Organizing Principles for Understanding Matter

Symmetry

- Conceptual simplification
- Conservation laws
- Distinguish phases of matter by pattern of broken symmetries

symmetry group p4

symmetry group p31m

Topology

- Properties insensitive to smooth deformation
- Quantized topological numbers
- Distinguish *topological* phases of matter

genus = 0

genus = 1

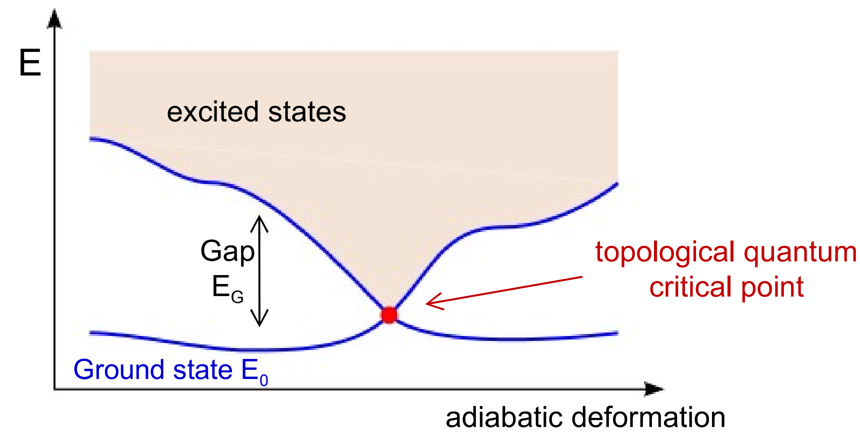
Interplay between symmetry and topology has led to a new understanding of electronic phases of matter.

Topology and Quantum Phases

Topological Equivalence : Principle of Adiabatic Continuity

Quantum phases with an energy gap are topologically equivalent if they can be smoothly deformed into one another without closing the gap.

Topologically distinct phases are separated by quantum phase transition.

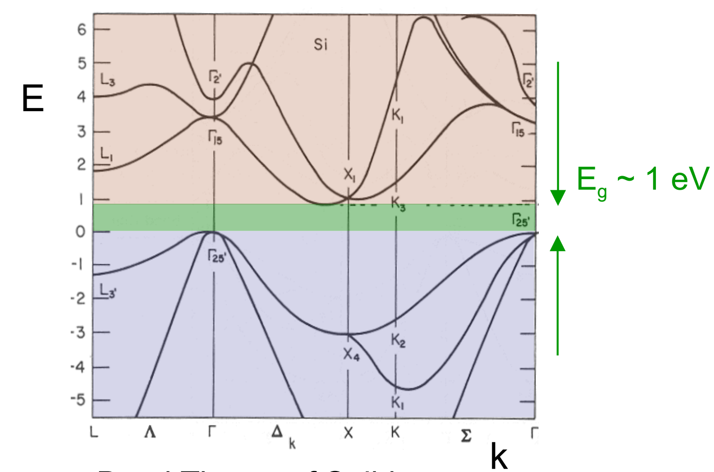


Topological Band Theory

Describe states that are adiabatically connected to non interacting fermions

Classify single particle Bloch band structures

$H(\mathbf{k})$: Brillouin zone (torus) $\mapsto$ Bloch Hamiltonans with energy gap



Band Theory of Solids
e.g. Silicon

Topological Electronic Phases

Many examples of topological band phenomena

States adiabatically connected to independent electrons:

- Quantum Hall (Chern) insulators
- Topological insulators
- Weak topological insulators
- Topological crystalline insulators
- Topological (Fermi, Weyl and Dirac) semimetals …..

Many real materials and experiments

Topological Superconductivity

Proximity induced topological superconductivity

Majorana bound states, quantum information

Tantalizing recent experimental progress

Beyond Band Theory: Strongly correlated states

State with intrinsic topological order

- fractional quantum numbers
- topological ground state degeneracy
- quantum information

- Symmetry protected topological states
- Surface topological order ……

Much recent conceptual progress, but theory is still far from the real electrons

Topological Band Theory

I. Introduction
- Insulating State, Topology and Band Theory

II. Band Topology in One Dimension
- Berry phase and electric polarization
- Su Schrieffer Heeger model :
  domain wall states and Jackiw Rebbi problem
- Thouless Charge Pump

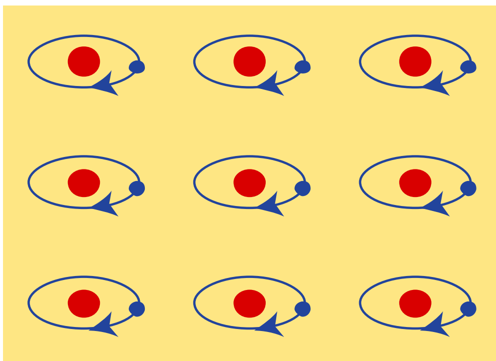
III. Band Topology in Two Dimensions
- Integer quantum Hall effect
- TKNN invariant
- Edge States, chiral Dirac fermions

IV. Generalizations
- 3D Quantum Hall Effect
- Topological Defects
- Weyl Semimetal

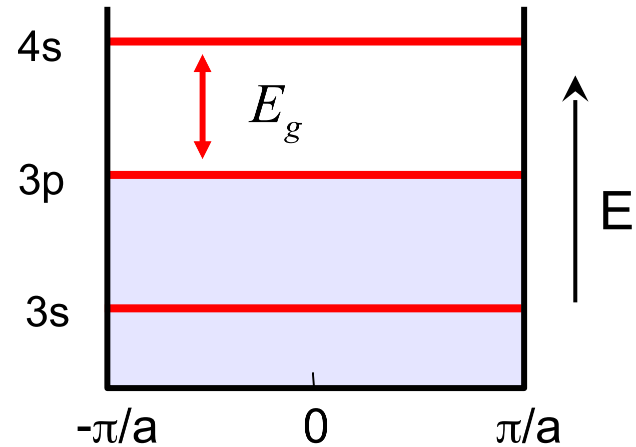
Insulator vs Quantum Hall state

The Insulating State

atomic insulator

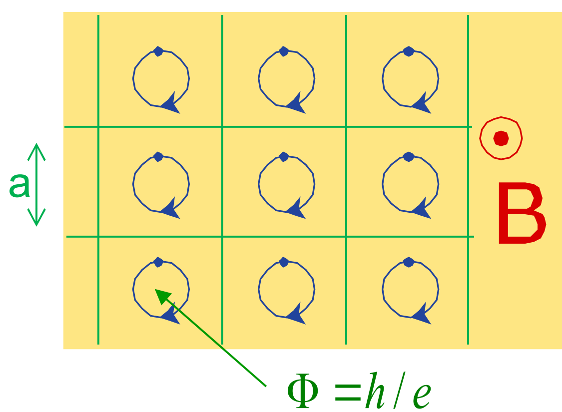


atomic energy levels

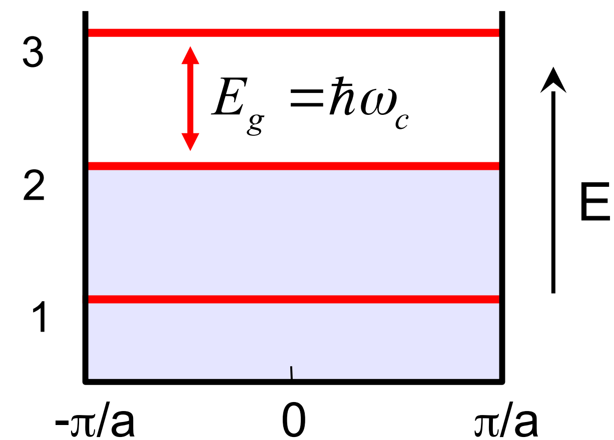


The Integer Quantum Hall State

2D Cyclotron Motion, $\sigma_{xy} = e^2/h$



Landau levels



What's the difference?

Distinguished by Topological Invariant

Topology

The study of geometrical properties that are insensitive to smooth deformations

Example: 2D surfaces in 3D

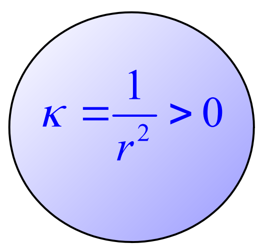
A closed surface is characterized by its genus, g = # holes

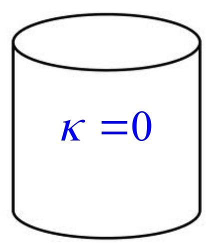
g=0

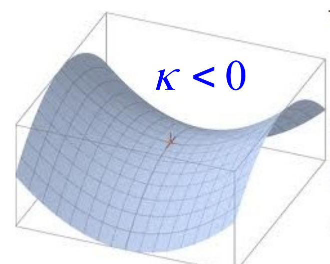
g=1

g is an integer topological invariant that can be expressed in terms of the gaussian curvature κ that characterizes the local radii of curvature

$$\kappa = \frac{1}{r_1 r_2}$$







Gauss Bonnet Theorem : $\int_S \kappa \, dA = 4\pi(1 - g)$

A good math book : Nakahara, 'Geometry, Topology and Physics'

Band Theory of Solids

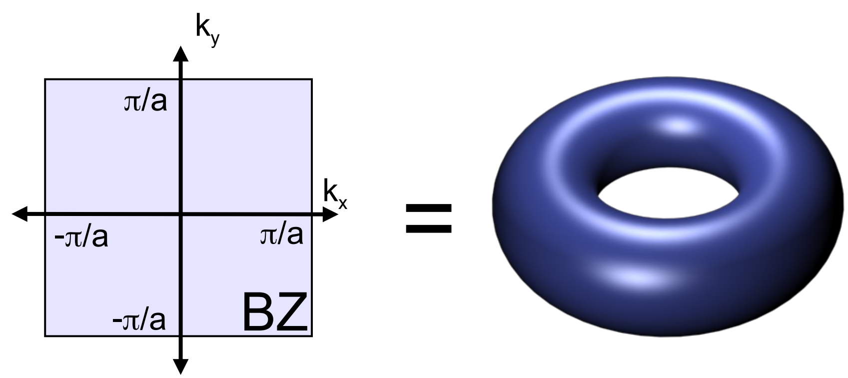
Bloch Theorem :

Lattice translation symmetry $T(\mathbf{R})|\psi\rangle = e^{i\mathbf{k}\cdot\mathbf{R}}|\psi\rangle$ $\qquad |\psi\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}|u(\mathbf{k})\rangle$

Bloch Hamiltonian $H(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{r}} \mathrm{H} e^{i\mathbf{k}\cdot\mathbf{r}}$ $\qquad H(\mathbf{k})|u_n(\mathbf{k})\rangle = E_n(\mathbf{k})|u_n(\mathbf{k})\rangle$

$\mathbf{k} \in$ Brillouin Zone

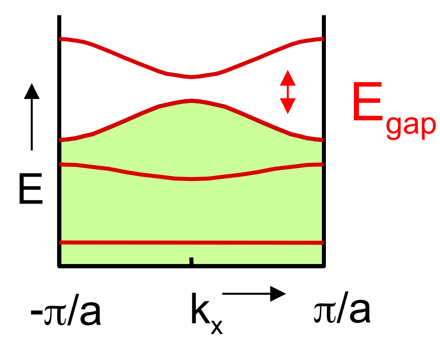
$=$ Torus, T^d



Band Structure :

A mapping $\mathbf{k} \mapsto H(\mathbf{k})$

(or equivalently to $E_n(\mathbf{k})$ and $|u_n(\mathbf{k})\rangle$)



Berry Phase

Phase ambiguity of quantum mechanical wave function

$$|u(\mathbf{k})\rangle \rightarrow e^{i\phi(\mathbf{k})}|u(\mathbf{k})\rangle$$

Berry connection : like a vector potential $\mathbf{A} = -i\langle u(\mathbf{k})|\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla_{\mathbf{k}}\phi(\mathbf{k})$$

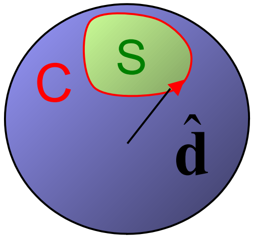
Berry phase : change in phase on a closed loop C $\gamma_C = \oint_C \mathbf{A}\cdot d\mathbf{k}$

Berry curvature : $\mathbf{F} = \nabla_{\mathbf{k}} \times \mathbf{A}$ $\gamma_C = \int_S \mathbf{F}\, d^2k$

Famous example : eigenstates of 2 level Hamiltonian

$$H(\mathbf{k}) = \mathbf{d}(\mathbf{k})\cdot\vec{\sigma} = \begin{pmatrix} d_z & d_x - id_y \\ d_x + id_y & -d_z \end{pmatrix}$$

$$H(\mathbf{k})|u(\mathbf{k})\rangle = +|\mathbf{d}(\mathbf{k})||u(\mathbf{k})\rangle$$



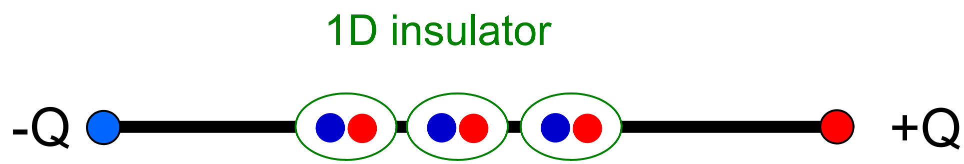
$$\gamma_C = \frac{1}{2}\left(\text{Solid Angle swept out by } \hat{\mathbf{d}}(\mathbf{k})\right)$$

Topology in one dimension : Berry phase and electric polarization

see, e.g. Resta, RMP 66, 899 (1994)

Classical electric polarization :

$$P = \frac{\text{dipole moment}}{\text{length}}$$



Bound charge density $\rho_{bound} = \nabla \cdot P$

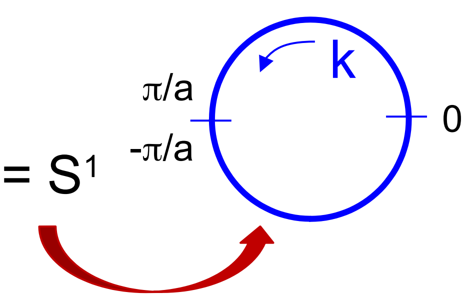
End charge $Q_{end} = P \cdot \hat{n}$

Proposition: The quantum polarization is a Berry phase

$$P = \frac{e}{2\pi} \oint_{BZ} A(k)\,dk$$

$$\mathbf{A} = -i \langle u(\mathbf{k}) | \nabla_{\mathbf{k}} | u(\mathbf{k}) \rangle$$

BZ = 1D Brillouin Zone = S^1

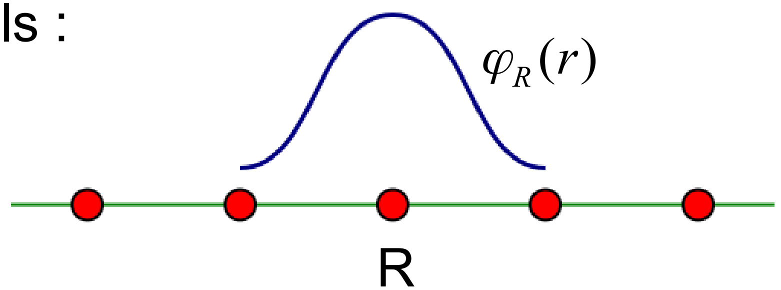


Quantum Polarization

Bloch states $\psi_k(r) = e^{ikr} u_k(r)$ are defined for periodic boundary conditions

Construct Localized Wannier Orbitals :

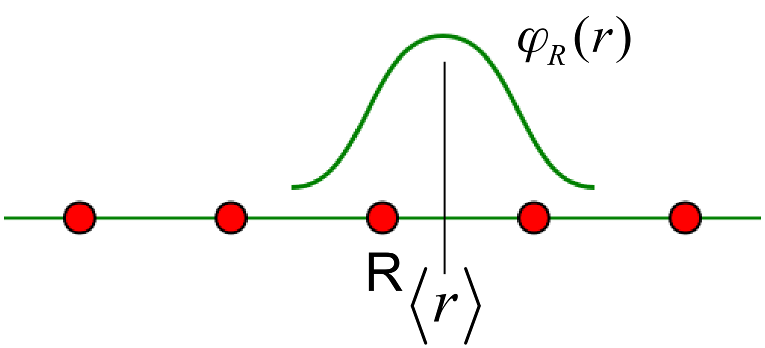
$$|\varphi(R)\rangle = \oint_{BZ} \frac{dk}{2\pi} e^{-ik(R-r)} |u(k)\rangle$$



Wannier states are gauge dependent, but for a sufficiently smooth gauge, they are localized states associated with a Bravais Lattice point R

$$P = e\langle\varphi(R)|r - R|\varphi(R)\rangle$$

$$= \frac{ie}{2\pi} \oint_{BZ} \langle u(k)|\nabla_k|u(k)\rangle$$



Gauge invariance and intrinsic ambiguity of P

- The end charge is not completely determined by the bulk polarization P because integer charges can be added or removed from the ends :

$$Q_{\text{end}} = P \text{ mod } e$$

- The Berry phase is gauge invariant under continuous gauge transformations, but is not gauge invariant under "large" gauge transformations.

