Topological order and many-body entanglement

Xiao-Gang Wen, MIT (2019/06, Quantum Frontiers)

Our world is very rich with all kinds of materials





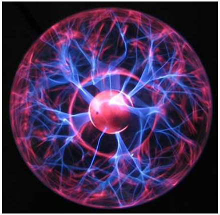

In middle school, we learned ...

there are four states of matter:

Solid

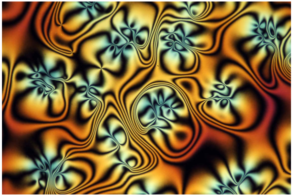
Liquid

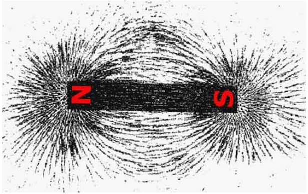
Gas

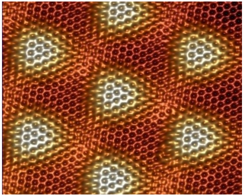


Plasma

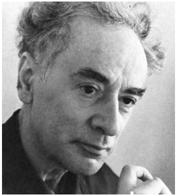
In university, we learned ... ...







- Rich forms of matter $\leftarrow$ rich types of **order**
- A deep insight from Landau: **different orders come from different symmetry breaking**.
- A corner stone of condensed matter physics



Classify phases of quantum matter ($T = 0$ phases)

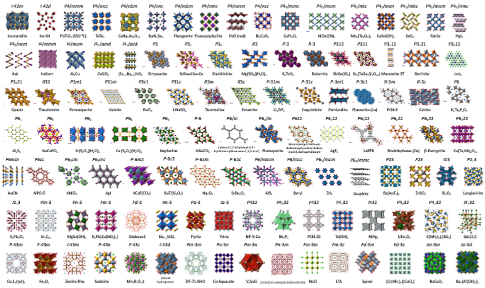
For a long time, we thought that Landau symmetry breaking classify all phases of matter

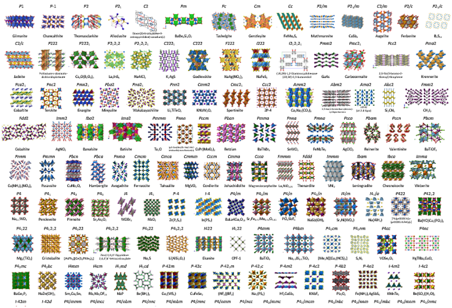
- **Symm. breaking phases are classified by a pair** $G_\Psi \subset G_H$

  G_H = symmetry group of the system.

  G_Ψ = symmetry group of the ground states.

- **230 crystals** from group theory





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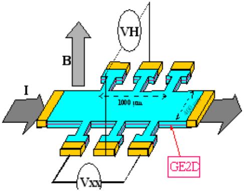
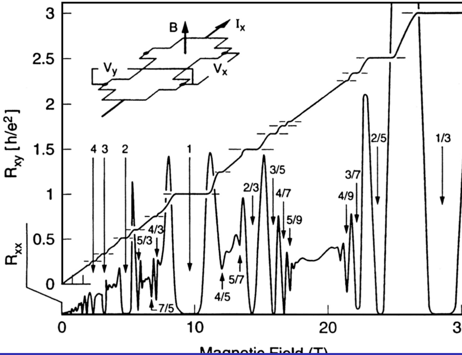


- FQH states have different phases even when there is no symm. and no symm. breaking.



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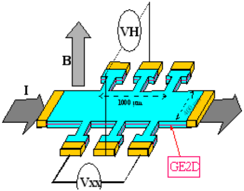
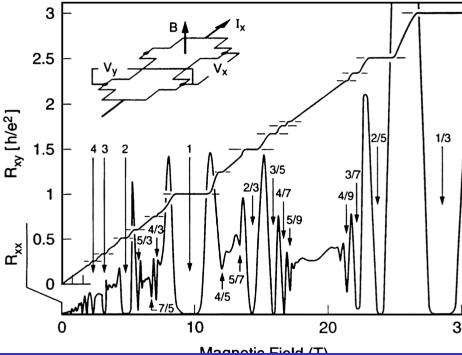


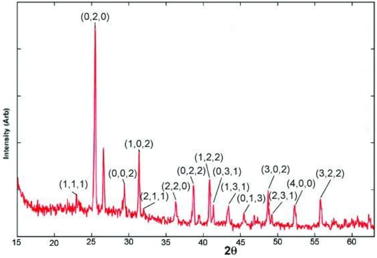
- FQH states have different phases even when there is no symm. and no symm. breaking.
- FQH states must contain a new kind of order, which was named **topological order**

