilde{\mathcal{S}}_i^z\tilde{\mathcal{S}}_j^z &\approx zN(\tilde{\mathcal{S}}_{00}^z)^2 + 2z\sum_{i,\alpha}\tilde{\mathcal{S}}_{00}^z\tilde{\mathcal{S}}_{\alpha 0}^z(b_{i\alpha}^{\dagger} + b_{i\alpha}) + 2z\sum_{\substack{i\\ \alpha,\beta}}\tilde{\mathcal{S}}_{00}^z(\tilde{\mathcal{S}}_{\alpha\beta}^z - \tilde{\mathcal{S}}_{00}^z\delta_{\alpha\beta}) b_{i\alpha}^{\dagger} b_{i\beta} \\
&+ \sum_{\substack{\langle i,j\rangle\\ \alpha\beta}}\tilde{\mathcal{S}}_{\alpha 0}^z\tilde{\mathcal{S}}_{0\beta}^z(b_{i\alpha}^{\dagger} b_{j\beta} + h.c.) + \sum_{\substack{\langle i,j\rangle\\ \alpha\beta}}\tilde{\mathcal{S}}_{\alpha 0}^z\tilde{\mathcal{S}}_{\beta 0}^z(b_{i\alpha}^{\dagger} b_{j\beta}^{\dagger} + h.c.), \quad \tilde{\mathcal{S}}_{\alpha\beta}^z = \tilde{\mathcal{S}}_{\beta\alpha}^z
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\langle i,j\rangle}\frac{1}{2}\left(\tilde{\mathcal{S}}_i^+\tilde{\mathcal{S}}_j^- + \tilde{\mathcal{S}}_i^-\tilde{\mathcal{S}}_j^-\right) &\approx zN(\tilde{\mathcal{S}}_{00}^+)^2 + 2z\sum_{i,\alpha}\tilde{\mathcal{S}}_{00}^+\left(\frac{\tilde{\mathcal{S}}_{\alpha 0}^+ + \tilde{\mathcal{S}}_{\alpha 0}^-}{2}\right)(b_{i\alpha}^{\dagger} + b_{i\alpha}) + 2z\sum_{\substack{i\\ \alpha,\beta}}\tilde{\mathcal{S}}_{00}^+\left(\frac{\tilde{\mathcal{S}}_{\alpha\beta}^+ + \tilde{\mathcal{S}}_{\alpha\beta}^-}{2} - \tilde{\mathcal{S}}_{00}\right. \\
&+ \sum_{\substack{\langle i,j\rangle\\ \alpha\beta}}\left(\frac{\tilde{\mathcal{S}}_{\alpha 0}^+\tilde{\mathcal{S}}_{0\beta}^- + \tilde{\mathcal{S}}_{\alpha 0}^-\tilde{\mathcal{S}}_{0\beta}^+}{2}\right)(b_{i\alpha}^{\dagger} b_{j\beta} + h.c.) + \sum_{\substack{\langle i,j\rangle\\ \alpha\beta}}\left(\frac{\tilde{\mathcal{S}}_{\alpha 0}^+\tilde{\mathcal{S}}_{\beta 0}^- + \tilde{\mathcal{S}}_{0\alpha}^+\tilde{\mathcal{S}}_{0\beta}^-}{2}\right)(b_{i\alpha}^{\dagger} b_{j\beta}^{\dagger} + h.c
\end{aligned}$$

And finally

$$\tilde{\mathbf{b}}_i^\dagger \tilde{\mathcal{A}} \tilde{\mathbf{b}}_i \approx \tilde{\mathcal{A}}_{00} + \sum_{\alpha=1,2} \tilde{\mathcal{A}}_{\alpha 0}(b_{i\alpha}^\dagger + b_{i\alpha}) + \sum_{\alpha,\beta}(\tilde{\mathcal{A}}_{\alpha\beta} - \tilde{\mathcal{A}}_{00}\delta_{\alpha\beta}) b_{i\alpha}^\dagger b_{i\beta},$$

Collecting all the terms, the Hamiltonian can be written as the following

$$H = e_0 N + \mathcal{H}_{sw} + H_2$$

Classical energy

Higher order terms

bilinear in bosonic operators

Linear terms vanish by virtue of minimization condition. This results in the following contraint

$$\frac{J}{2}\tilde{\mathcal{S}}_{00}^{+}\left(\frac{\tilde{\mathcal{S}}_{\alpha 0}^{+} + \tilde{\mathcal{S}}_{\alpha 0}^{-}}{2}\right) + J\tilde{\mathcal{S}}_{00}^{z}\tilde{\mathcal{S}}_{\alpha 0}^{z} + \tilde{\mathcal{A}}_{\alpha 0} = 0, \quad \alpha = 1, 2$$