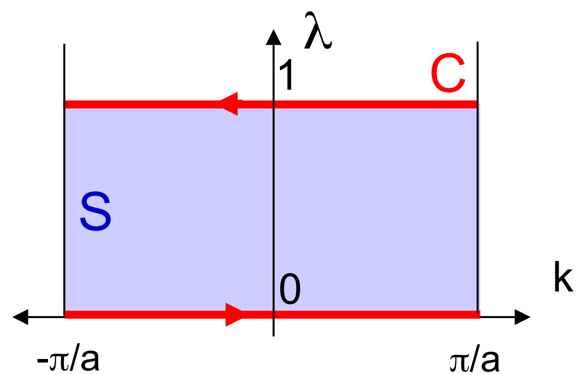
$$P \to P + en \quad \text{when} \quad |u(k)\rangle \to e^{i\phi(k)}|u(k)\rangle \quad \text{with} \quad \phi(\pi/a) - \phi(-\pi/a) = 2\pi n$$

Changes in P, due to adiabatic variation are well defined and gauge invariant

$$|u(k)\rangle \to |u(k,\lambda(t))\rangle$$

$$\Delta P = P_{\lambda=1} - P_{\lambda=0} = \frac{e}{2\pi}\oint_C \mathbf{A}dk = \frac{e}{2\pi}\int_S \mathbf{F}dkd\lambda$$

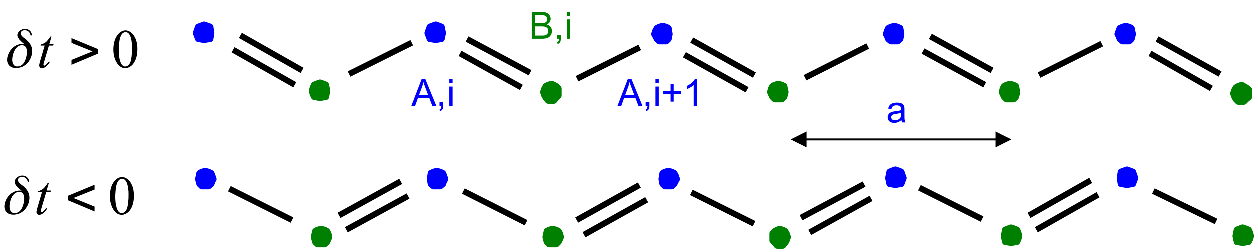
gauge invariant Berry curvature

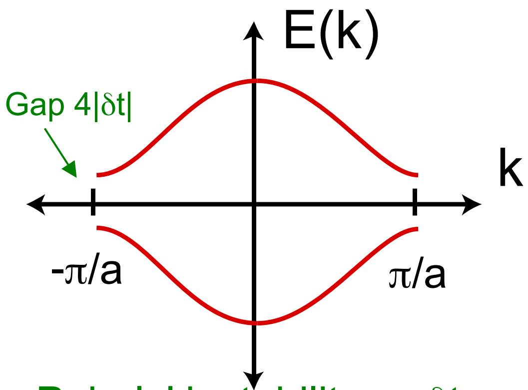


Su Schrieffer Heeger Model

model for polyacetylene
simplest "two band" model

$$H = \sum_i (t+\delta t) c^\dagger_{Ai} c_{Bi} + (t-\delta t) c^\dagger_{Ai+1} c_{Bi} + h.c.$$





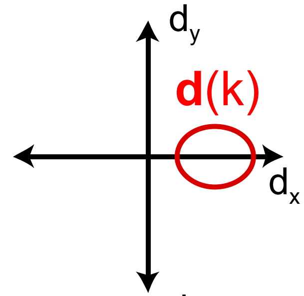
Peierls' instability → δt

$$H(k) = \mathbf{d}(k) \cdot \sigma$$

$$d_x(k) = (t+\delta t) + (t-\delta t)\cos ka$$

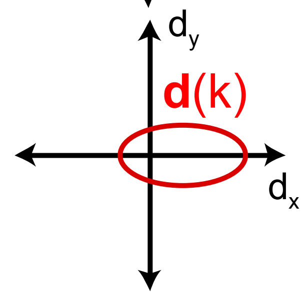
$$d_y(k) = (t-\delta t)\sin ka$$

$$d_z(k) = 0$$



δt>0 : Berry phase 0

P = 0



δt<0 : Berry phase π

P = e/2

Provided symmetry requires $d_z(k)=0$, the states with δt>0 and δt<0 are distinguished by an integer winding number. Without extra symmetry, all 1D band structures are topologically equivalent.

Symmetries of the SSH model

"Chiral" Symmetry : $\{H(k), \sigma_z\} = 0$ (or $\sigma_z H(k) \sigma_z = -H(k)$)

- Artificial symmetry of polyacetylene. Consequence of bipartite lattice with only A-B hopping: $c_{iA} \to c_{iA}$, $c_{iB} \to -c_{iB}$
- Requires $d_z(k)=0$: integer winding number
- Leads to particle-hole symmetric spectrum:

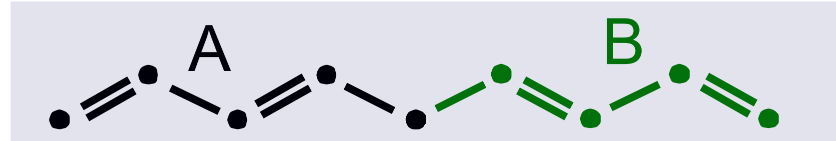
$$H\sigma_z |\psi_E\rangle = -E\sigma_z |\psi_E\rangle \quad \Rightarrow \quad \sigma_z |\psi_E\rangle = |\psi_{-E}\rangle$$

Reflection Symmetry : $H(-k) = \sigma_x H(k) \sigma_x$

- Real symmetry of polyacetylene.
- Allows $d_z(k)\neq 0$, but constrains $d_x(-k)= d_x(k)$, $d_{y,z}(-k)= -d_{y,z}(k)$
- No p-h symmetry, but polarization is quantized: Z_2 invariant

P = 0 or e/2 mod e

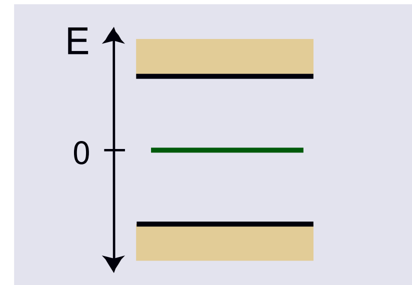
Domain Wall States



Single Particle Spectrum: zero mode

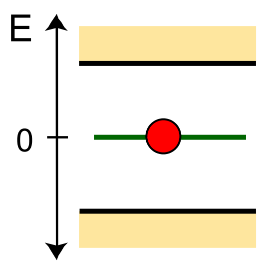
$t - \delta t = 0$: decoupled E=0 state

$t - \delta t > 0$: protected by particle hole symmetry

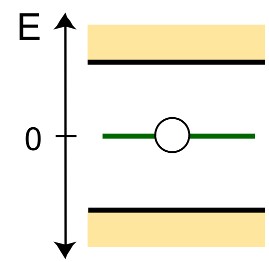


Many body ground state: charge fractionalization

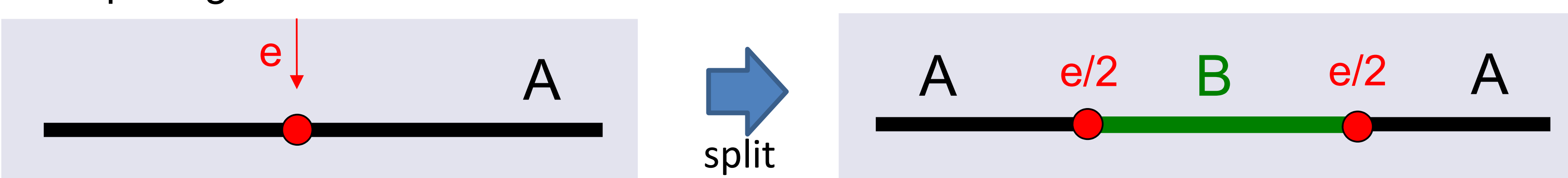
Occupied:
charge +e/2



Empty:
charge -e/2

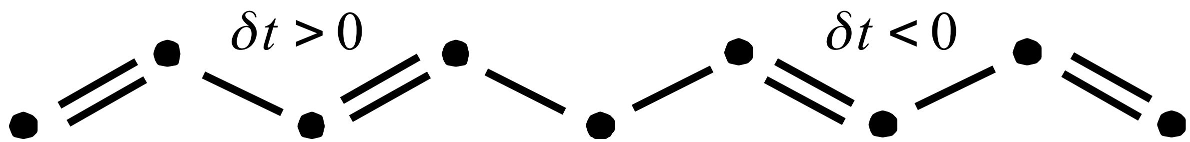


Splitting the electron:



Jackiw Rebbi Problem

(Jackiw and Rebbi 76,
Su Schrieffer, Heeger 79)



Low energy continuum theory for δt << t :

$$k \to \frac{\pi}{a} + q \quad ; \quad q \to -i\partial_x$$

Focus on low energy states with k~π/a

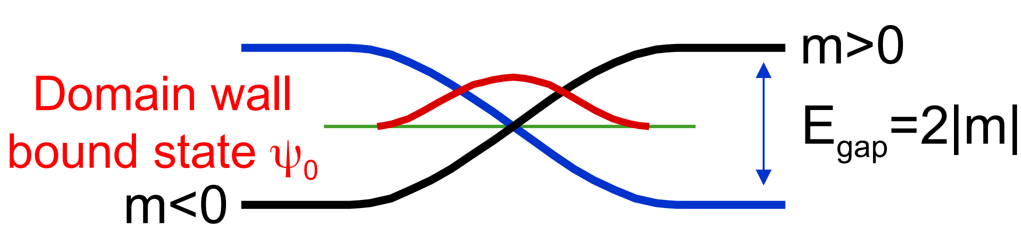
$$H = -i\mathrm{v}_F \sigma_x \partial_x + m(x)\sigma_y \qquad \mathrm{v}_F = ta \quad ; \quad m = 2\delta t$$

Massive 1+1 D Dirac Hamiltonian

$$E(q) = \pm\sqrt{(\mathrm{v}_F q)^2 + m^2}$$

Zero mode at boundary where m(x) changes sign :

Solve $(i/\mathrm{v}_F)\sigma_x H\psi_0 = 0 \quad \to \quad \partial_x \psi_0 = -m(x)\sigma_z \psi_0 / \mathrm{v}_F$

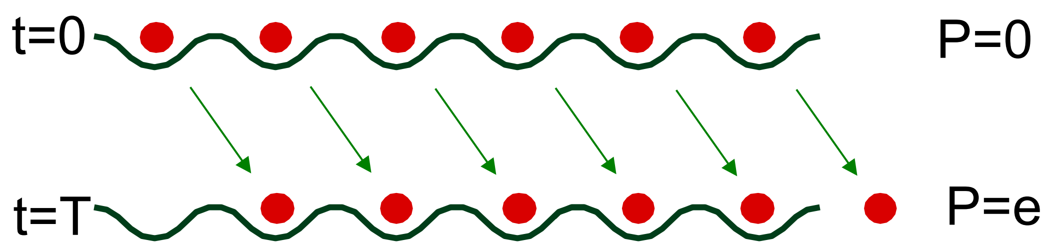


$$\psi_0(x) = e^{-\int_0^x m(x')dx'/\mathrm{v}_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

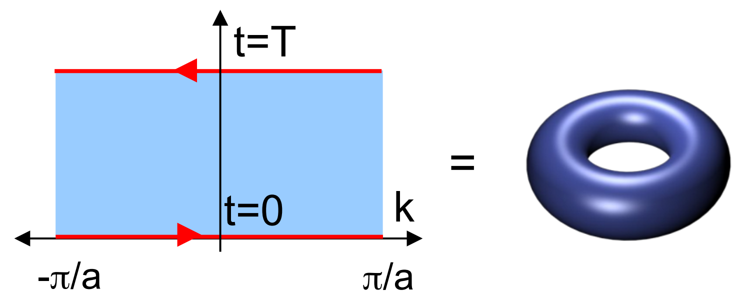
Thouless Charge Pump

The integer charge pumped across a 1D insulator in one period of an adiabatic cycle is a topological invariant that characterizes the cycle.

$$H(k,t+T) = H(k,t)$$



$$\Delta P = \frac{e}{2\pi}\left(\oint A(k,T)dk - \oint A(k,0)dk\right) = ne$$

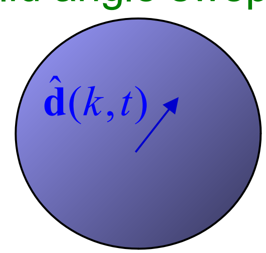


$$n = \frac{1}{2\pi}\int_{T^2} \mathbf{F}\, dk\, dt$$

The integral of the Berry curvature defines the first Chern number, n, an integer topological invariant characterizing the occupied Bloch states, $|u(k,t)\rangle$

In the 2 band model, the Chern number is related to the solid angle swept out by $\hat{\mathbf{d}}(k,t)$, which must wrap around the sphere an integer n times.

$$n = \frac{1}{4\pi}\int_{T^2} dk\, dt\; \hat{\mathbf{d}}\cdot(\partial_k \hat{\mathbf{d}} \times \partial_t \hat{\mathbf{d}})$$

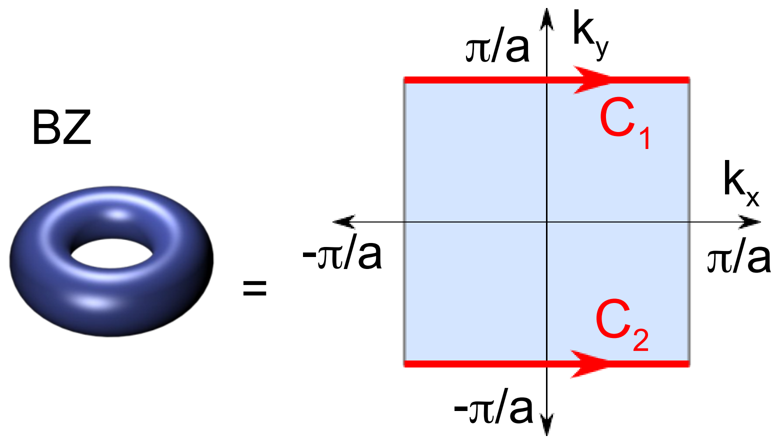


TKNN Invariant

Thouless, Kohmoto, Nightingale and den Nijs 82

For 2D band structure, define $\mathbf{A}(\mathbf{k}) = -i\langle u(\mathbf{k})|\nabla_{\mathbf{k}}|u(\mathbf{k})\rangle$

BZ

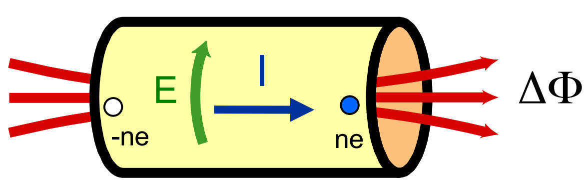


$$n = \frac{1}{2\pi}\oint_{C_1} \mathbf{A}\cdot d\mathbf{k} - \frac{1}{2\pi}\oint_{C_2} \mathbf{A}\cdot d\mathbf{k} \in \mathbb{Z}$$

$$= \frac{1}{2\pi}\int_{BZ} d^2k\,\mathbf{F}(\mathbf{k})$$

Physical meaning: Quantized Hall conductivity $\sigma_{xy} = n\frac{e^2}{h}$

Laughlin argument: Thread flux $\Delta\Phi = h/e$



$Ea = d\Phi / dt$ (Faraday's law)

$I = \sigma_{xy} Ea = \sigma_{xy} d\Phi / dt$

$\Delta P = \int I dt = \sigma_{xy}\Delta\Phi = \sigma_{xy} h / e$

Thouless pump: Cylinder with circumference 1 lattice constant (a)

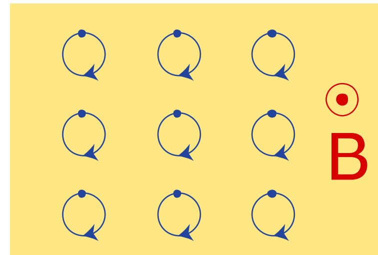
Φ plays role of k_y $\Delta\Phi = h/e \Rightarrow \Delta k_y = 2\pi / a$ $\Delta P = ne$

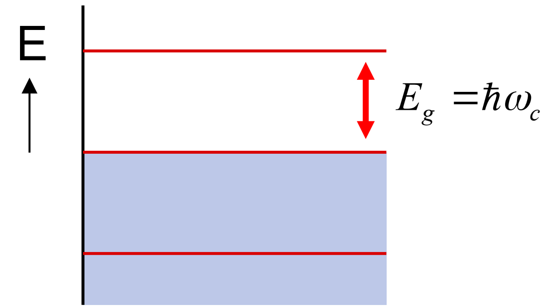
Alternative calculation: compute σ_{xy} via Kubo formula

Realizing a non trivial Chern number

Integer quantum Hall effect:

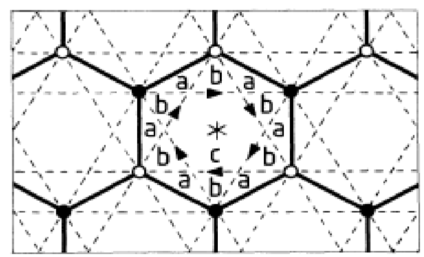
Landau levels

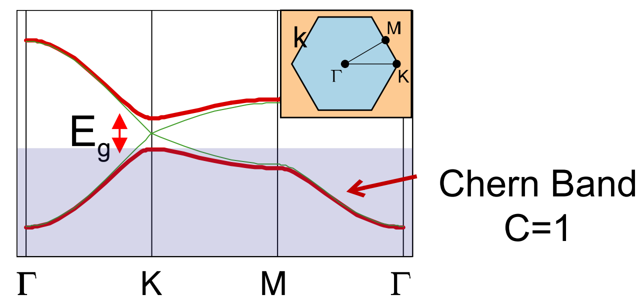




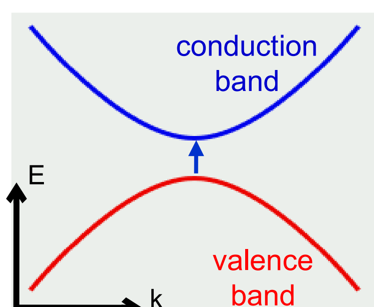
Chern insulator:

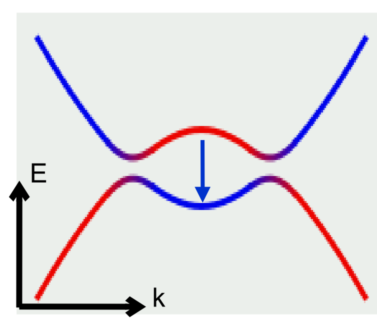
e.g. Haldane model





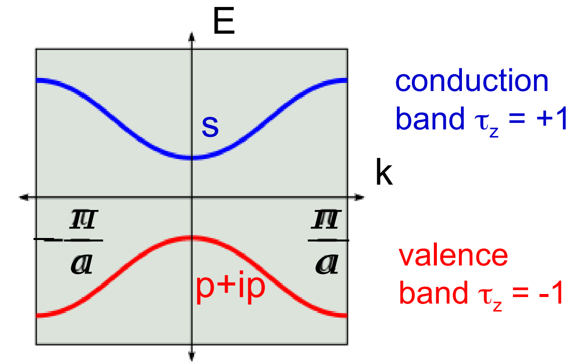
Band Inversion Paradigm





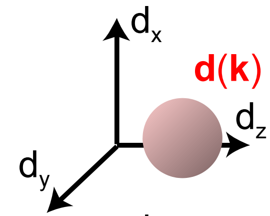
Lattice model for Chern insulator

$$H(\mathbf{k}) = \tau_z\left(2t_0\left[\cos k_x a + \cos k_y a\right] + \Delta E_{sp}\right) + 2t_{sp}\left(\tau_x \sin k_x a + \tau_y \sin k_y a\right) = \mathbf{d}(k)\cdot\vec{\tau}$$