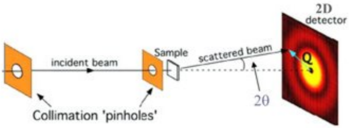
  Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)

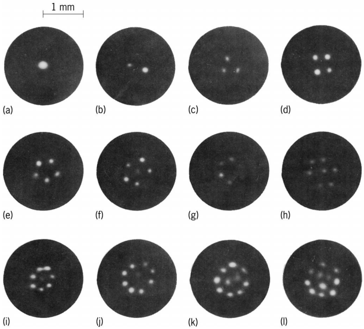


Every physical concept is defined by experiment

- The concept of crystal order is defined via X-ray scattering







- The concept of superfuild order is defined via zero-viscosity and quantization of vorticity

What measurable quantities define topo. order?

- There are three kinds of quantum matter:
  (1) no low energy excitations (Insulator)
  (2) some low energy excitations (Superfluid)
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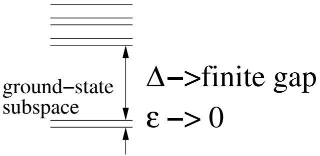

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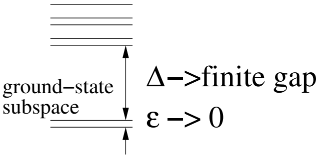
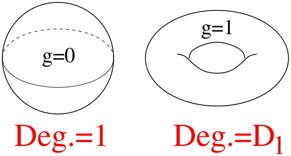




- The only non-trivial measurable low enery quantity is the ground state degeneracy, which may depend on the topology of space. Wen, PRB **40** 7387 (89); IJMP **4** 239 (90)

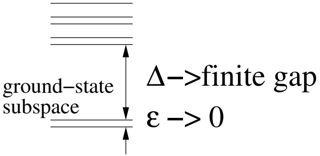
Topo. order is defined by topological degeneracy

- *But, the ground state degeneracy of FQH states appears to a finite-size effect (which depends on "boundary conditions" ie topologies), rather than a thermodynamic property. How can it defines a new phases of quantum matter?*

- The ground state degeneracies are robust against any local perturbations that can break any symmetries. The ground state degeneracies have nothing to do with symmetry.
  → **topological degeneracy**

Wen Niu PRB **41** 9377 (90)

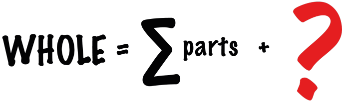
- The ground state degeneracies can change by but some large changes of Hamiltonian
  → gap-closing phase transition.



Many-body entanglement → Topo. degeneracy

- For a highly entangled many-body quantum systems: **knowing every parts still cannot determine the whole**

- In other words, there are different "wholes", that their every local parts are identical.



- Local perturbations can only see the parts → those different "wholes" (the whole quantum states) have the same energy.

- Those kinds of many-body quantum systems have

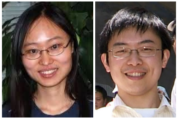
topological entanglement entropy

Kitaev-Preskill hep-th/0510092

Levin-Wen cond-mat/0510613

and **long range quantum entanglement**

Chen-Gu-Wen arXiv:1004.3835



Macroscopic characterization $\rightarrow$ microscopic origin

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  $\rightarrow$ microscopic origin (**long range entanglement**)
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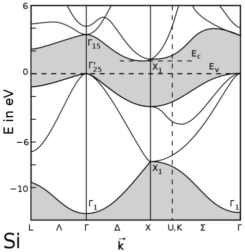
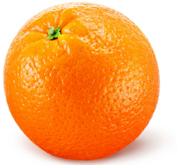
- From macroscopic characterization of **superconductivity** (zero-resistivity, quantized vorticity)
  → microscopic origin (**BSC electron-pairing**)
  took 46 years.

This **topology** is not that *topology*





Topology in topological insulator/superconductor (2005) corresponds to the twist in the band structure of orbitals, which is similar to the topological structure that distinguishes a sphere from a torus. This kind of topology is *classical topology*.

Kane-Mele cond-mat/0506581

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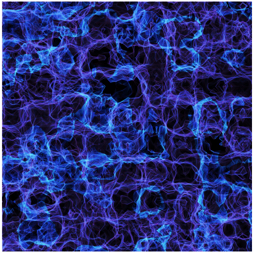
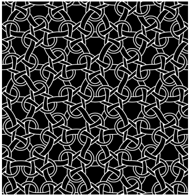
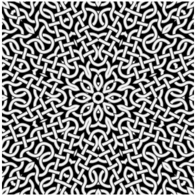

Topology in topological order (1989) corresponds to pattern of many-body entanglement in many-body wave function $\Psi(m_1, m_2, \cdots, m_N)$, that is robust against any local perturbations that can break any symmetry. Such robustness is the meaning of **topological** in topological order. This kind of topology is **quantum topology**.