Spinwave Hamiltonian

The general form for the spinwave Hamiltonian is

$$\mathcal{H}_{sw} = \sum_{\substack{\langle i,j\rangle \\ \alpha\beta}} t_{\alpha\beta}\left[(b^\dagger_{i\alpha}b_{j\beta} + h.c.) + \Delta_{\alpha\beta}(b^\dagger_{i\alpha}b^\dagger_{j\beta} + h.c.)\right] + \sum_{\substack{i \\ \alpha\beta}} \lambda_{\alpha\beta} b^\dagger_{i\alpha} b_{i\beta}$$

where

$$t_{\alpha\beta} = -\frac{J}{2}(\tilde{\mathcal{S}}^+_{\alpha 0}\tilde{\mathcal{S}}^-_{0\beta} + \tilde{\mathcal{S}}^-_{\alpha 0}\tilde{\mathcal{S}}^+_{0\beta}) + J\tilde{\mathcal{S}}^z_{\alpha 0}\tilde{\mathcal{S}}^z_{0\beta}$$

$$\Delta_{\alpha\beta} = -\frac{J}{2}(\tilde{\mathcal{S}}^+_{\alpha 0}\tilde{\mathcal{S}}^-_{\beta 0} + \tilde{\mathcal{S}}^-_{\alpha 0}\tilde{\mathcal{S}}^+_{\beta 0}) + J\tilde{\mathcal{S}}^z_{\alpha 0}\tilde{\mathcal{S}}^z_{\beta 0}$$

$$\lambda_{\alpha\beta} = -2zJ\tilde{\mathcal{S}}^+_{00}\left(\frac{\tilde{\mathcal{S}}^+_{\alpha\beta} + \tilde{\mathcal{S}}^-_{\alpha\beta}}{2} - \tilde{\mathcal{S}}^-_{00}\delta_{\alpha\beta}\right) + 2zJ\tilde{\mathcal{S}}^z_{00}(\tilde{\mathcal{S}}^z_{\alpha\beta} - \tilde{\mathcal{S}}^z_{00}\delta_{\alpha\beta})$$
$$-D(\tilde{\mathcal{A}}_{\alpha\beta} - \tilde{\mathcal{A}}_{00}\delta_{\alpha\beta})$$

Momentum space representation

Fourier transform the bosonic operators

$$\hat{b}^{\dagger}_{\mathbf{k}\alpha} = \frac{1}{\sqrt{N_s}} \sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i} \tilde{b}^{\dagger}_{i\alpha}$$

to obtain the spinwave Hamiltonian in the momentum representation

$$\mathcal{H}_{sw} = \sum_{\substack{\alpha,\beta \\ \mathbf{k}}} \left[\epsilon_{\alpha\beta}(\mathbf{k}) \hat{b}^{\dagger}_{\mathbf{k}\alpha} \hat{b}_{\mathbf{k}\beta} + \frac{\gamma_{\alpha\beta}(\mathbf{k})}{2} (\hat{b}^{\dagger}_{\mathbf{k}\alpha} \hat{b}^{\dagger}_{-\mathbf{k}\beta} + \hat{b}_{\mathbf{k}\alpha} \hat{b}_{-\mathbf{k}\beta}) \right]$$

where

$$\begin{aligned} \epsilon_{\alpha\beta}(\mathbf{k}) &= \lambda_{\alpha\beta} + t_{\alpha\beta} \sum_{\nu} \cos(k_\nu) \\ \gamma_{\alpha\beta}(\mathbf{k}) &= \Delta_{\alpha\beta} \sum_{\nu} \cos(k_\nu) \end{aligned}$$

Diagonalization of the spinwave Hamiltonian

Define the matrices

$$\mathcal{E}_{\mathbf{k}} = \begin{pmatrix} \epsilon_{11}(\mathbf{k}) & \epsilon_{12}(\mathbf{k}) \\ \epsilon_{21}(\mathbf{k}) & \epsilon_{22}(\mathbf{k}) \end{pmatrix}, \quad \Gamma_{\mathbf{k}} = \begin{pmatrix} \gamma_{11}(\mathbf{k}) & \gamma_{12}(\mathbf{k}) \\ \gamma_{21}(\mathbf{k}) & \gamma_{22}(\mathbf{k}) \end{pmatrix}$$