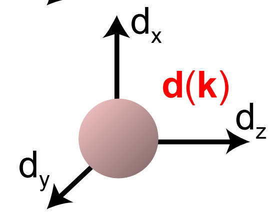
Square lattice model with inversion of bands with s and p_x+ip_y symmetry near Γ

$|\Delta E_{sp}| > 4t$: Uninverted Trivial Insulator



Chern number 0

$|\Delta E_{sp}| < 4t$: Inverted Chern Insulator



Chern number 1

Regularized continuum model for Chern insulator

$$H(\mathbf{k}) = \tau_z\left(m + ak^2\right) + \mathrm{v}\left(k_x\tau_x + k_y\tau_y\right) = \mathbf{d}(k)\cdot\tau$$

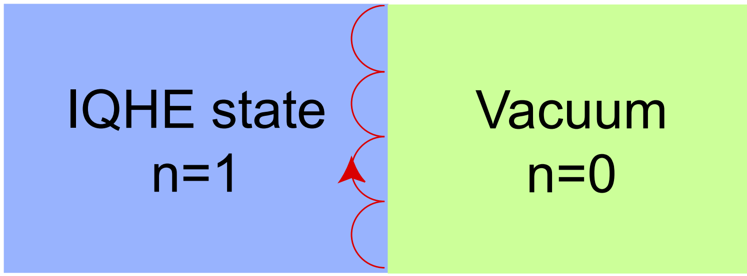
$m = 4t_0 - \Delta E_{sp}$
$a = t_0\, a$
$v = 2t_{sp}\, a$

Inverted near k=0 for m<0. Uninverted for $k \to \infty$

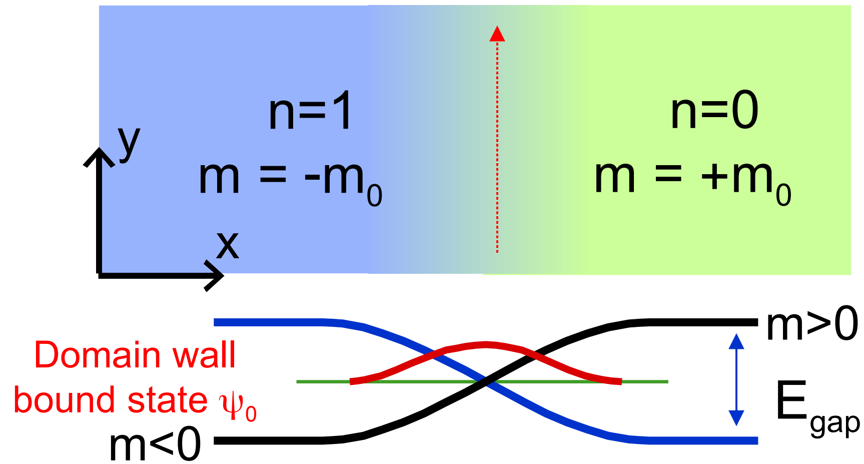
Inverted near k=0 for m<0. Uninverted for

Edge States

Gapless states at the interface between topologically distinct phases



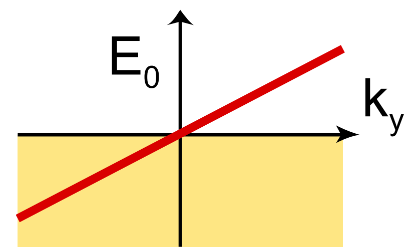
Edge states ~ skipping orbits
Lead to quantized transport



Band inversion transition : Dirac Equation

$$H = v_F(-i\sigma_x\partial_x + \sigma_y k_y) + m(x)\sigma_z$$

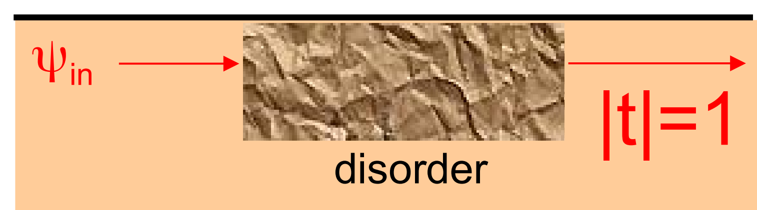
$$\psi_0(x) \sim e^{ik_y y} e^{-\int_0^x m(x')dx'/v_F} \qquad E_0(k_y) = v_F k_y$$



Chiral Dirac Fermions

Chiral Dirac fermions are unique 1D states :

"One way" ballistic transport, responsible for quantized conductance. Insensitive to disorder, impossible to localize



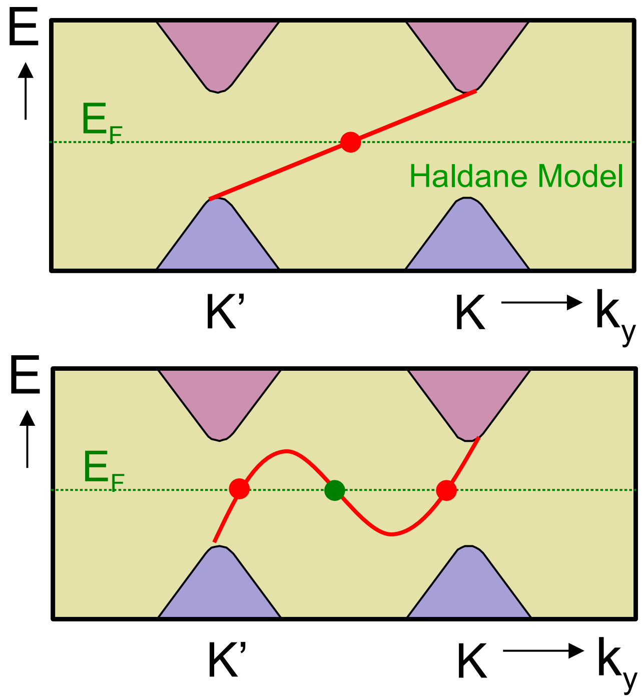
Fermion Doubling Theorem :

Chiral Dirac Fermions can not exist in a purely 1D system.

Bulk - Boundary Correspondence

$\Delta N = N_R - N_L$ is a topological invariant characterizing the boundary.

N_R (N_L) = # Right (Left) moving chiral fermion branches intersecting E_F



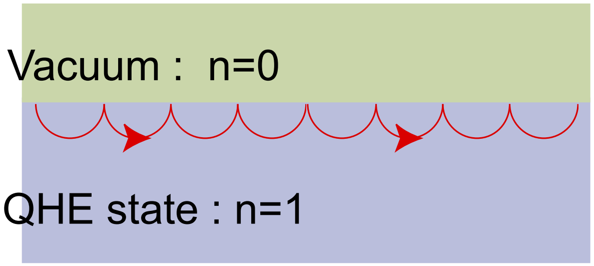
$\Delta N = 1 – 0 = 1$

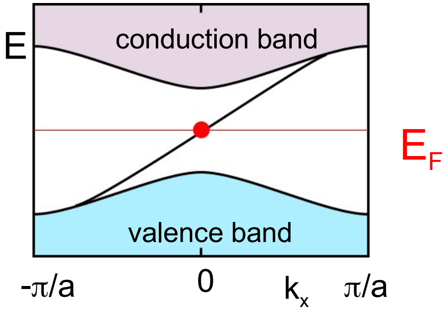
$\Delta N = 2 – 1 = 1$

Bulk – Boundary Correspondence :

The boundary topological invariant ΔN characterizing the gapless modes = Difference in the topological invariants Δn characterizing the bulk on either side

Single particle edge spectrum : "one way edge states"





Many body edge spectrum : "chiral Fermi liquid"

- Free Dirac fermion conformal field theory $H = -iv\psi^\dagger \partial_x \psi$
- Quantized electrical conductance: $G = \frac{dI}{dV} = \nu \frac{e^2}{h}$ $\quad \nu = 1$
- Quantized thermal conductance: $\kappa = \frac{dI_Q}{dT} = c \frac{\pi^2}{3} \frac{k_B^2}{h} T$ $\quad c = 1$ chiral central charge

Chiral Anomaly :

In the presence of an electric field, the charge at the edge is not conserved

$$\frac{dQ_+}{dt} = \frac{e}{2\pi}\frac{dk}{dt} = \frac{e}{2\pi}\frac{eE}{\hbar} = \sigma_{xy} E$$

Generalizations

d=4 : 4 dimensional generalization of IQHE   Zhang, Hu '01

$$\mathbf{A}_{ij} = \langle u_i(\mathbf{k}) | \nabla_{\mathbf{k}} | u_j(\mathbf{k}) \rangle \cdot d\mathbf{k}$$

Non-Abelian Berry connection 1-form

$$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$$

Non-Abelian Berry curvature 2-form

$$n = \frac{1}{8\pi^2} \int_{T^4} \mathrm{Tr}[\mathbf{F} \wedge \mathbf{F}] \in \mathbb{Z}$$

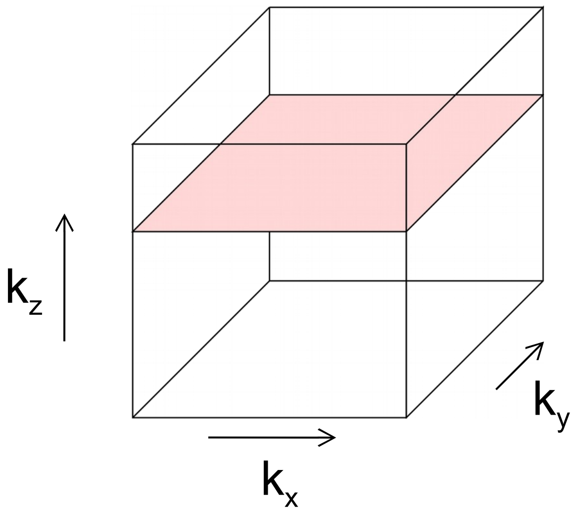
2nd Chern number = integral of 4-form over 4D BZ

Boundary states : 3+1D Chiral Dirac fermions

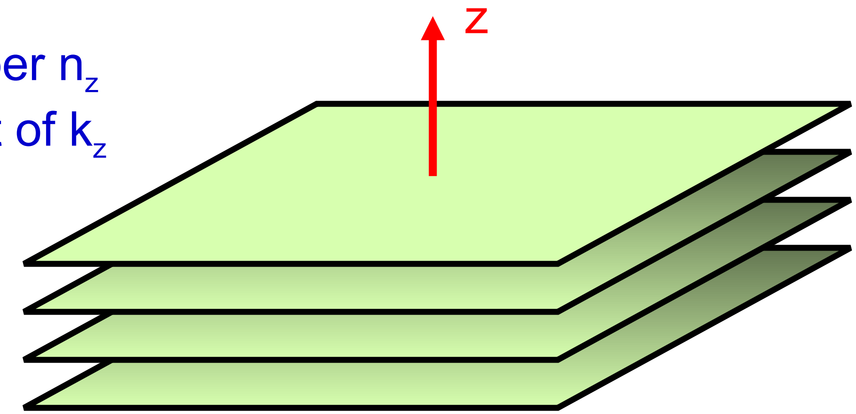
Higher Dimensions : "Bott periodicity" d → d+2

	d							
	1	2	3	4	5	6	7	8
no symmetry	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
chiral symmetry	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

3D Quantum Hall Effect



Chern number n_z
independent of k_z



More generally, the 3 independent Chern numbers (n_x,n_y,n_z) define a reciprocal lattice vector **G** that characterizes a family of lattice planes.

$$\sigma_{ij} = \frac{e^2}{2\pi h}\varepsilon_{ijk}G_k$$

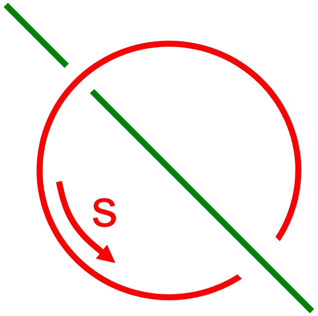
Topological Defects

Consider insulating Bloch Hamiltonians that vary slowly in real space

Teo, Kane '10

$$H = H(\mathbf{k}, s)$$

1 parameter family of 3D Bloch Hamiltonians



2nd Chern number : $n = \frac{1}{8\pi^2} \int_{T^3 \times S^1} \mathrm{Tr}[\mathbf{F} \wedge \mathbf{F}]$

Generalized bulk-boundary correspondence :

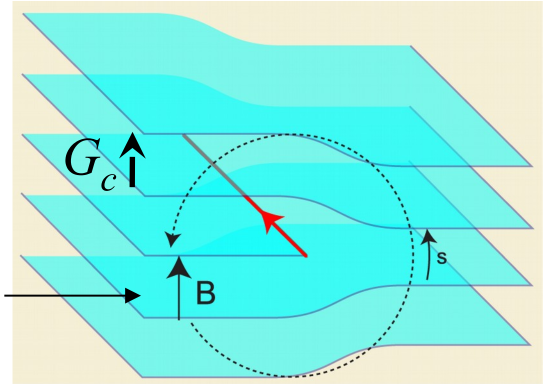
n specifies the number of chiral Dirac fermion modes bound to defect line

Example : dislocation in 3D layered IQHE

$$n = \frac{1}{2\pi} \mathbf{G}_c \cdot \mathbf{B}$$

3D Chern number
(vector ⊥ layers)

Burgers' vector

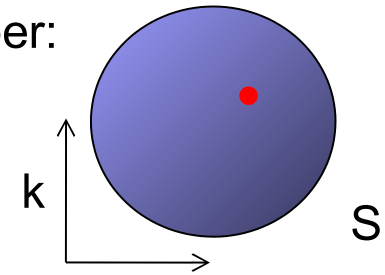


Are there other ways to engineer
1D chiral dirac fermions?

Weyl Semimetal

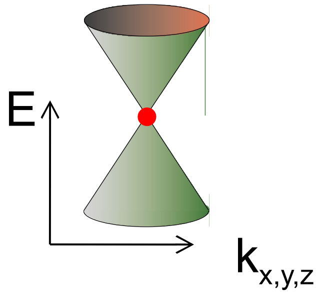
Gapless “Weyl points” in momentum space are topologically protected in 3D

A sphere in momentum space can have a Chern number:



$$n_S = \int_S d^2k \mathbf{F}(\mathbf{k}) \in \mathbb{Z}$$

n_S=+1: S must enclose a degenerate Weyl point:
Magnetic monopole for Berry flux

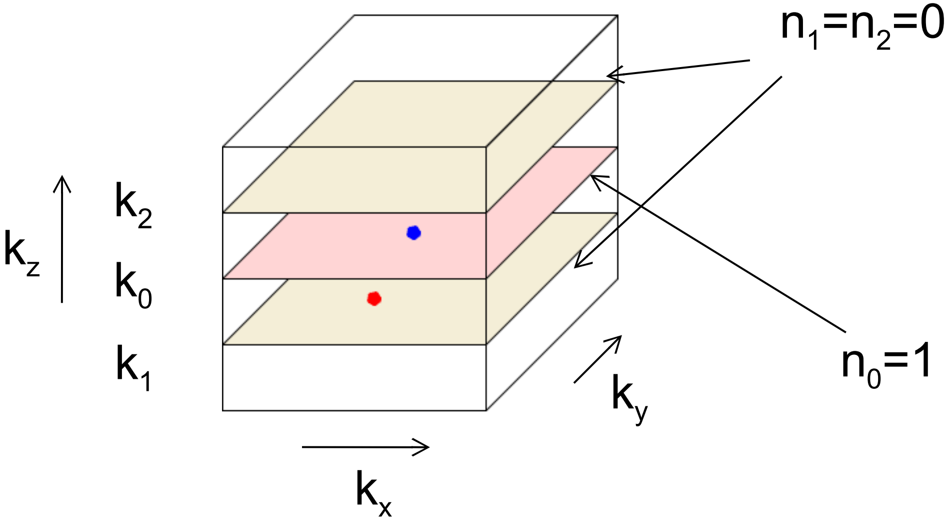


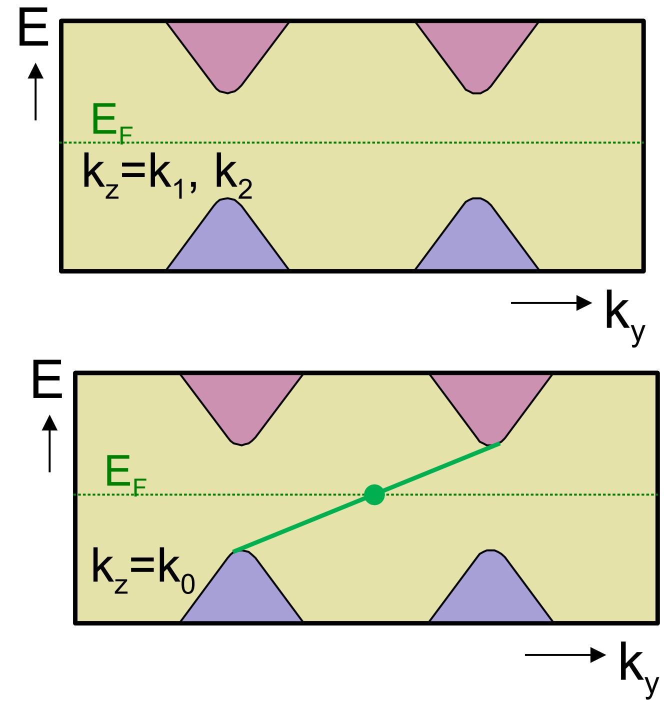
$$H(k_0 + q) = \mathrm{v}(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z)$$

(or $\mathrm{v}_{ia} q_i \sigma_a$ with $\det[\mathrm{v}_{ia}] > 0$)

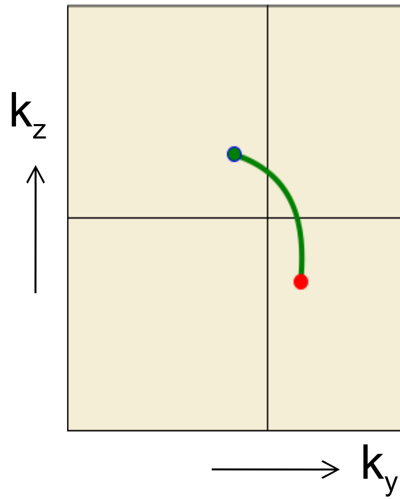
Total magnetic charge in Brillouin zone must be zero: Weyl points must come in +/- pairs.