Wen PRB **40** 7387 (1989)

Entanglement through examples

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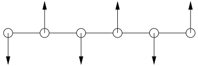
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$
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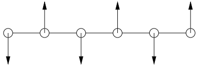
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ $\to$
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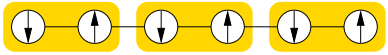
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ $\to$
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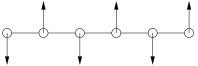
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\to$
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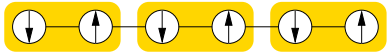
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ $\to$
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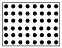
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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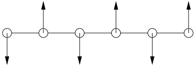
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ →
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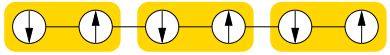
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ →
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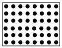
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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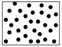
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

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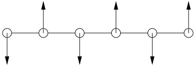
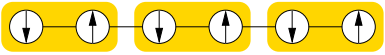
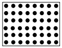
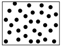
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots$ → short-range entangled (SRE) entangled
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$ = direct-product state → unentangled state (classical)
- Particle condensation (superfluid) $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..)..$ = direct-product state → unentangled state (classical)

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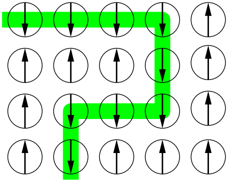
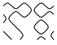
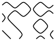
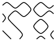
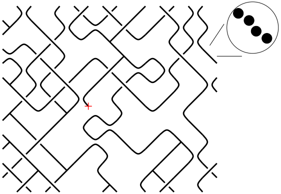


- *sum* over a subset of spin configurations:

$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum$ |  $\rangle$

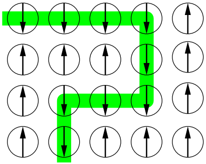
$|\Phi^{DS}_{\text{loops}}\rangle = \sum(-)^{\#\text{ of loops}}$ |  $\rangle$

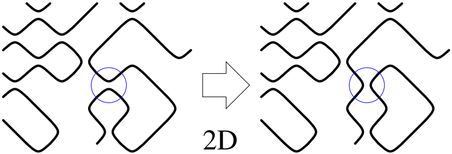
$|\Phi^{\theta}_{\text{loops}}\rangle = \sum(e^{i\theta})^{\#\text{ of loops}}$ |  $\rangle$

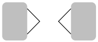


- Can the above wavefunction be the ground states of local Hamiltonians?

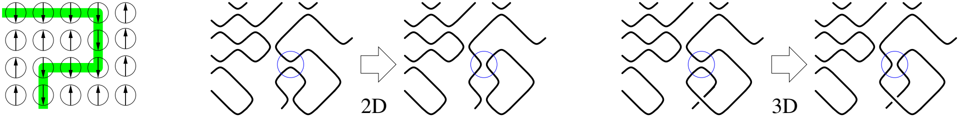
Local dance rule → global dance pattern





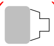
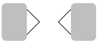
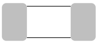
- Local rules of a string liquid (for ground state):
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str} () $=\Phi_{\text{str}}$ () , Φ_{str} () $=\Phi_{\text{str}}$ ()
  → Global wave function of loops Φ_{str} () $= 1$
- There is a Hamiltonian H:
  (1) Open ends cost energy
  (2) string can hop and reconnect freely.
  The ground state of H gives rise to the above string lqiuid wave function.

Local dance rule $\to$ global dance pattern



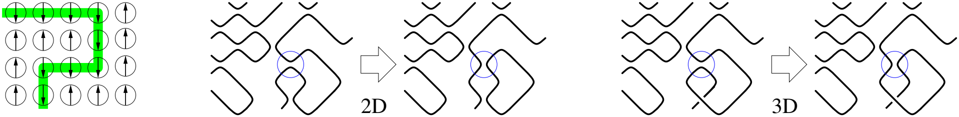
- Local rules of another string liquid (ground state):

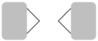
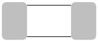
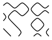
  (1) Dance while holding hands (no open ends)

  (2) Φ_{str} () $= \Phi_{\text{str}}$ () , Φ_{str} () $= -\Phi_{\text{str}}$ ()