The spinwave Hamiltonian can be diagonalized by a Bogoliubov transformation to get

$$\mathcal{H}^B_{sw} = \sum_{\substack{\alpha \\ \mathbf{k}}} \left[\omega_{\mathbf{k}\alpha} (c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \frac{1}{2}) - \frac{\epsilon_{\alpha\alpha}(\mathbf{k})}{2} \right]$$

The square of the quasiparticle energies, $\omega^2_{\mathbf{k}\alpha}$, are given by the eigenvalues of the matrix

$$\mathcal{C}_{\mathbf{k}} = (\mathcal{E}_{\mathbf{k}} - \Gamma_{\mathbf{k}})(\mathcal{E}_{\mathbf{k}} + \Gamma_{\mathbf{k}})$$

The XY-AFM state

The mean field ground state is given by the local spin configuration

$$|\psi\rangle_i = \cos\theta|0\rangle_i + \sin\theta\cos\phi|+1\rangle_i + \sin\theta\sin\phi|-1\rangle_i$$

The unitary transformation to the new basis is defined as

$$\begin{aligned}
\tilde{b}_0^\dagger &= \cos\theta\ b_0^\dagger + \sin\theta\cos\phi\ b_1^\dagger + \sin\theta\sin\phi\ b_2^\dagger \\
\tilde{b}_1^\dagger &= -\sin\theta\ b_0^\dagger + \cos\theta\cos\phi\ b_1^\dagger + \cos\theta\sin\phi\ b_2^\dagger \\
\tilde{b}_2^\dagger &= -\sin\phi\ b_1^\dagger + \cos\phi\ b_2^\dagger
\end{aligned}$$

In matrix form

$$\tilde{\mathbf{b}}^\dagger = \mathbf{b}^\dagger \mathcal{U}^T \Rightarrow \mathbf{b}^\dagger = \tilde{\mathbf{b}}^\dagger \mathcal{U}$$

and the transformation matrix

$$\mathcal{U} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

The XY-AFM state

The spin matrices in the transformed basis

$$\begin{aligned}\tilde{\mathcal{S}}^z &= \mathcal{U}\mathcal{S}^z\mathcal{U}^T \\ &= \begin{pmatrix} \sin^2\theta\cos 2\phi & \frac{1}{2}\sin 2\theta\cos 2\phi & -\sin\theta\sin 2\phi \\ \frac{1}{2}\sin 2\theta\cos 2\phi & \cos^2\theta\cos 2\phi & -\cos\theta\sin 2\phi \\ -\sin\theta\sin 2\phi & -\cos\theta\sin 2\phi & -\cos 2\phi \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{S}}^+ &= \mathcal{U}\mathcal{S}^+\mathcal{U}^T \\ &= \sqrt{2}\begin{pmatrix} \frac{1}{2}\sin 2\theta(\cos\phi+\sin\phi) & -\sin^2\theta\cos\phi+\cos^2\theta\sin\phi & \cos\theta\cos\phi \\ \cos^2\theta\cos\phi-\sin^2\theta\sin\phi & -\frac{1}{2}\sin 2\theta(\cos\phi+\sin\phi) & -\sin\theta\cos\phi \\ -\cos\theta\sin\phi & \sin\theta\sin\phi & 0 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{S}}^- &= \mathcal{U}\mathcal{S}^-\mathcal{U}^T \\ &= \sqrt{2}\begin{pmatrix} \frac{1}{2}\sin 2\theta(\cos\phi+\sin\phi) & \cos^2\theta\cos\phi-\sin^2\theta\sin\phi & -\cos\theta\sin\phi \\ -\sin^2\theta\cos\phi+\cos^2\theta\sin\phi & -\frac{1}{2}\sin\theta(\cos\phi+\sin\phi) & \sin\theta\sin\phi \\ \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{A}} &= \mathcal{U}\mathcal{A}\mathcal{U}^T \\ &= \begin{pmatrix} \cos^2\theta & -\frac{1}{2}\sin 2\theta & 0 \\ \frac{1}{2}\sin 2\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

The XY-AFM state

The energy of the classical ground state $|gs\rangle = \prod_i \tilde{b}^{\dagger}_{i0}|vac\rangle$

is obtained as

$$
\begin{aligned}
e &= \frac{1}{N}(E_{exch} + E_{anis} + E_{zman}) \\
\frac{E_{exch}}{N} &= zJ\left[\sin^4\theta\cos^2 2\phi - 2\cos^2\theta\sin^2\theta(\cos\phi + \sin\phi)^2\right] \\
\frac{E_{anis}}{N} &= D\sin^2\theta \\
\frac{E_{zman}}{N} &= -h\sin^2\theta\cos 2\phi
\end{aligned}
$$