Surface Fermi Arc





Surface BZ



Chiral Anomaly

In the presence of E and B, the charge at one (or the other) Weyl point is not conserved:

$$\frac{dn_+}{dt} = -\frac{dn_-}{dt} = \frac{e^2}{h^2}\mathbf{E}\cdot\mathbf{B}$$

Topological Band Theory II: Time reversal symmetry

I. Graphene
- Haldane model
- Time reversal symmetry and Kramers' theorem

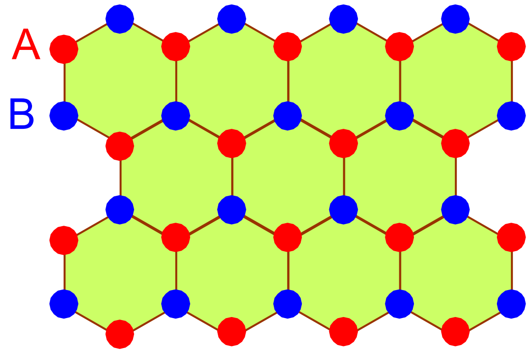
II. 2D quantum spin Hall insulator
- Z_2 topological invariant
- Edge states
- HgCdTe quantum wells, expts

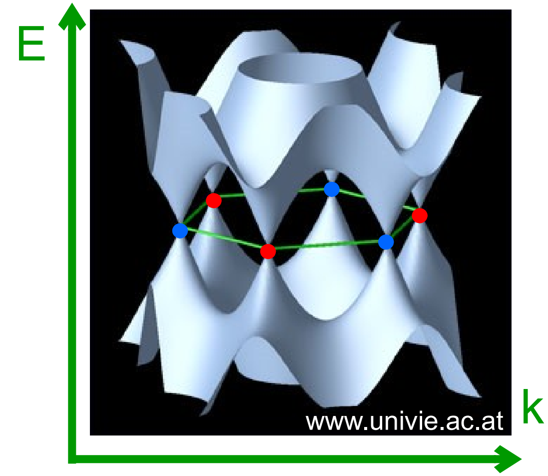
III. Topological Insulators in 3D
- Weak vs strong
- Topological invariants from band structure

IV. The surface of a topological insulator
- Dirac Fermions
- Absence of backscattering and localization
- Quantum Hall effect
- θ term and topological magnetoelectric effect

Graphene

Novoselov et al. '05



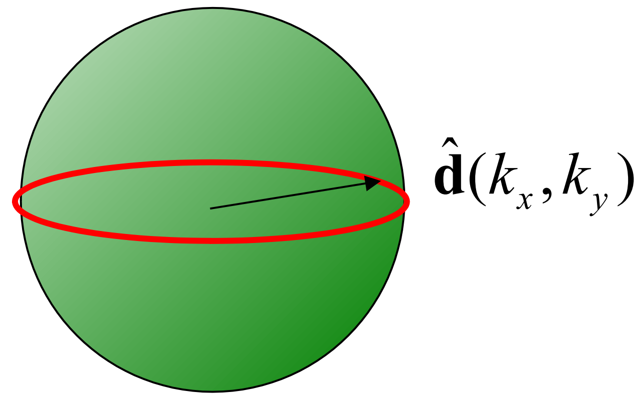


Two band model $H = -t \sum_{<ij>} c^{\dagger}_{Ai} c_{Bj}$

$$H(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \sigma$$

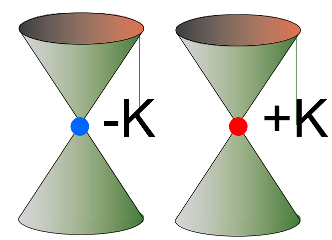
$$E(\mathbf{k}) = \pm |\mathbf{d}(\mathbf{k})|$$

$$\mathbf{d}(\mathbf{k}) = \sum_{j=1}^{3} -t\left(\hat{x}\cos\mathbf{k}\cdot\mathbf{r}_j + \hat{y}\sin\mathbf{k}\cdot\mathbf{r}_j\right)$$



Inversion and Time reversal symmetry require $d_z(\mathbf{k}) = 0$

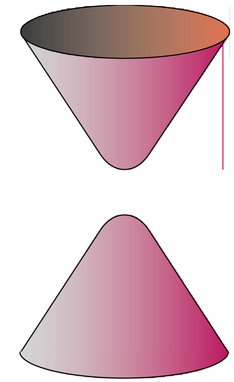
2D Dirac points at $\mathbf{k} = \pm\mathbf{K}$ point vortices in (d_x, d_y)



$$H(\pm\mathbf{K} + \mathbf{q}) = v\,\sigma \cdot \mathbf{q}$$ Massless Dirac Hamiltonian

Berry's phase π around Dirac point

Topological gapped phases in Graphene



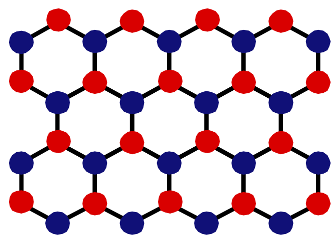
Break P or T symmetry : $H(\pm\mathbf{K}+\mathbf{q}) = v\mathbf{q}\cdot\sigma + m_{\pm}\sigma_z$

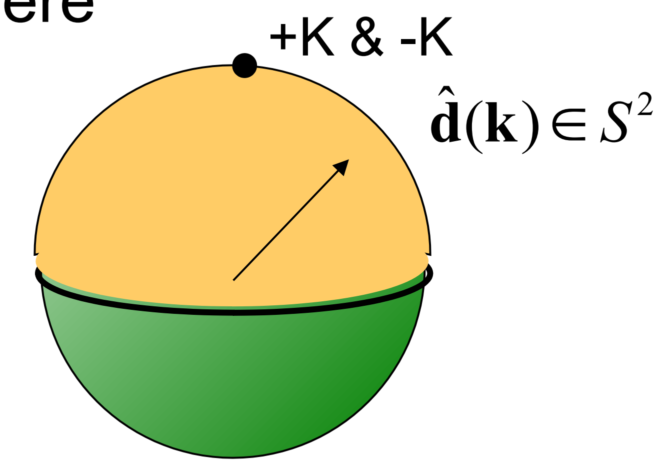
$$E(\mathbf{q}) = \pm\sqrt{v^2|\mathbf{q}|^2 + m_{\pm}^2}$$

$n = \#$ times $\hat{\mathbf{d}}(\mathbf{k})$ wraps around sphere

1. Broken P : eg Boron Nitride

$m_+ = m_-$

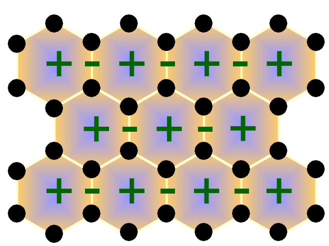


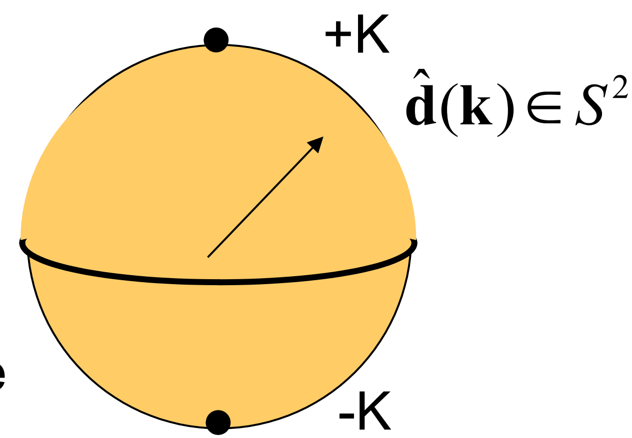


Chern number n=0 : Trivial Insulator

2. Broken T : Haldane Model '88

$m_+ = - m_-$





Chern number n=1 : Quantum Hall state

Energy gaps in graphene:

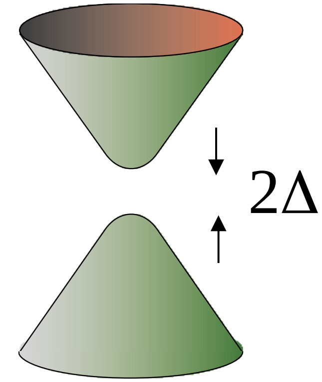
$\sigma_z \sim$ sublattice

$\tau_z \sim$ valley

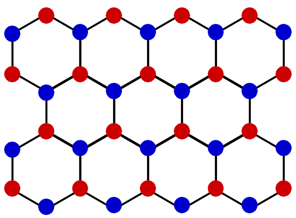
$s_z \sim$ spin

$$H = v_F \sigma \cdot p + V$$

$$E(p) = \pm\sqrt{v_F^2 p^2 + \Delta^2}$$



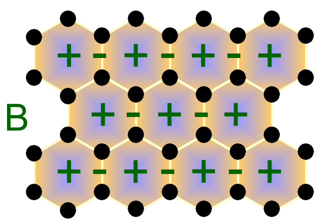
1. Staggered Sublattice Potential (e.g. BN)



$$V = \Delta_{CDW} \sigma^z$$

Broken Inversion Symmetry

2. Periodic Magnetic Field with no net flux (Haldane PRL '88)



$$V = \Delta_{\text{Haldane}} \sigma^z \tau^z$$

Broken Time Reversal Symmetry

Quantized Hall Effect $\sigma_{xy} = \text{sgn}\,\Delta \frac{e^2}{h}$

3. Intrinsic Spin Orbit Potential

$$V = \Delta_{SO} \sigma^z \tau^z s^z$$

Respects ALL symmetries

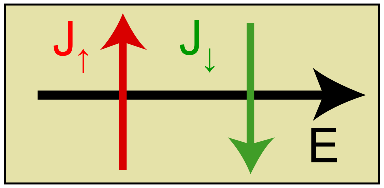
Quantum Spin-Hall Effect

Quantum Spin Hall Effect in Graphene

The intrinsic spin orbit interaction leads to a small (~10mK-1K) energy gap

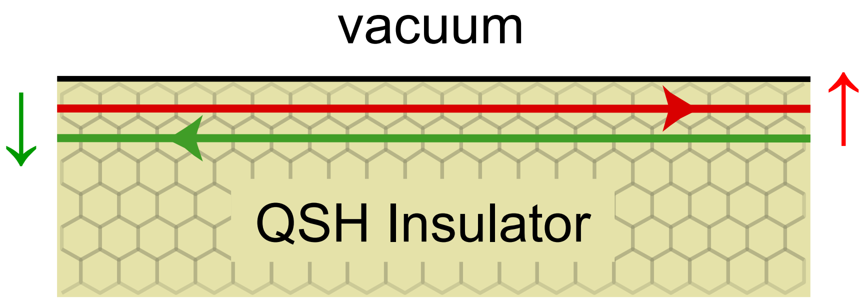
Simplest model:
|Haldane|^2
(conserves S_z)

$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix} = \begin{pmatrix} H_{\text{Haldane}} & 0 \\ 0 & H^*_{\text{Haldane}} \end{pmatrix}$$

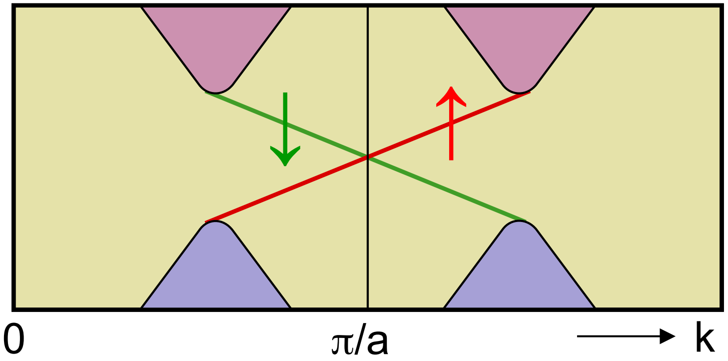


Bulk energy gap, but gapless edge states

“Spin Filtered” or “helical” edge states



Edge band structure



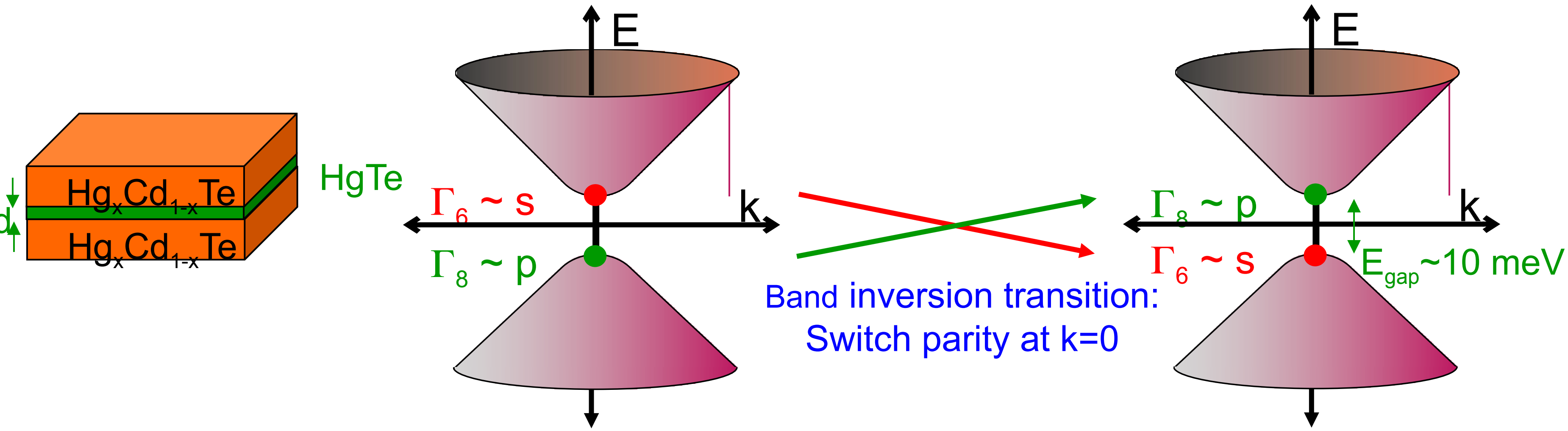
Edge states form a unique 1D electronic conductor

- HALF an ordinary 1D electron gas
- Protected by Time Reversal Symmetry

Quantum Spin Hall Effect in HgTe quantum wells

Theory: Bernevig, Hughes and Zhang, Science '06

d < 6.3 nm : Normal band order d > 6.3 nm : Inverted band order



Conventional Insulator

$$\prod \xi_{2n}(\Lambda_a) = +1$$

Quantum spin Hall Insulator with topological edge states

$$\prod \xi_{2n}(\Lambda_a) = -1$$

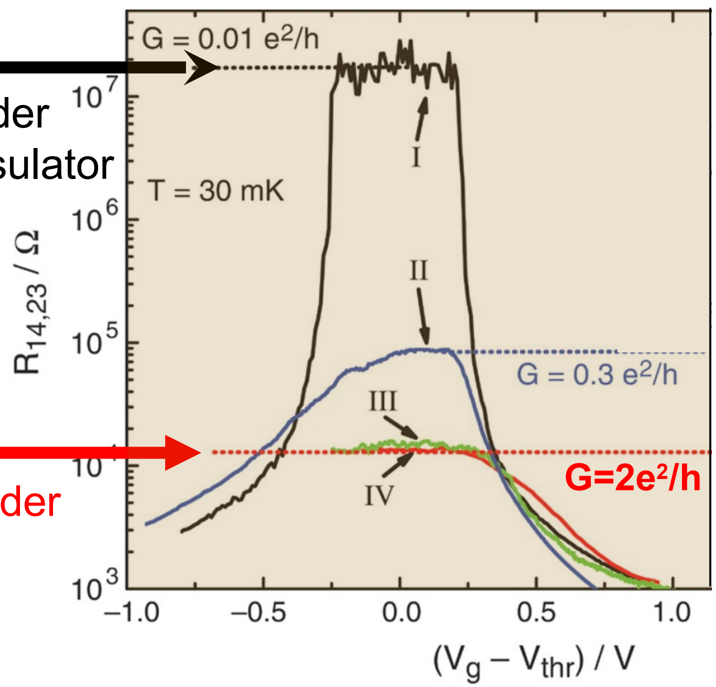
BHZ Model : 4 band T-invariant band inversion model

$$H(\mathbf{k}) = \tau_z\left(m + ak^2\right) + v\left(k_x\tau_x\sigma_x + k_y\tau_x\sigma_y\right)$$

Experiments on HgCdTe quantum wells

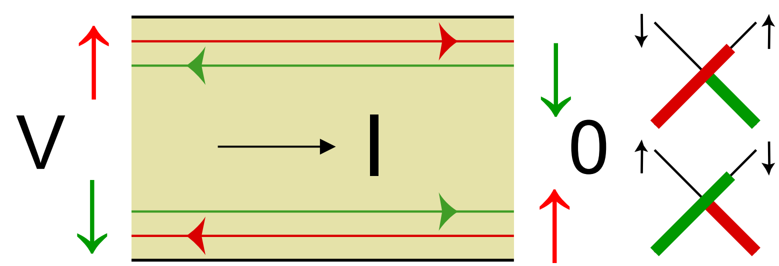
Expt: Konig, Wiedmann, Brune, Roth, Buhmann, Molenkamp, Qi, Zhang Science 2007

d< 6.3 nm
normal band order
conventional insulator



d> 6.3nm
inverted band order
QSH insulator

Landauer Conductance $G=2e^2/h$



Measured conductance $2e^2/h$ independent of W for short samples ($L<L_{in}$)

Time Reversal Symmetry : $[H,\Theta]=0$

Anti Unitary time reversal operator : $\Theta\psi = e^{i\pi S^y/\hbar}\psi^*$

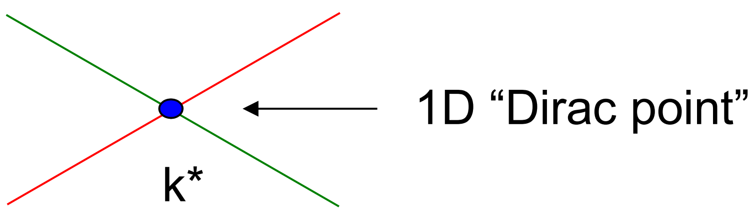
Spin ½ : $\Theta\begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix} = \begin{pmatrix}\psi^*_\downarrow \\ -\psi^*_\uparrow\end{pmatrix}$ $\qquad \Theta^2 = -1$

Kramers' Theorem: for spin ½ all eigenstates are at least 2 fold degenerate

Proof : for a non degenerate eigenstate $\Theta|\chi\rangle = c|\chi\rangle$, $\Theta^2|\chi\rangle = |c|^2|\chi\rangle$ $\qquad \Theta^2 = |c|^2 \neq -1$

Consequences for edge states :

States at "time reversal invariant momenta" k*=0 and k*=π/a (=-π/a) are degenerate.