  $\to$ Global wave function of loops Φ_{str} () $= (-)^{\#\text{ of loops}}$

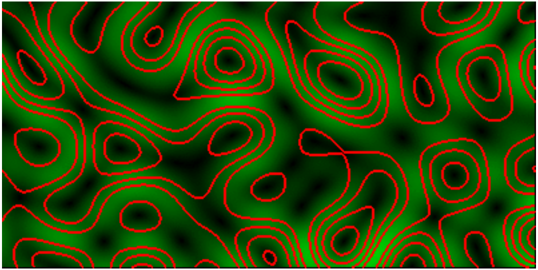
Local dance rule → global dance pattern



- Local rules of another string liquid (ground state):
  (1) Dance while holding hands (no open ends)
  (2) $\Phi_{str}\Big($$\Big) = \Phi_{str}\Big($$\Big)$, $\Phi_{str}\Big($$\Big) = -\Phi_{str}\Big($$\Big)$
  → Global wave function of loops $\Phi_{str}\Big($$\Big) = (-)^{\#\text{ of loops}}$
- The second string liquid $\Phi_{str}\Big($$\Big) = (-)^{\#\text{ of loops}}$ can exist only in 2-dimensions.
  The first string liquid $\Phi_{str}\Big($$\Big) = 1$ can exist in both 2- and 3-dimensions.

Knowing all the parts $\neq$ knowing the whole

- Do those two string liquids really have topological order? Do they have topo. ground state degenercy?



Knowing all the parts $\neq$ knowing the whole

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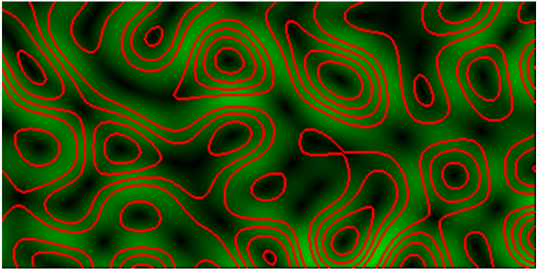
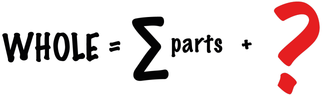
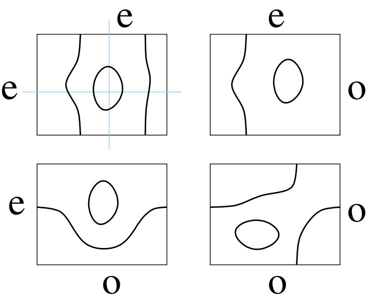




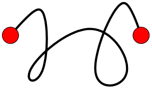
- 4 locally indistinguishable states on torus for both liquids → **topo. order**
- Ground state degeneracy cannot distinguish them.

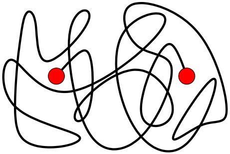


$D^{tor}=4$

Topological excitations

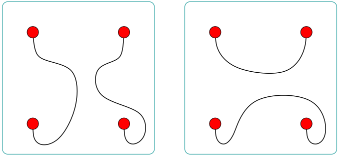
- **Ends of strings** behave like point objects.
- They cannot be created alone → **topological**



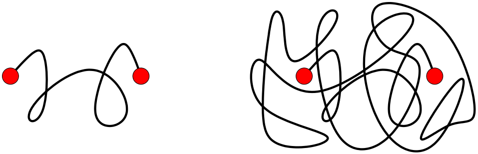


- Let us fix 4 ends of string on a sphere S^2. *How many locally indistinguishable states are there?*

\- There are 2 sectors → 2 states.



Topological excitations



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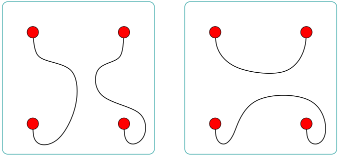
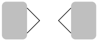
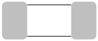


- Let us fix 4 ends of string on a sphere S^2. *How many locally indistinguishable states are there?*

- ~~There are 2 sectors → 2 states.~~
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations Φ_{str} () $= \pm\Phi_{str}$ ().

- In general, fixed $2N$ ends of string → 1 state. Each end of string has no degeneracy → no internal degrees of freedom.

Topological excitations

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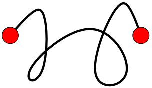
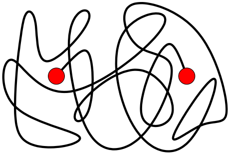
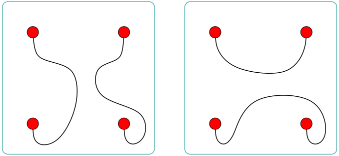




- Let us fix 4 ends of string on a sphere S^2. *How many locally indistinguishable states are there?*



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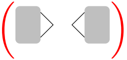
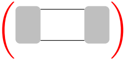
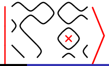
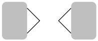
- In fact, there is only 1 sector → 1 state, due to the string reconnection fluctuations Φ_{str}  $= \pm\Phi_{\text{str}}$ .