The parameters θ and ϕ defining the classical ground state is obtained by minimization of the classical energy

$$\frac{\partial e}{\partial \phi} = 0$$

$$\Rightarrow 2h\sin^2\theta\sin 2\phi = zJ(2\sin^4\theta\cos 2\phi\sin 2\phi - \sin^2 2\theta\cos 2\phi)$$

$$\frac{\partial e}{\partial \theta} = 0$$

$$\Rightarrow \sin\theta = \frac{4zJ(1+\sin 2\phi) + 2h\cos 2\phi - 2D}{zJ(4\cos^2 2\phi + 8\sin 2\phi + 8)}$$

The XY-AFM state

Let us the consider the ground state for $h=0$. In this case it is reasonable to assume that the weights of the $|+1\rangle$ and the $|-1\rangle$ are equal in the ground state, that is $\phi = \frac{\pi}{4}$. The parameter θ in this case reduces to the simple form

$$\sin^2\theta = \frac{1}{2} - \frac{D}{8zJ}$$

The expression is valid only for

$$D < 4zJ \equiv D_c$$ boundary of the XY-AFM phase

The XY-AFM state

The spin matrices in this limit $(h=0)$ simplifies to

$$\tilde{\mathcal{S}}^z = \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix}$$

$$\tilde{\mathcal{S}}^+ = \begin{pmatrix} \sin 2\theta & \cos 2\theta & \cos\theta \\ \cos 2\theta & -\sin 2\theta & -\sin\theta \\ -\cos\theta & \sin\theta & 0 \end{pmatrix}$$

$$\tilde{\mathcal{S}}^- = \begin{pmatrix} \sin 2\theta & \cos 2\theta & -\cos\theta \\ \cos 2\theta & -\sin 2\theta & \sin\theta \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\tilde{\mathcal{A}} = \begin{pmatrix} \cos^2\theta & -\frac{1}{2}\sin 2\theta & 0 \\ -\frac{1}{2}\sin 2\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the matrices in the spinwave Hamiltonian take the form

$$t = J\begin{pmatrix} -\cos^2 2\theta & 0 \\ 0 & -\cos 2\theta \end{pmatrix}$$

$$\Delta = J\begin{pmatrix} -\cos^2 2\theta & 0 \\ 0 & 1 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} 4zJ\sin^2 2\theta + D\cos 2\theta & 0 \\ 0 & 2zJ\sin^2 2\theta + D\cos^2\theta \end{pmatrix}$$

The XY-AFM state

Transforming to the momentum representation, the energy matrices turn out to have simple diagonal forms

$$\mathcal{E}_{\mathbf{k}} = \begin{pmatrix} 4zJ\sin^2 2\theta + D\cos 2\theta - 2J\eta_{\mathbf{k}}\cos^2 2\theta & 0 \\ 0 & 2zJ\sin^2 2\theta + D\cos^2\theta - 2J\eta_{\mathbf{k}}\cos 2\theta \end{pmatrix}$$

$$\Gamma_{\mathbf{k}} = -2J\eta_{\mathbf{k}}\begin{pmatrix} \cos^2 2\theta & 0 \\ 0 & -1 \end{pmatrix}, \qquad \eta_{\mathbf{k}} = \sum_{\nu}\cos k_{\nu}$$

The dispersion for the two modes are given by

$$\omega_{\mathbf{k}1}^2 = (4zJ\sin^2 2\theta + D\cos 2\theta)(4zJ\sin^2 2\theta + D\cos 2\theta - 4J\eta_{\mathbf{k}}\cos^2 2\theta)$$

$$\omega_{\mathbf{k}2}^2 = (2zJ\sin^2 2\theta + D\cos^2\theta - 4J\eta_{\mathbf{k}}\cos^2\theta)(2zJ\sin^2 2\theta + D\cos^2\theta + 4J\eta_{\mathbf{k}}\sin^2\theta)$$

In the low energy limit $k \to 0$

$$\omega_{\mathbf{k}1} \approx \sqrt{D_c^2 - D^2} + \frac{D^2}{4z\sqrt{D_c^2 - D^2}}k^2$$

$$\omega_{\mathbf{k}2} \approx \sqrt{J(D_c + D)}k$$