The crossing of the edge states is protected, even if spin conservation is volated.

Absence of backscattering, even for strong disorder. No Anderson localization

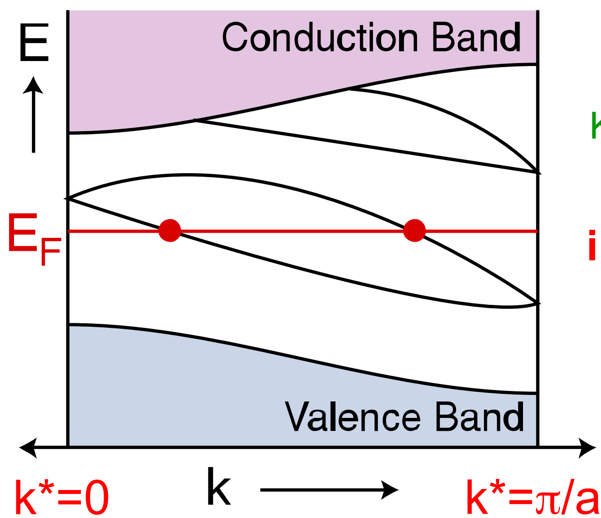
Time Reversal Invariant $\mathbb{Z}_2$ Topological Insulator

2D Bloch Hamiltonians subject to the T constraint $\Theta H(\mathbf{k})\Theta^{-1} = H(-\mathbf{k})$

with $\Theta^2 = -1$ are classified by a $\mathbb{Z}_2$ topological invariant ($\nu = 0,1$)

Understand via Bulk-Boundary correspondence : Edge States for $0<k<\pi/a$

$\nu=0$: Conventional Insulator

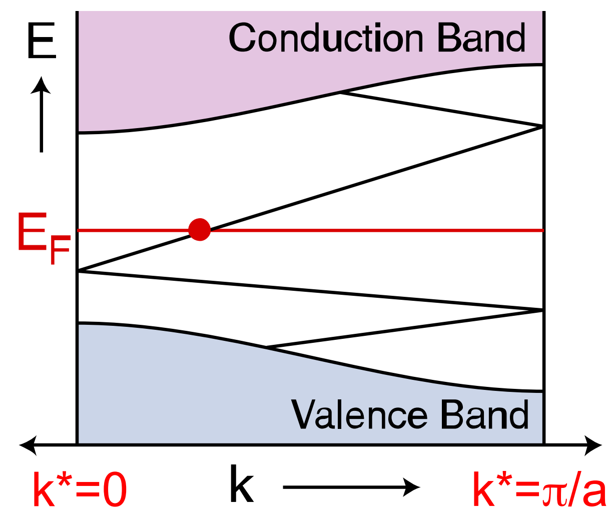


Even number of bands crossing Fermi energy

Kramers degenerate at **time reversal invariant momenta**

k* = -k* + G

$\nu=1$: Topological Insulator

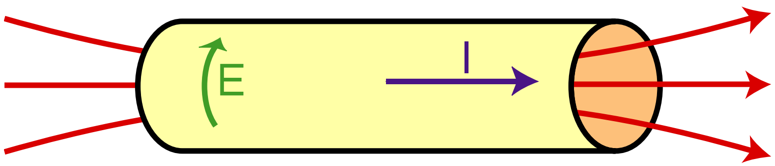


Odd number of bands crossing Fermi energy

Physical Meaning of $\mathbb{Z}_2$ Invariant

Sensitivity to boundary conditions in a multiply connected geometry

ν=N IQHE on cylinder: Laughlin Argument

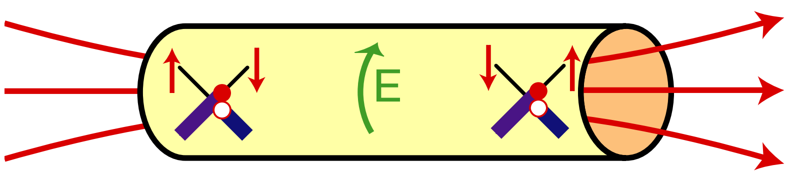


$\Delta\Phi = \phi_0 = h/e$

$\Delta Q = N e$

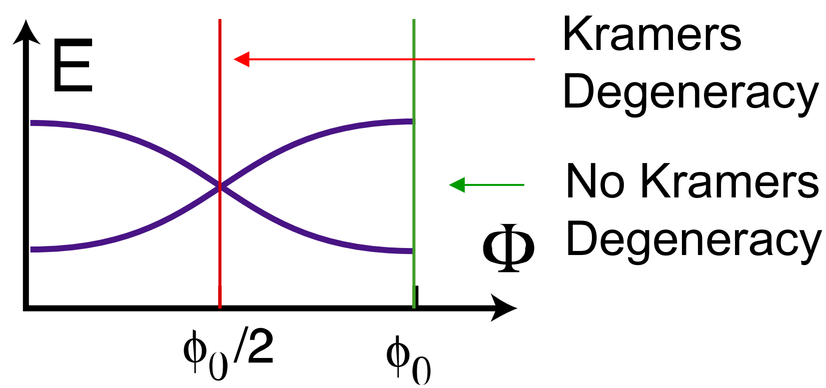
Flux $\phi_0 \Rightarrow$ Quantized change in Electron Number at the end.

Quantum Spin Hall Effect on cylinder



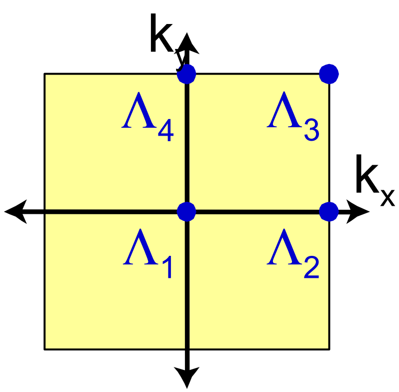
$\Delta\Phi = \phi_0 / 2$

Flux ϕ_0 /2 $\Rightarrow$ Change in Electron Number Parity at the end, signaling change in Kramers degeneracy.



Formula for the $\mathbb{Z}_2$ invariant

- Bloch wavefunctions : $|u_n(\mathbf{k})\rangle$ (N occupied bands)
- T - Reversal Matrix : $w_{mn}(\mathbf{k}) = \langle u_m(\mathbf{k})|\Theta|u_n(-\mathbf{k})\rangle \in U(\mathrm{N})$
- Antisymmetry property : $\Theta^2 = -1 \Rightarrow w(\mathbf{k}) = -w^T(-\mathbf{k})$
- T - invariant momenta : $\mathbf{k} = \Lambda_a = -\Lambda_a \Rightarrow w(\Lambda_a) = -w^T(\Lambda_a)$



Bulk 2D Brillouin Zone

- Pfaffian : $\det[w(\Lambda_a)] = \left(\mathrm{Pf}[w(\Lambda_a)]\right)^2$ e.g. $\det\begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix} = z^2$
- Fixed point parity : $\delta(\Lambda_a) = \dfrac{\mathrm{Pf}[w(\Lambda_a)]}{\sqrt{\det[w(\Lambda_a)]}} = \pm 1$
- Gauge dependent product : $\delta(\Lambda_a)\delta(\Lambda_b)$

  "time reversal polarization" analogous to $\frac{e}{2\pi}\oint A(k)dk$

- Z_2 invariant : $(-1)^\nu = \prod_{a=1}^{4} \delta(\Lambda_a) = \pm 1$

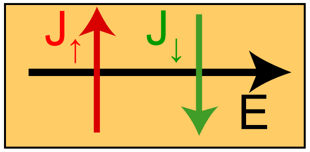
  Gauge invariant, but requires continuous gauge

ν is easier to determine if there is extra symmetry:

1. S_z conserved : independent spin Chern integers :

$$n_\uparrow = - n_\downarrow \quad \text{(due to time reversal)}$$

Quantum spin Hall Effect :



$$\nu = n_{\uparrow,\downarrow} \mod 2$$

2. Inversion (P) Symmetry : determined by Parity of occupied 2D Bloch states

$$P|\psi_n(\Lambda_a)\rangle = \xi_n(\Lambda_a)|\psi_n(\Lambda_a)\rangle$$

$$\xi_n(\Lambda_a) = \pm 1$$

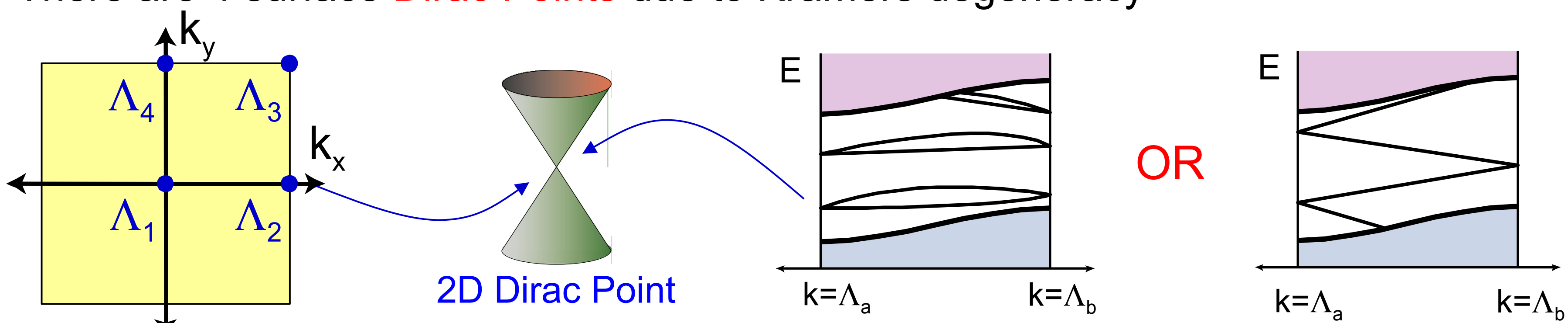
$$(-1)^\nu = \prod_{a=1}^{4} \prod_n \xi_{2n}(\Lambda_a)$$

In a special gauge: $\delta(\Lambda_a) = \prod_n \xi_n(\Lambda_a)$

Allows a straightforward determination of ν from band structure calculations.

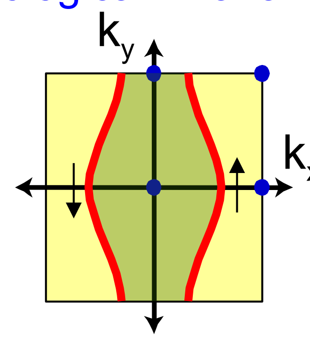
3D Topological Insulators

There are 4 surface Dirac Points due to Kramers degeneracy



Surface Brillouin Zone

How do the Dirac points connect? Determined by 4 bulk Z_2 topological invariants $\nu_0 ; (\nu_1\nu_2\nu_3)$

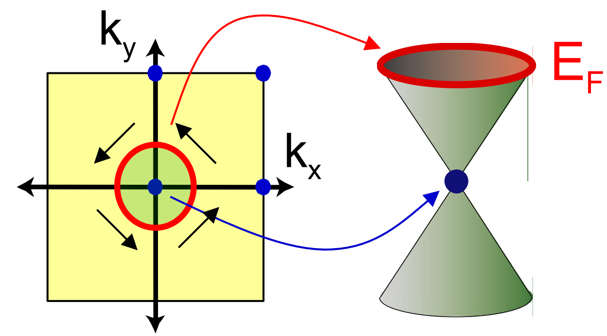


$\nu_0 = 0$: Weak Topological Insulator

Related to layered 2D QSHI ; $(\nu_1\nu_2\nu_3)$ ~ Miller indices

Fermi surface encloses even number of Dirac points

$\nu_0 = 1$: Strong Topological Insulator



Fermi circle encloses odd number of Dirac points

Topological Metal :

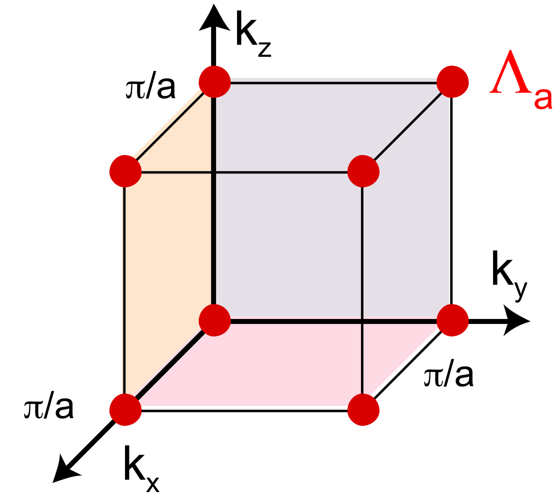
- 1/4 graphene
- Berry's phase π
- Robust to disorder: impossible to localize

Topological Invariants in 3D

1. 2D → 3D : Time reversal invariant planes

The 2D invariant

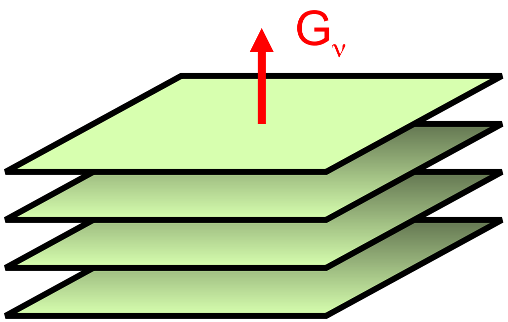
$$(-1)^{\nu} = \prod_{a=1}^{4} \delta(\Lambda_a) \qquad \delta(\Lambda_a) = \frac{\mathrm{Pf}[w(\Lambda_a)]}{\sqrt{\det[w(\Lambda_a)]}}$$



Each of the time reversal invariant planes in the 3D Brillouin zone is characterized by a 2D invariant.

Weak Topological Invariants (vector):

$$(-1)^{\nu_i} = \prod_{a=1}^{4} \delta(\Lambda_a) \Big|_{k_i=0 \text{ plane}} \qquad \mathbf{G}_\nu = \frac{2\pi}{a}(\nu_1, \nu_2, \nu_3)$$



"mod 2" reciprocal lattice vector indexes lattice planes for layered 2D QSHI

Strong Topological Invariant (scalar)

$$(-1)^{\nu_o} = \prod_{a=1}^{8} \delta(\Lambda_a)$$

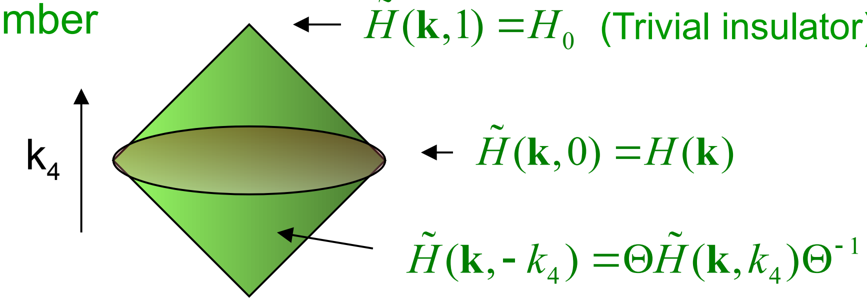
Topological Invariants in 3D

2. 4D → 3D : Dimensional Reduction

Add an extra parameter, k_4, that smoothly connects the topological insulator to a trivial insulator (while breaking time reversal symmetry)

H(**k**,k_4) is characterized by its second Chern number

$$n = \frac{1}{8\pi^2}\int d^4k \,\mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}]$$



n depends on how H(**k**) is connected to H_0, but due to time reversal, the difference must be even.

$$\nu_0 = n \bmod 2$$

Express in terms of Chern Simons 3-form : $\mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}] = dQ_3$

$$\nu_0 = \frac{1}{4\pi^2}\int d^3k\, Q_3(\mathbf{k}) \bmod 2 \qquad Q_3(\mathbf{k}) = \mathrm{Tr}[\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}]$$

Gauge invariant up to an even integer.