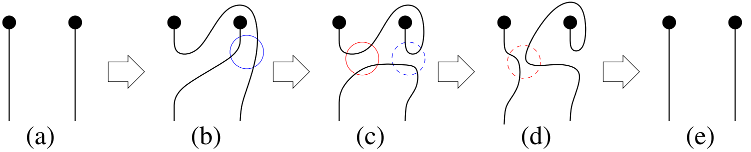
- In general, fixed $2N$ ends of string → 1 state. Each end of string has no degeneracy → no internal degrees of freedom.
- Another type of topological excitation **vortex** at $\times$:

$|m\rangle = \sum(-)^{\#\text{ of loops around }\times}$ 

Emergence of fractional spin

- Ends of strings are point-like. Are they bosons or fermions?
  Two ends = a small string = a boson, but each end can still be a fermion. Fidkowski-Freedman-Nayak-Walker-Wang cond-mat/0610583
- Φ_{str} () $= 1$ string liquid Φ_{str} () $= \Phi_{\text{str}}$ ()
- End of string wave function: $|\text{end}\rangle =$  $+ c$  $+ c$  $+ \cdots$
  The string near the end is totally fixed, since the end is determined by a trapping Hamiltonian δH which can be chosen to fix the string. The string alway from the end is not fixed, since they are determined by the bluk Hamiltonian H which gives rise to a string liquid.
- 360° rotation:  $\to$  and  $=$  $\to$ : $R_{360^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- We find two types of topological exitations
  (1) $|e\rangle =$  $+$  spin 0. (2) $|f\rangle =$  $-$  spin $1/2$.

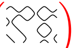
Spin-statistics theorem: Emergence of Fermi statistics



- (a) $\to$ (b) = exchange two string-ends.
- (d) $\to$ (e) = 360° rotation of a string-end.
- Amplitude (a) = Amplitude (e)
- Exchange two string-ends plus a 360° rotation of one of the string-end generate no phase.

$\to$ **Spin-statistics theorem**

Z_2 topological order and its physical properties

Φ_{str} () $= 1$ string liquid has Z_2-topological order.

- 4 **types** of topological excitations: *(f is a fermion)*

  (1) $|e\rangle =$  $+$  spin 0.

  (2) $|f\rangle =$  $-$  spin $1/2$.

  (3) $|m = e \otimes f\rangle =$  $-$  spin 0.

  (4) $|1\rangle =$  $+$  spin 0.

- The type-1 excitation is the tirivial excitation, that can be created by local operators.
  The type-e, type-m, and type-f excitations are non-tirivial excitation, that cannot be created by local operators.
- 1, e,m are bosons and f is a fermion. e,m, and f have π mutual statistics between them.
- **Fusion rule**:

  $e \otimes e = 1$; $f \otimes f = 1$; $m \otimes m = 1$;
  $e \otimes m = f$; $f \otimes e = m$; $m \otimes f = e$;
  $1 \otimes e = e$; $1 \otimes m = m$; $1 \otimes f = f$;

Z_2 topo. order is described by Z_2 gauge theory

Physical properties of Z_2 gauge theory
= Physical properties of Z_2 topological order

- Z_2-charge (a representatiosn of Z_2) and Z_2-vortex (π-flux) as two bosonic point-like excitations.
- Z_2-charge and Z_2-vortex bound state $\to$ a fermion (f), since Z_2-charge and Z_2-vortex has a π mutual statistics between them (charge-1 around flux-π).
- Z_2-charge, Z_2-vortex, and their bound state has a π mutual statistics between them.
- Z_2-charge $\to$ e, Z_2-vortex $\to$ m, bound state $\to$ f.
- Z_2 gauge theory on torus also has 4 degenerate ground states

Emergence of fractional spin and semion statistics

Φ_{str} () $= (-)^{\#\text{ of loops}}$ string liquid. Φ_{str} ( ) $= -\Phi_{\text{str}}$ ()

- End of string wave function: $|\text{end}\rangle =$  $+ c$  $- c$  $+ \cdots$
- 360° rotation:  $\to$  and  $= -$  $\to -$ : $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- **Types** of topological excitations: *($s_\pm$ are semions)*

  (1) $|s_+\rangle =$  $+ \mathrm{i}$  spin $\frac{1}{4}$. (2) $|s_-\rangle =$  $- \mathrm{i}$  spin $-\frac{1}{4}$

  (3) $|m = s_- \otimes s_+\rangle = {}^\times -$  spin 0. (4) $|1\rangle = {}^\times +$  spin 0.
- **double-semion topo. order** $= U^2(1)$ Chern-Simon gauge theory $L(a_\mu) = \frac{2}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{2}{4\pi} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \epsilon^{\mu\nu\lambda}$

Emergence of fractional spin and semion statistics

$\Phi_{\text{str}}(\;) = (-)^{\#\text{ of loops}}$ string liquid. $\Phi_{\text{str}}(\;) = -\Phi_{\text{str}}(\;)$

- End of string wave function: $|\text{end}\rangle = \; + c \; - c \; + \cdots$
- 360° rotation: $\; \to \;$ and $\; = - \; \to - \;$: $R_{360^\circ} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- **Types** of topological excitations: *($s_\pm$ are semions)*

  (1) $|s_+\rangle = \; + \mathrm{i} \;$ spin $\frac{1}{4}$. (2) $|s_-\rangle = \; - \mathrm{i} \;$ spin $-\frac{1}{4}$

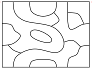
  (3) $|m = s_- \otimes s_+\rangle = \times - \;$ spin 0. (4) $|1\rangle = \times + \;$ spin 0.
- **double-semion topo. order** = $U^2(1)$ Chern-Simon gauge theory $L(a_\mu) = \frac{2}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{2}{4\pi} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \epsilon^{\mu\nu\lambda}$
- Two string lqiuids → Two topological orders:

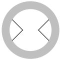
  Z_2 **topo. order** Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91), Moessner-Sondhi PRL 86 1881 (01) and **double-semion topo. order** Freedman etal cond-mat/0307511, Levin-Wen cond-mat/0404617

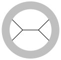
String-net liquid

Levin-Wen cond-mat/0404617

Ground state:

- String-net liquid: allow three strings to join, but do not allow a string to end $\Phi_{\text{str}}\Big($  $\Big)$
- The dancing rule : $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

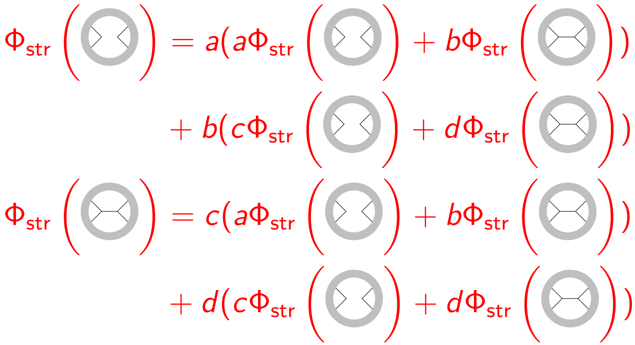
$\Phi_{\text{str}}\Big($  $\Big) = a\ \Phi_{\text{str}}\Big($  $\Big) + b\ \Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = c\ \Phi_{\text{str}}\Big($  $\Big) + d\ \Phi_{\text{str}}\Big($  $\Big)$

- The above is a relation between two orthogonal basis: two local resolutions of how four strings join (quantum geometry)

$$a^2 + b^2 = 1, \quad ac + bd = 0, \quad ca + db = 0, \quad c^2 + d^2 = 1$$

Self consistent dancing rule



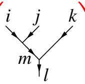
We find

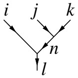
$$a^2 + bc = 1, \quad ab + bd = 0, \quad ac + dc = 0, \quad bc + d^2 = 1$$

$$\to d = -a, \quad b = c, \quad a^2 + b^2 = 1.$$

More self consistency condition

- Rewrite the string reconnection rule (0→no-string, 1→string)

$$\Phi\left(\right.$$ 

$$\left.\right) = \sum_{n=0}^{1} F^{ijm}_{kln} \Phi \left(\right.$$ 

$$\left.\right), \qquad i,j,k,l,m,n = 0,1$$

The 2-by-2 matrix $F^{ij}_{kl} \to (F^{ij}_{kl})^m_{nl}$ is unitary.

We have

F^{000}_{000}  $= 1$

F^{000}_{111}  $= (F^{011}_{100}$  $)^* = (F^{101}_{010}$  $)^* = F^{110}_{001}$  $= 1$

F^{011}_{011}  $= (F^{101}_{101}$  $)^* = 1$

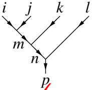
F^{011}_{111}  $= (F^{101}_{111}$  $)^* = F^{111}_{011}$  $= (F^{111}_{101}$  $)^* = 1$

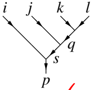
F^{110}_{110}  $= a$

F^{110}_{111}  $= b = (F^{111}_{110}$  $)^* = c^*$

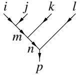
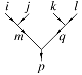
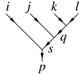
F^{111}_{111}  $= d = -a,$

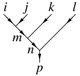
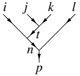
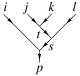
More self consistency condition

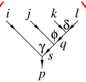
- 

can be trans. to


through two different paths:

$\Phi\Bigg($  $\Bigg) = \sum_q F^{mkn,\beta\chi}_{lpq} \Phi\Bigg($  $\Bigg) = \sum_{q,s} F^{mkn}_{lpq} F^{ijm}_{qps} \Phi\Bigg($  $\Bigg),$

$\Phi\Bigg($  $\Bigg) = \sum_t F^{ijm}_{knt} \Phi\Bigg($  $\Bigg) = \sum_{t,s} F^{ijm}_{knt} F^{itn}_{lps} \Phi\Bigg($  $\Bigg)$

$= \sum_{t,s,q} F^{ijm}_{knt} F^{itn}_{lps} F^{jkt}_{lsq} \Phi\Bigg($  $\Bigg).$

- The two paths should lead to the same relation

$$\sum_t F^{ijm}_{knt} F^{itn}_{lps} F^{jkt}_{lsq} = F^{mkn}_{lpq} F^{ijm}_{qps}$$

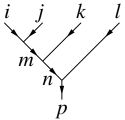
Such a set of non-linear algebraic equations is the famous pentagon identity.

The pentagon identity

- $i, j, k, l, p, m, n, q, s = 0, 1 \to$
  $2^9 = 512+$ non-linear equations with $2^6 = 64$ unknowns.

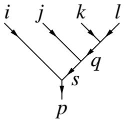
- Solving the pentagon identity: choose $i, j, k, l, p = 1$

$$\sum_{t=0,1} F^{11m}_{1nt} F^{1tn}_{11s} F^{11t}_{1sq} = F^{m1n}_{11q} F^{11m}_{q1s}$$



choose $n, q, s = 1, m = 0$

$$\sum_{t=0,1} F^{110}_{11t} F^{1t1}_{111} F^{11t}_{111} = F^{011}_{111} F^{110}_{111}$$



$$\to a \times 1 \times b + b \times (-a) \times (-a) = 1 \times b$$

$$\to a + a^2 = 1, \qquad \to a = (\pm\sqrt{5} - 1)/2$$

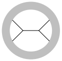
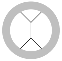
Since $a^2 + b^2 = 1$, we find

$$a = (\sqrt{5} - 1)/2 \equiv \gamma, \quad b = \sqrt{a} = \sqrt{\gamma}$$

String-net dancing rule

- The dancing rule : $\Phi_{\text{str}}\Big($  $\Big) = \Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = \gamma\Phi_{\text{str}}\Big($  $\Big) + \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big)$

$\Phi_{\text{str}}\Big($  $\Big) = \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big) - \gamma\Phi_{\text{str}}\Big($  $\Big)$

String-net dancing rule

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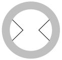
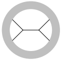
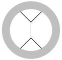
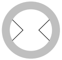
$\Phi_{\text{str}}\Big($  $\Big) = \gamma\Phi_{\text{str}}\Big($  $\Big) + \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big)$

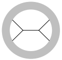
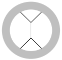
$\Phi_{\text{str}}\Big($  $\Big) = \sqrt{\gamma}\Phi_{\text{str}}\Big($  $\Big) - \gamma\Phi_{\text{str}}\Big($  $\Big)$

- **Topological excitations**:
  For fixed 4 ends of string-net on a sphere S^2, how many locally indistinguishable states are there?

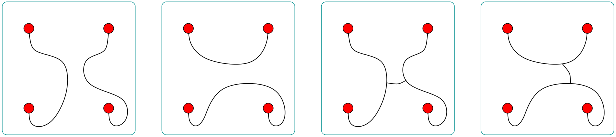
String-net dancing rule

- The dancing rule : $\Phi_{str}\Big($  $\Big) = \Phi_{str}\Big($  $\Big)$

$\Phi_{str}\Big($  $\Big) = \gamma\Phi_{str}\Big($  $\Big) + \sqrt{\gamma}\Phi_{str}\Big($  $\Big)$

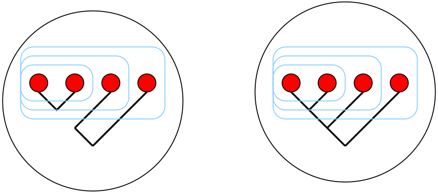
$\Phi_{str}\Big($  $\Big) = \sqrt{\gamma}\Phi_{str}\Big($  $\Big) - \gamma\Phi_{str}\Big($  $\Big)$

- **Topological excitations**:
For fixed 4 ends of string-net on a sphere S^2, how many locally indistinguishable states are there? **four states?**



Topo. degeneracy with 4 fixed ends of string-net

To get linearly independent states, we fuse the end of the string-net in a particular order:



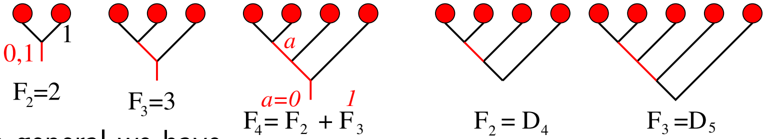
→ There are only **two** locally indistinguishable states
= a qubit

This is a quantum memory that is robust angainst any environmental noise.