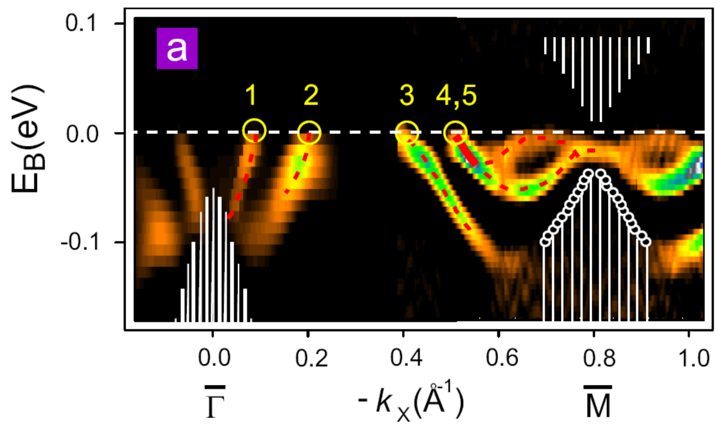
Topological Superconductivity

0. … from last time: The surface of a topological insulator
1. Bogoliubov de Gennes Theory
2. Majorana bound states, Kitaev model
3. Topological superconductor
4. Periodic Table of topological insulators and superconductors
5. Topological quantum computation
6. Proximity effect devices

$Bi_{1-x}Sb_x$

Theory: Predict $Bi_{1-x}Sb_x$ is a topological insulator by exploiting inversion symmetry of pure Bi, Sb (Fu,Kane PRL'07)

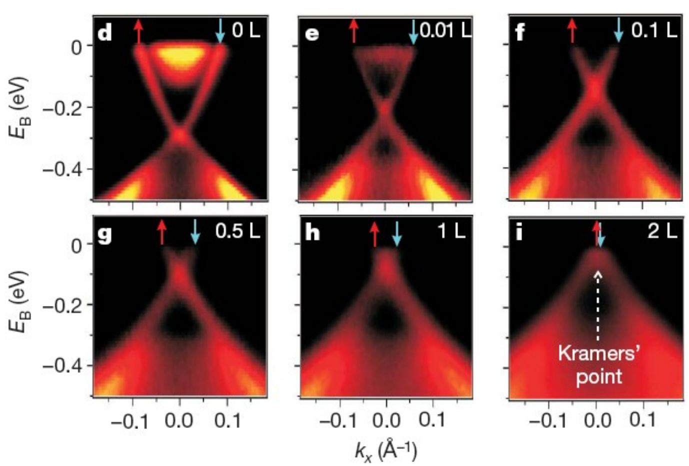
Experiment: ARPES (Hsieh et al. Nature '08)



- $Bi_{1-x}Sb_x$ is a Strong Topological Insulator $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(111)$
- 5 surface state bands cross E_F between Γ and M

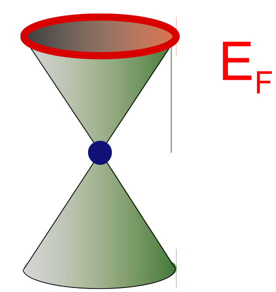
Bi_2Se_3

ARPES Experiment : Y. Xia et al., Nature Phys. (2009).
Band Theory : H. Zhang et. al, Nature Phys. (2009).



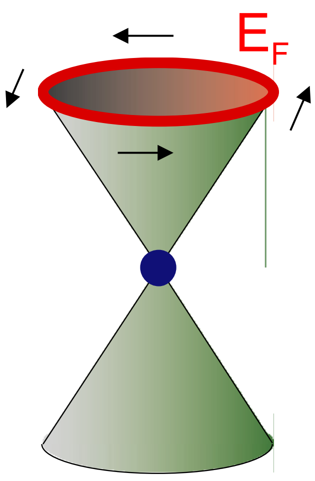
Control E_F on surface by exposing to NO_2

- $\nu_0;(\nu_1,\nu_2,\nu_3) = 1;(000)$: Band inversion at Γ
- Energy gap: $\Delta \sim .3$ eV : A room temperature topological insulator
- Simple surface state structure : Similar to graphene, except only a single Dirac point



Unique Properties of Topological Insulator Surface States

"Half" an ordinary 2DEG ; ¼ Graphene



Spin polarized Fermi surface

- Charge Current ~ Spin Density
- Spin Current ~ Charge Density

π Berry's phase

- Robust to disorder
- Weak Antilocalization
- Impossible to localize, Klein paradox

Broken symmetry can lead to surface energy gap:

- Quantum Hall state, topological magnetoelectric effect (broken Time Reversal)

  Fu, Kane '07; Qi, Hughes, Zhang '08, Essin, Moore, Vanderbilt '09

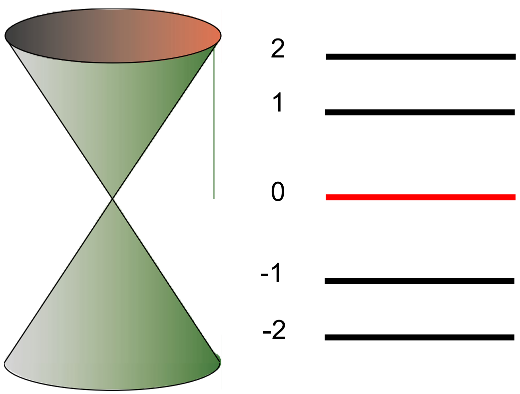
- Superconducting state (broken gauge symmetry)

  Fu, Kane '08

Surface Quantum Hall Effect

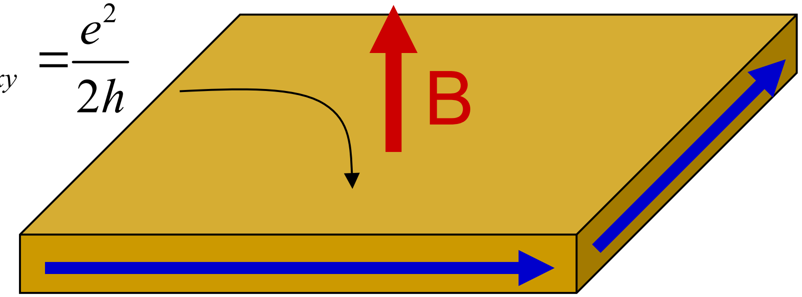
Orbital QHE :

E=0 Landau Level for Dirac fermions. “Fractional” IQHE



$$\sigma_{xy} = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

$$\sigma_{xy} = \frac{e^2}{2h}$$



$$\sigma_{xy} = \frac{e^2}{2h}$$

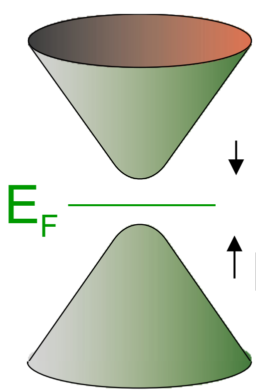
ν=1 chiral edge state

Anomalous QHE :

Induce a surface gap by depositing magnetic material

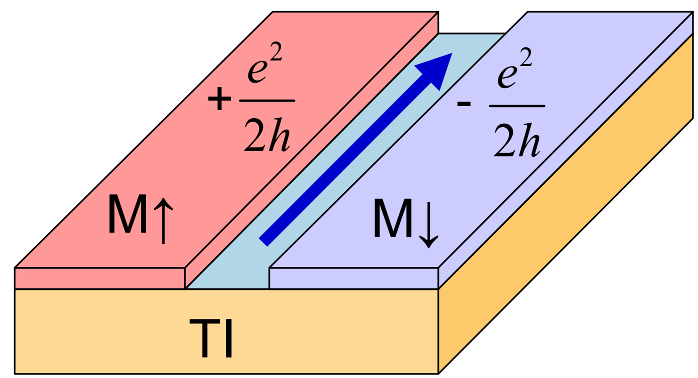
$$H_0 = \psi^\dagger(-iv\sigma\cdot\nabla - \mu + \Delta_M \sigma_z)\psi$$

Mass due to Exchange field



$E_{gap} = 2|\Delta_M|$

$$\sigma_{xy} = \mathrm{sgn}(\Delta_M)\frac{e^2}{2h}$$

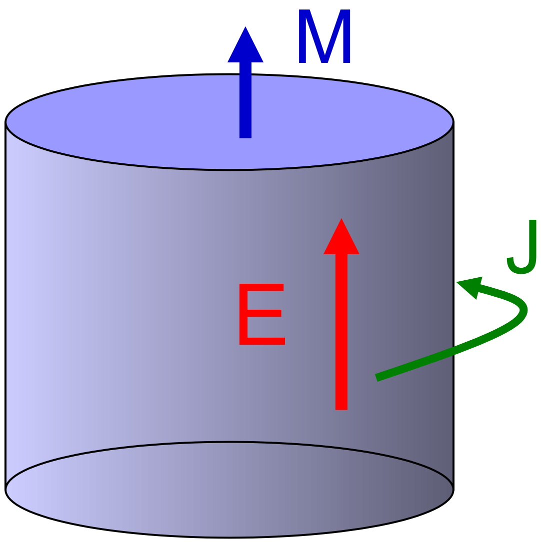


Chiral Edge State at Domain Wall : $\Delta_M \leftrightarrow -\Delta_M$

Topological Magnetoelectric Effect

Qi, Hughes, Zhang '08; Essin, Moore, Vanderbilt '09

Consider a solid cylinder of TI with a magnetically gapped surface



$$J = \sigma_{xy} E = \frac{e^2}{h}\left(n + \frac{1}{2}\right) E = M$$

Magnetoelectric Polarizability

$$M = \alpha E \qquad \alpha = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

topological "θ term"

$$\Delta S = \alpha \int d^3x dt \; \mathbf{E} \cdot \mathbf{B}$$

$$\alpha = \theta \frac{e^2}{2\pi h}$$

TR sym. : $\theta = 0$ or π mod 2π

The fractional part of the magnetoelectric polarizability is determined by the bulk, and independent of the surface (provided there is a gap)
Analogous to the electric polarization, P, in 1D.

	ΔL	formula	"uncertainty quantum"
d=1 : Polarization P	$P \cdot \mathbf{E}$	$\frac{e}{2\pi}\int_{BZ} \mathrm{Tr}[\mathbf{A}]$	e (extra end electron)
d=3 : Magnetoelectric poliarizability α	$\alpha \mathbf{E} \cdot \mathbf{B}$	$\frac{e^2}{4\pi^2 h}\int_{BZ} \mathrm{Tr}[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}]$	e^2/h (extra surface quantum Hall layer)

Strong Interactions

Topological Insulator coupled to compact U(1) gauge field, A

For a compact gauge field, magnetic monopoles are excitations in the theory.
Useful diagnostic for strongly interacting theories.

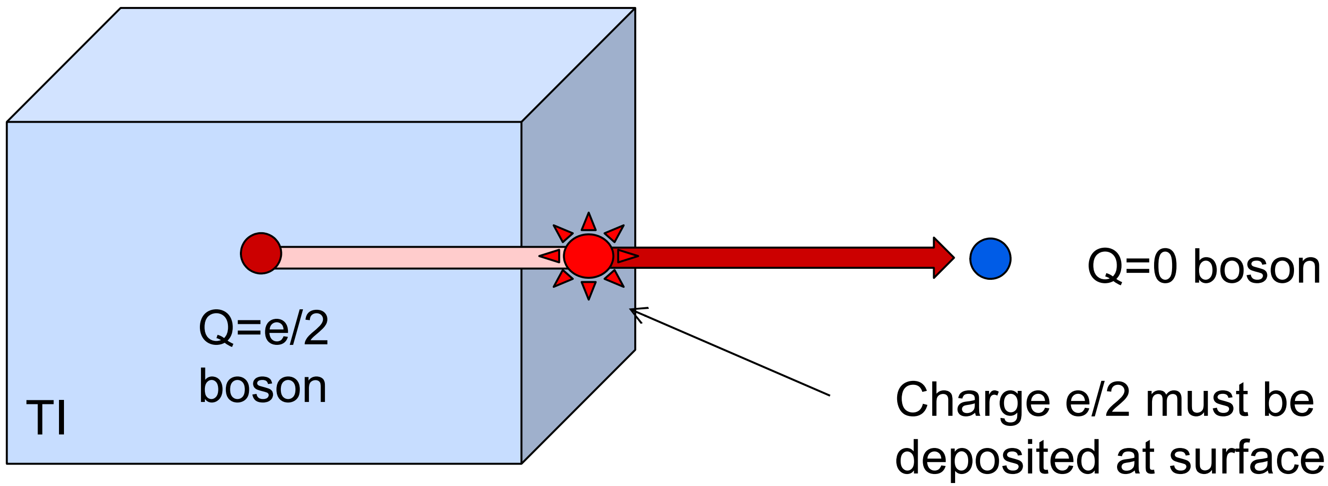
Low energy theory for A : θ term   Qi, Hughes and Zhang '08

$$S = i\theta N \qquad\qquad N = \frac{1}{32\pi^2}\int d^3x\, d\tau\, \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \in \mathbb{Z}$$

- Time reversal symmetry : $\theta = 0$ or π mod 2π. $\theta = \pi$ mod 2π for electron TI
- Witten Effect: Magnetic monopoles are charged $Q = \frac{\theta}{2\pi} e$
- Monopoles (or dyons) are bosons with half integer charge $Q = e\left(n + \frac{1}{2}\right)$

Can the surface of a TI be gapped without breaking symmetry ?

Pass monopole from inside to outside of TI :



Break T at surface:

$\sigma_{xy} = e^2/2h$: charge e/2 flows away on surface

Superconductor at surface:

Charge conservation is violated at surface

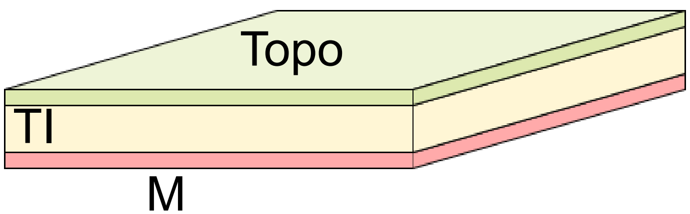
Keep U(1) and T at surface :

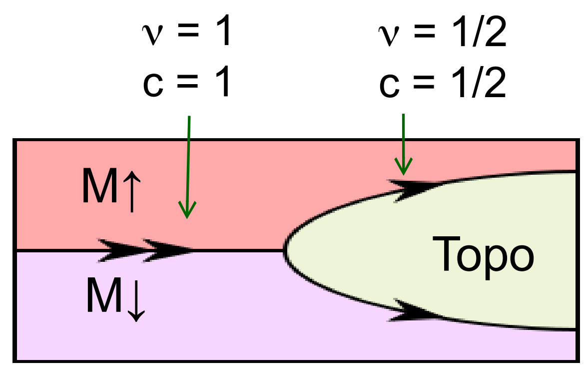
Charge e/2 stays at surface : Requires a topologically ordered surface state with e/2 quasiparticle

Requirements for a Topological Surface Phase on TI

It should be impossible in 2D if symmetry is preserved, but if symmetry is broken there should be a 2D state with the same topological order

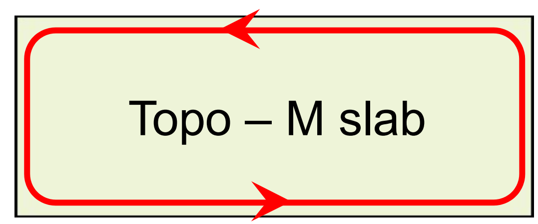
Broken T: Topo – M slab





A 2D Non-Abelian quantum Hall state with

- Hall conductance $\nu\ e^2/h$; $\nu = 1/2$
- Thermal Hall cond. $c\ \pi^2 k_B^2/6h$; $c = 1/2$



ν = 1/2
c = 1/2

Theories of symmetry preserving gapped state:

Related to Moore-Read (Pfaffian) state of FQHE at ν=1/2

- "T-Pfaffian state" Bonderson, Nayak, Qi '13 ; Chen, Fidkowski, Vishwanath '13
- "Moore-Read/antisemion state" Metlitski, Kane, Fisher '13 ; Wang, Potter, Senthil '13

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \Rightarrow \langle\Psi^\dagger\Psi^\dagger\rangle\Psi\Psi = \Delta^*\Psi\Psi$

$$H = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix}\Psi^\dagger & \Psi\end{pmatrix} H_{BdG} \begin{pmatrix}\Psi \\ \Psi^\dagger\end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

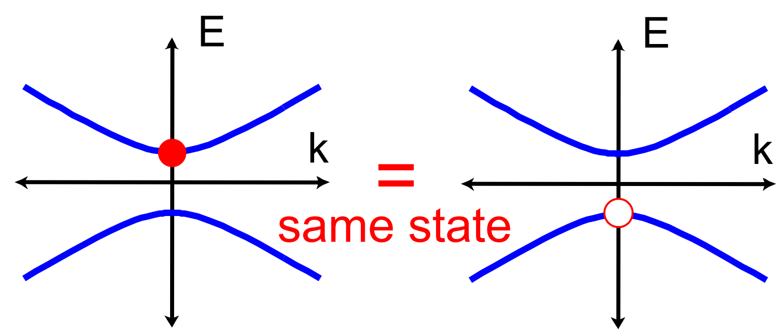
Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG} \qquad \Xi\varphi = \tau_x \varphi^* \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Xi^2 = +1$$

Particle – hole redundancy

$$\varphi_{-E} = \Xi\varphi_E \quad \Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



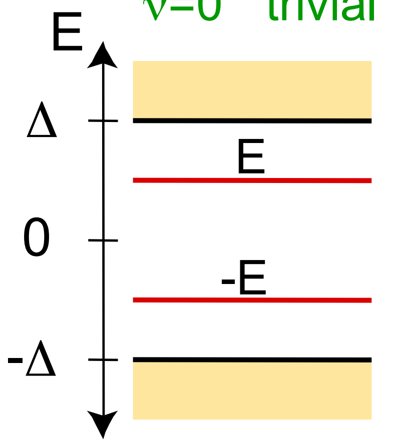
Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

1D $\mathbb{Z}_2$ Topological Superconductor : ν = 0,1 (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum

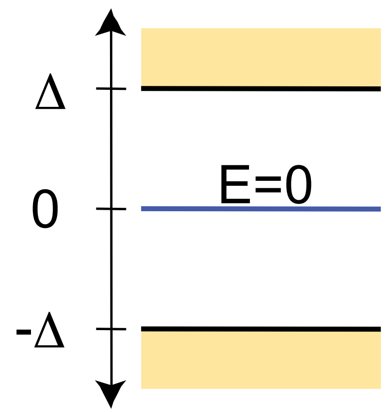
ν=0 "trivial"



Γ_E

$\Gamma_{-E} = \Gamma_E^\dagger$

ν=1 "topological"



Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion : Particle = Antiparticle $\gamma = \gamma^\dagger$

Real part of a Dirac fermion :

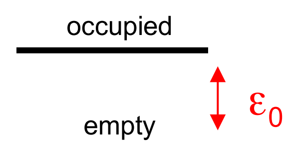
$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases}$$

$$\gamma_i^2 = 1 \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

"Half a state"

Two Majorana fermions define a single two level system

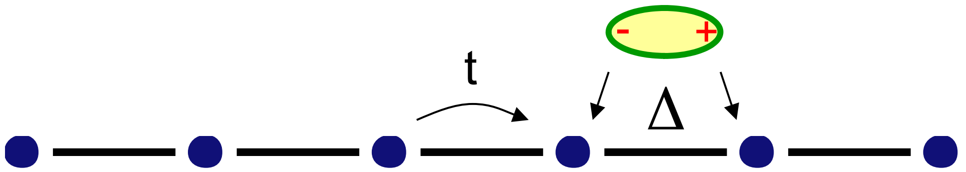
$$H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$



Kitaev Model for 1D p wave superconductor

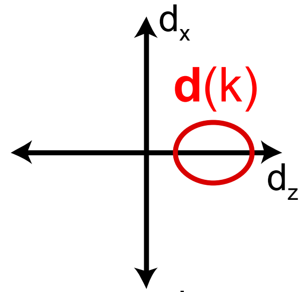
$$H - \mu N = \sum_i t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i + \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger)$$

$$= \sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} H_{BdG}(k) \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

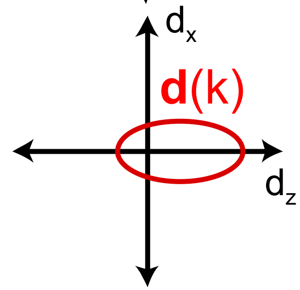


$$H_{BdG}(k) = \tau_z(2t\cos k - \mu) + \tau_x \Delta \sin k = \mathbf{d}(k) \cdot \tau$$

$|\mu| > 2t$: Strong pairing phase
trivial superconductor



$|\mu| < 2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x, d_y, d_z)\big|_k = (-d_x, -d_y, d_z)\big|_{-k}$

Majorana Chain

$$c_i \rightarrow \gamma_{1i} + i\gamma_{2i}$$

$$H = 2i \sum_i t_1 \gamma_{1i}\gamma_{2i} + t_2 \gamma_{2i}\gamma_{1i+1}$$

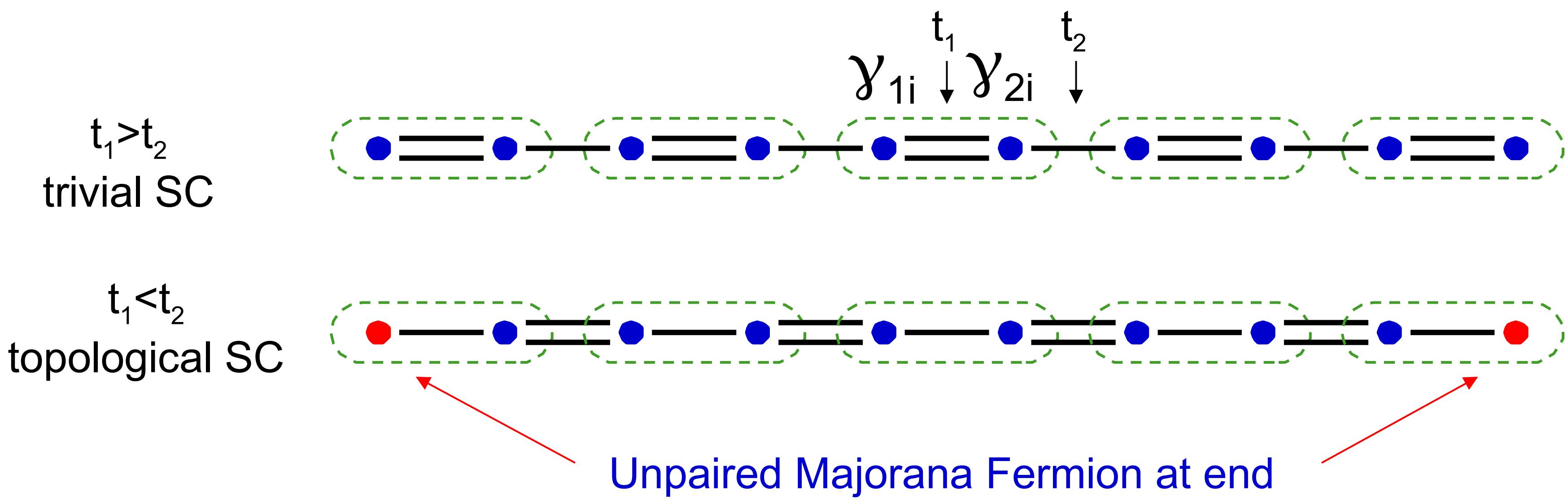
$$\mu c_i^\dagger c_i \rightarrow 2i\mu\gamma_{1i}\gamma_{2i}$$

$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \rightarrow 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \rightarrow 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

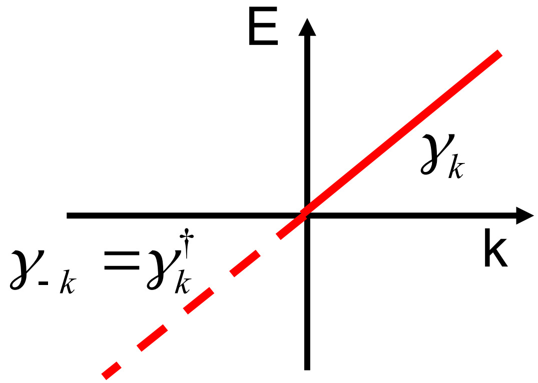
For $\Delta = t$: nearest neighbor Majorana chain

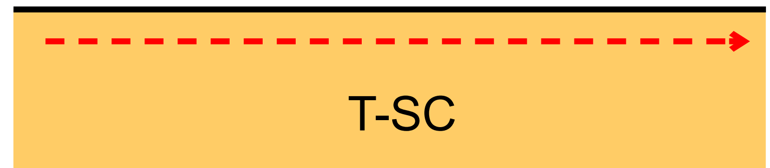
$$t_1 = \mu, \; t_2 = 2t$$



2D topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





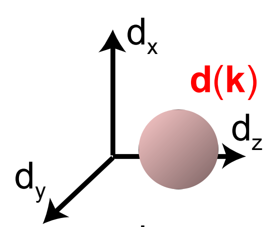
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

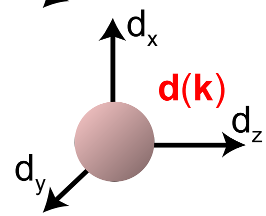
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \tau$

$|\mu| > 4t$: Strong pairing phase
trivial superconductor



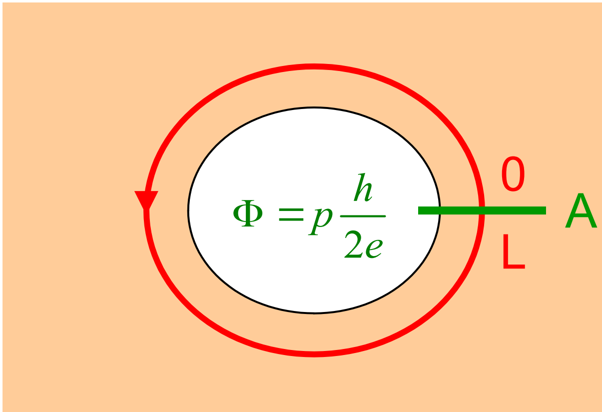
Chern number 0

$|\mu| < 4t$: Weak pairing phase
topological superconductor



Chern number 1

Majorana zero mode at a vortex

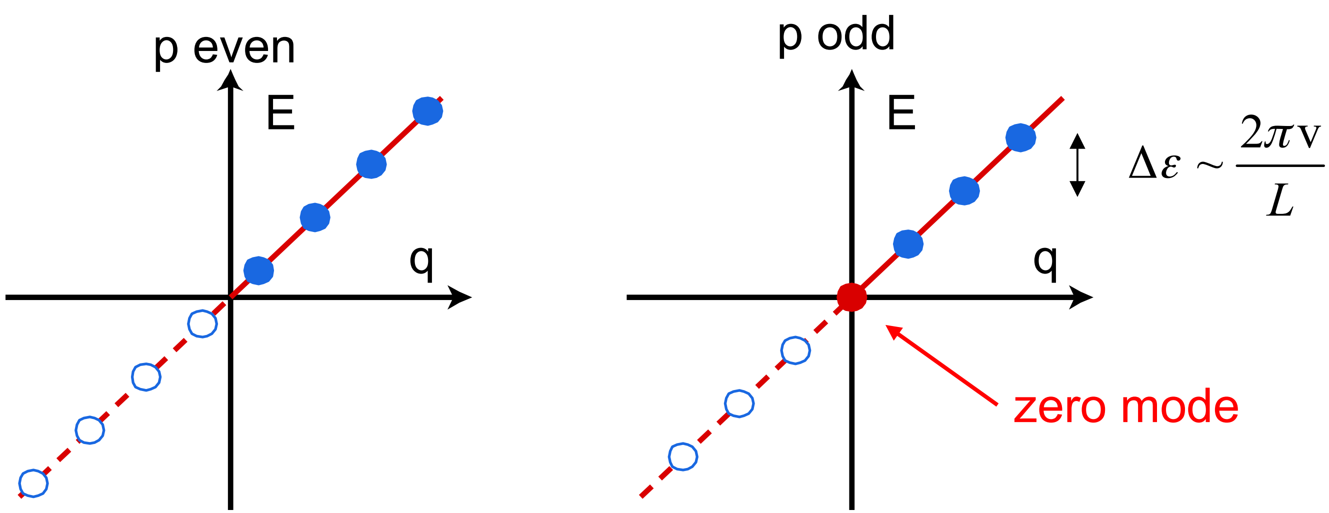


Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}(2m + 1 + p)$$



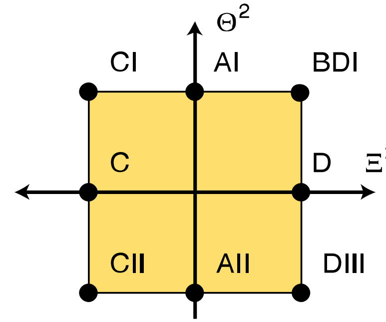
Without the hole : Caroli, de Gennes, Matricon theory ('64) $\Delta\varepsilon \sim \frac{\Delta^2}{E_F}$

Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})\ ;\quad \Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})\ ;\quad \Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})\ ;\quad \Pi \propto \Theta\Xi$



8 antiunitary symmetry classes

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Complex K-theory

Real K-theory

Altland-Zirnbauer Random Matrix Classes

Bott Periodicity $d \rightarrow d+8$

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

Majorana Fermions and Topological Quantum Computing

(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

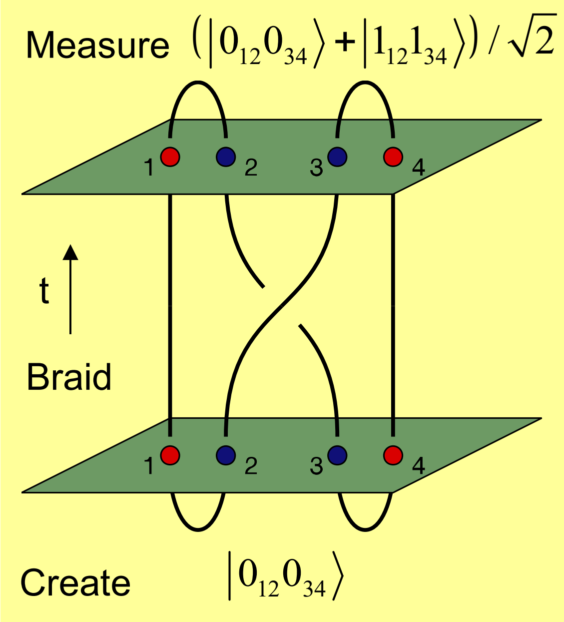
Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \rightarrow \gamma_j$$

$$\gamma_j \rightarrow -\gamma_i$$

These operations, however, are not sufficient to make a universal quantum computer



Potential condensed matter hosts for topological superconductivity

- Quasiparticles in fractional Quantum Hall effect at ν=5/2 Moore Read '91
- Unconventional superconductors

  - Sr_2RuO_4 Das Sarma, Nayak, Tewari '06
  - Fermionic atoms near feshbach resonance Gurarie '05
  - $Cu_xBi_2Se_3$?

- Proximity Effect Devices using ordinary superconductors

  - Topological Insulator devices Fu, Kane '08
  - 2D Semiconductor/Magnet devices Sau, Lutchyn, Tewari, Das Sarma '09, Lee '09
  - 1D Semiconductor devices:

    eg In As quantum wires Oreg, von Oppen, Alicea, Fisher '10
    Lutchyn, Sau, Das Sarma '10
    Expt : Maurik et al. (Kouwenhoven) '12

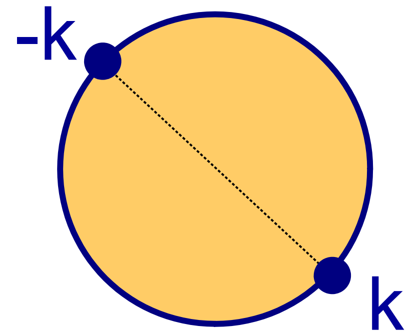
  - 1D Ferromagnetic atomic chains on superconductors

    Expt : Nadj-Perg et al. (Yazdani) '14

Topological Superconductors

Spinless p-wave superconductor:

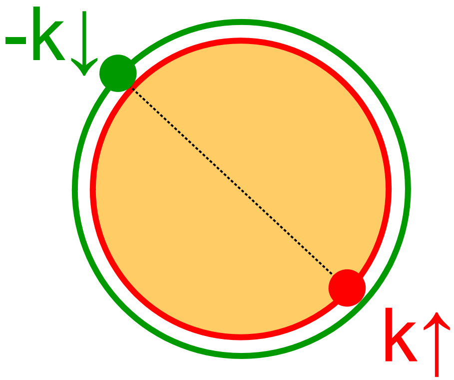
$$\left\langle c_k^\dagger c_{-k}^\dagger \right\rangle \propto \Delta e^{i\varphi}(k_x + ik_y)$$



Ordinary Superconductor :

$$\left\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right\rangle \propto \Delta e^{i\varphi}$$

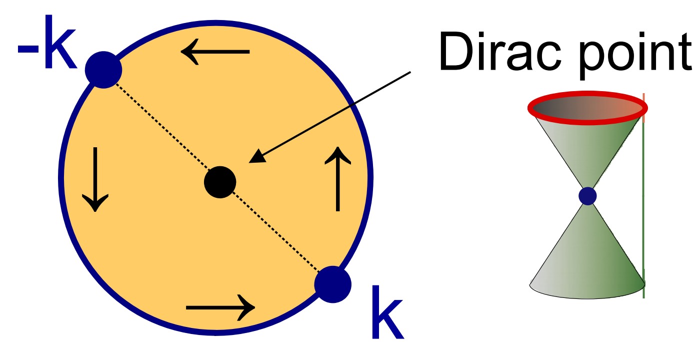
(s-wave, singlet pairing)



Surface of topological insulator

$$\left\langle c_k^\dagger c_{-k}^\dagger \right\rangle \propto \Delta_{\text{surface}} e^{i\varphi}$$

(s-wave, singlet pairing)

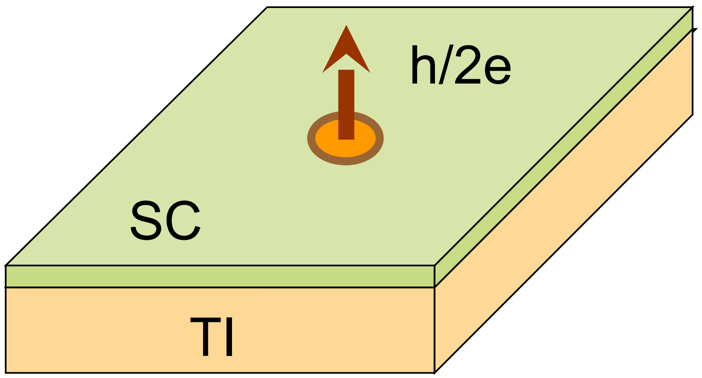


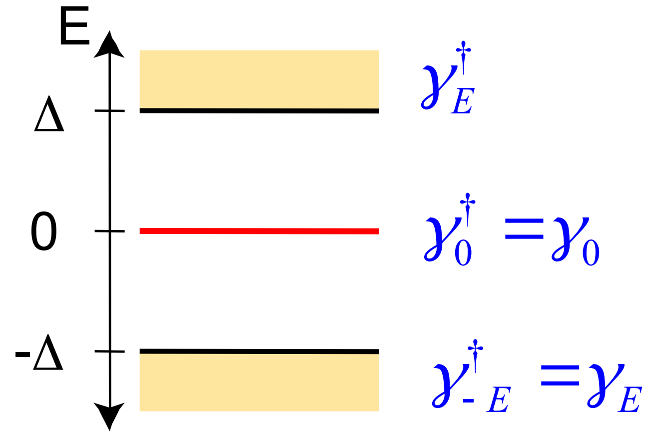
Half an ordinary superconductor

Nontrivial ground state supports Majorana fermions at vortices

Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

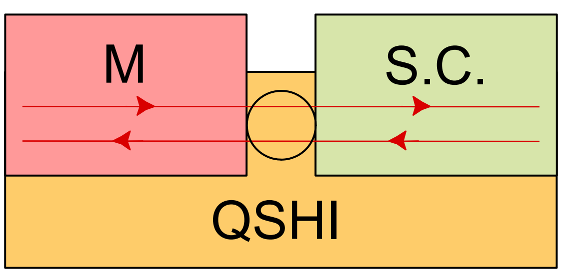




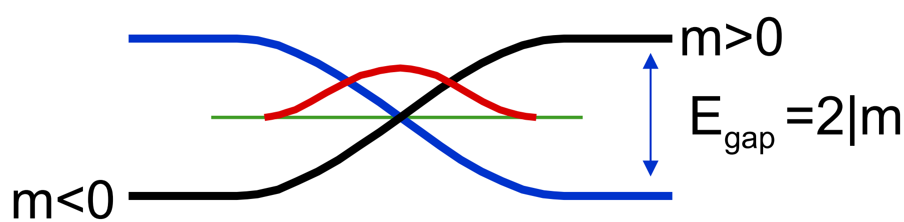
Quasiparticle Bound state at E=0

Majorana Fermion γ_0 "Half a State"

2. Superconductor-magnet interface at edge of 2D QSHI



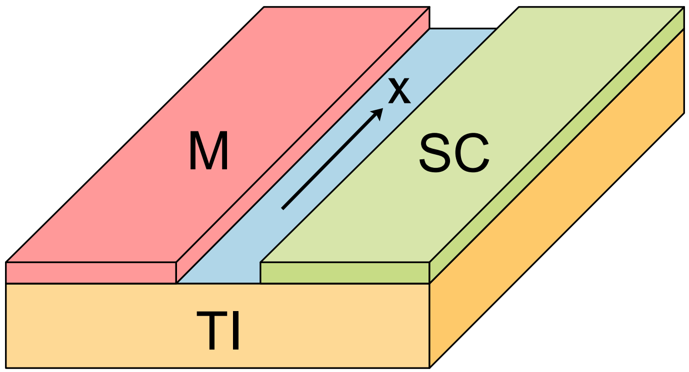
$$m = |\Delta_S| - |\Delta_M|$$

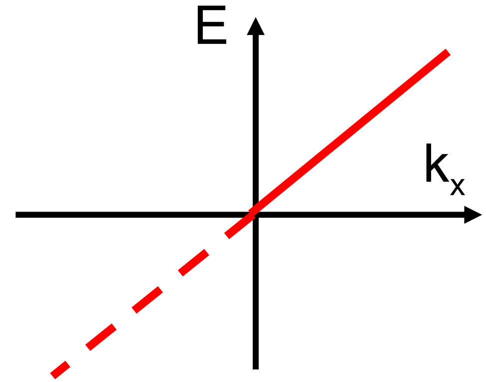


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

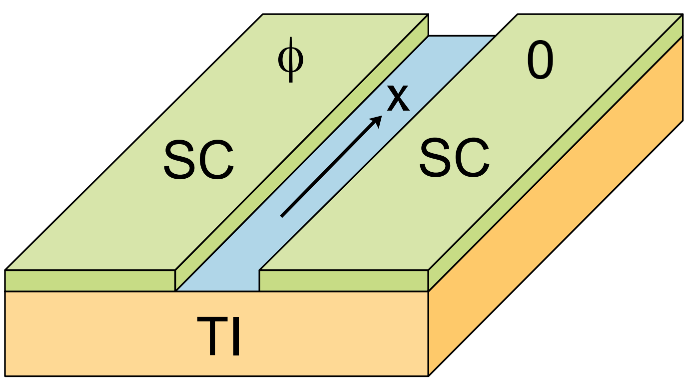


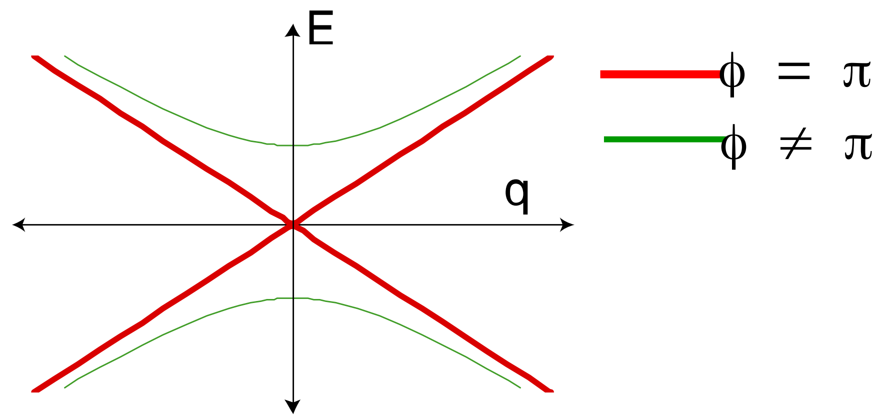


$\gamma_k = \gamma^\dagger_{-k}$: "Half" a 1D chiral Dirac fermion

$$H = -i\hbar v_F \, \gamma \partial_x \gamma$$

2. S-TI-S Josephson Junction





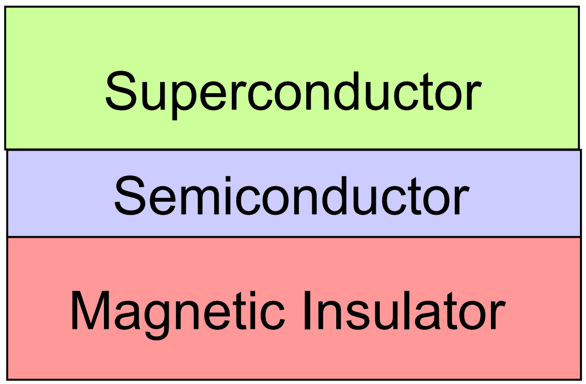
Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

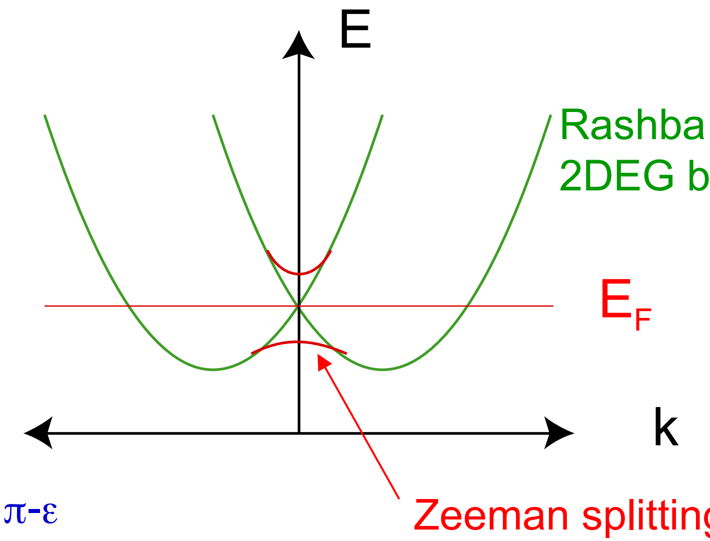
$$H = -i\hbar v_F \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R \right) + i\Delta \cos(\phi/2) \gamma_L \gamma_R$$

Another route to the 2D p+ip superconductor

Semiconductor - Magnet - Superconductor structure

Sau, Lutchyn, Tewari, Das Sarma '09



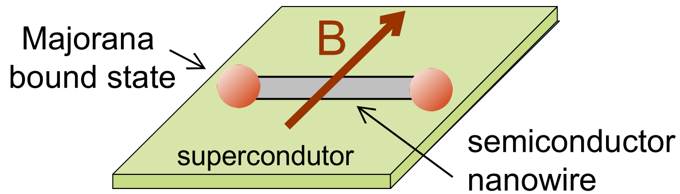


- Single Fermi circle with Berry phase π-ε
- Topological superconductor with Majorana edge states and Majorana bound states at vortices.
- Variants :
  - use applied magnetic field to lift Kramers degeneracy (Alicea '10)
  - Use 1D quantum wire (eg InSb). A route to 1D p wave superconductor with Majorana end states. (Oreg, von Oppen, Alicea, Fisher '10

Experiments semiconductor nanowires

- Superconductor + Semiconductor nanowire

  Sau, Lutchyn, Tewari, Das Sarma '09;
  Alicea '10;
  Oreg, Refael, von Oppen '10

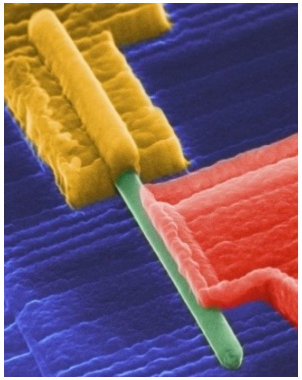


First experiment: 1D Semiconductor quantum wire on superconductor

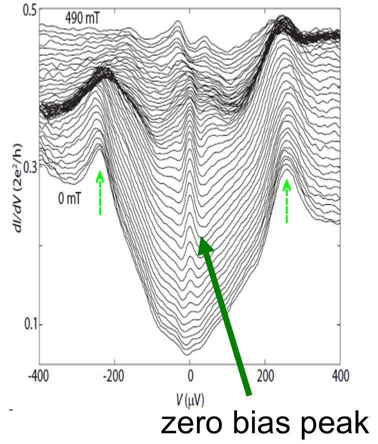
Mourik, …, Kouwenhoven, et al. '12

Observe zero bias peak in tunneling conductance:

Attributed to Majorana end state.

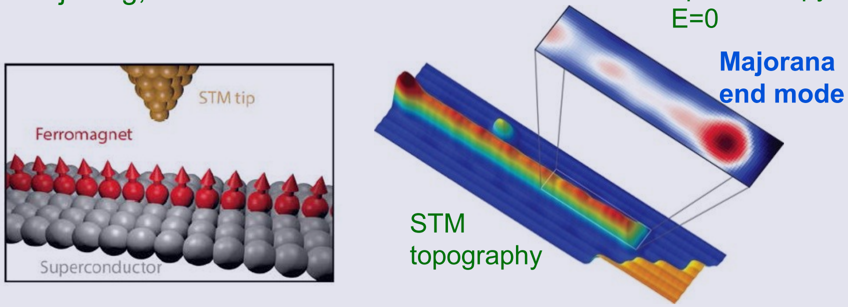


InSb nanowire



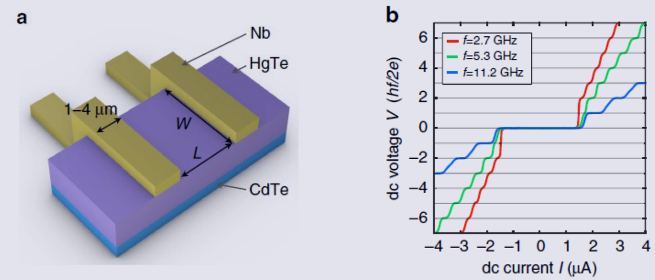
Ferromagnetic Atomic Chains on Superconductors (Iron on lead)

Nadj-Perg, Science '14



4π periodic Josephson effect in HgTe SC – 2D TI – SC junctions

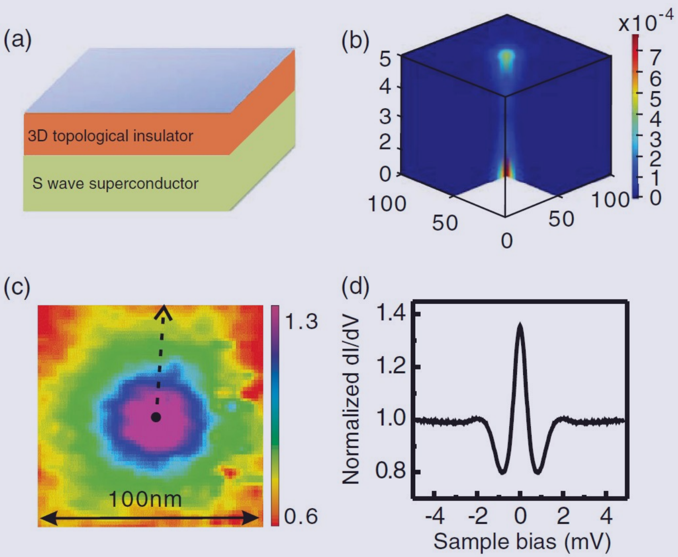
Wiedenmann, et al., Nature Comm. 7:10303 (2016).



Vortex at superconductor TI interface

J-P Xu, et al. PRL 114, 017001 (2015)

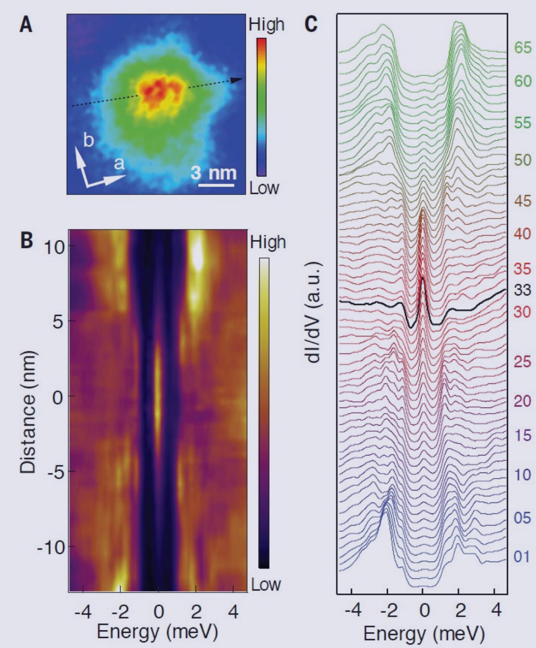
HH Sun, et al. PRL 116, 257003 (2016)



Vortex in Iron based superconductor

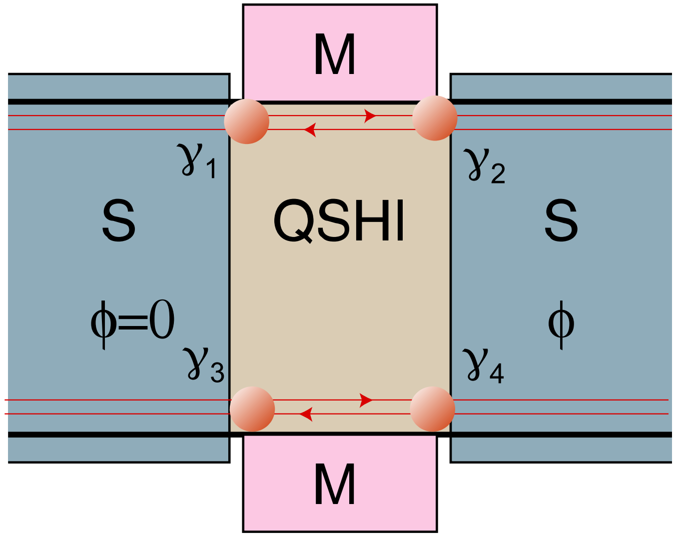
Wang et al. Science 362, 333 (2018).

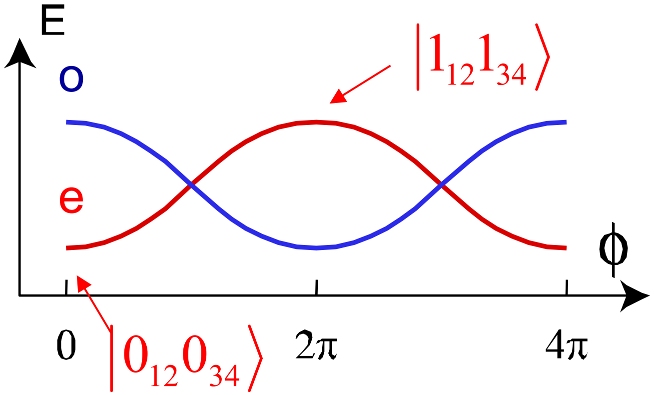
"TI with SC on insided"



Fractional Josephson Effect

Kitaev '01
Kwon, Sengupta, Yakovenko '04
Fu, Kane '08

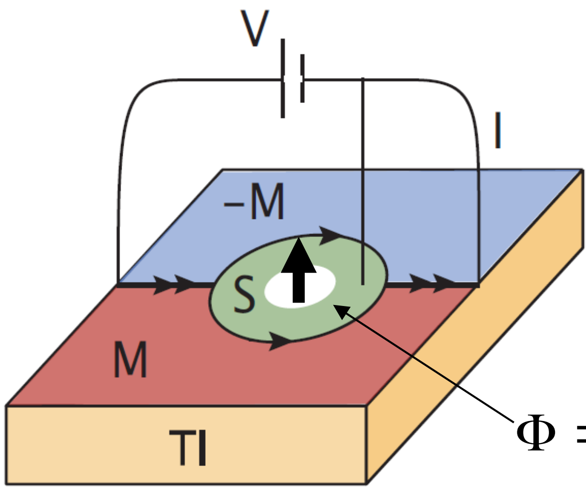




- 4π perioidicity of $E(\phi)$ protected by local conservation of fermion parity.
- AC Josephson effect with half the usual frequency: $f = eV/h$

A Z_2 Interferometer for Majorana Fermions

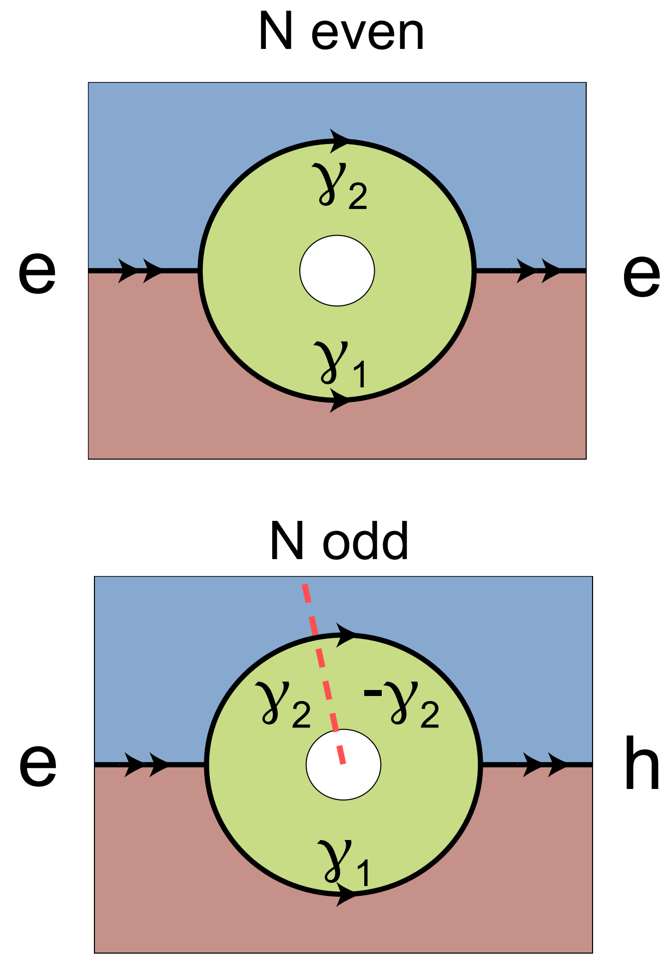
A signature for neutral Majorana fermions probed with charge transport



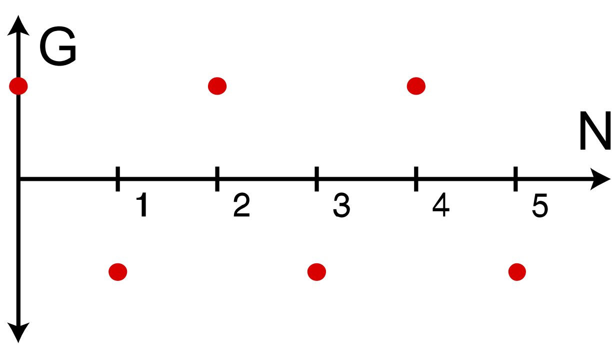
$$\Phi = N\frac{h}{2e}$$

$$c^\dagger = \gamma_1 - i\gamma_2$$

$$c = \gamma_1 + i\gamma_2$$



- Chiral electrons on magnetic domain wall split into a pair of chiral Majorana fermions
- "Z_2 Aharonov Bohm phase" converts an electron into a hole



Akhmerov, Nilsson, Beenakker, PRL '09
Fu and Kane, PRL '09