→ The defining character of topological order:
a material with robust quantum memory.

Topo. degeneracy with n fixed ends of string-net

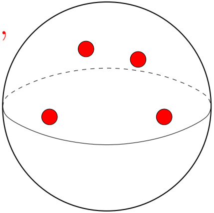
- Let D_n is the number of locally indistinguishable states with n fixed ends of string-net, on a sphere S^2. (We know $D_4 = 2$)
- To compute D_n, we count different linearly independent ways to fuse n ends of string-net



- In general we have

$$F_n = F_{n-1} + F_{n-2} \text{ (Fibonacci numbers) }, \qquad D_n = F_{n-2}$$

$\to D_0 = 1,\ D_1 = 0,\ D_2 = 1,\ D_3 = 1,\ D_4 = 2,$
$D_5 = 3,\ D_6 = 5,\ D_7 = 8,\ D_8 = 13, \cdots$



- An end of string-net is called a **Fibonacci anyon**

Internal degrees of freedom of a Fibonacci anyon

- To obtain the **internal degrees of freedom** of a Fibonacci anyon, we consider the number of linearly independent states with n fixed Fibonacci anyons in large n limit: $D_n \sim \big|_{n\to\infty} d^n$
- The number degrees of freedom = **quantum dimension**:

$$d = \lim_{n\to\infty} D_n^{1/n}$$

- To compute d, we note that $d = \lim_{n\to\infty} \frac{D_n}{D_{n-1}} = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$
We obtain $d = 1 + d^{-1}$ from $D_n = D_{n-1} + D_{n-2} \to$

$$d = \frac{\sqrt{5}+1}{2} = 1.618 = 2^{0.6942 \text{ qubits}}$$

- A spin-1/2 particle has a quantum dimension $d = 2 = 2^{1 \text{ qubit}}$
$d \neq$ integer $\to$ fractionalized degrees of freedom.

Double-Fibonacci topological order $= $ double G_2 Chern-Simon theory at level 1

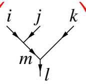
$$
\begin{aligned}
L(a_\mu, \tilde{a}_\mu) &= \frac{1}{4\pi}\mathrm{Tr}(a_\mu \partial_\nu a_\lambda + \frac{\mathrm{i}}{3} a_\mu a_\nu a_\lambda)\epsilon^{\mu\nu\lambda} \\
&\quad - \frac{1}{4\pi}\mathrm{Tr}(\tilde{a}_\mu \partial_\nu \tilde{a}_\lambda + \frac{\mathrm{i}}{3} \tilde{a}_\mu \tilde{a}_\nu \tilde{a}_\lambda)\epsilon^{\mu\nu\lambda}
\end{aligned}
$$

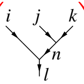
a_μ and $\tilde{a}_\mu$ are G_2 gauge fields.

String-net liquid can also realize gauge theory of finite group G

- Trivial type-0 string $\to$ trivial representation of G
- Type-i string $\to$ irreducible representation R_i of G
- Triple-string join rule If $R_i \otimes R_j \otimes R_k$ contain trivial representation $\to$ type-i type-j type-k strings can join.
- String reconnection rule:

$$\Phi\left(\text{diagram}_1\right) = \sum_{n=0}^{1} F^{ijm}_{kln} \Phi\left(\text{diagram}_2\right), \qquad i,j,k,l,m,n = 0,1$$

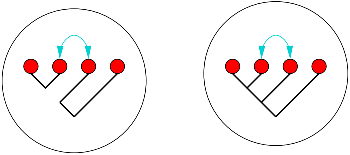




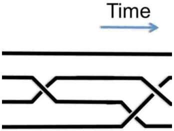
with F^{ijm}_{kln} given by the 6-j simple of G.

Topo. qubits and topo. quantum computation

- Four fixed Fibonacci anyons on S^2 has 2-fold **topological degeneracy** (two locally indistinguishable states) → **topological qubit**



- Exchange two Fibonacci anyons induce a 2×2 unitary matrix acting on the topological qubit → **non-Abelian statistics** *also appear in $\chi^3_{\nu=2}(z_i)$ FQH state, and the non-Abelian statistics is described by $SU_2(3)$ CS theory* Wen PRL **66** 802 (91) → universal **Topo. quantum computation** (via CS theory)



Freedman-Kitaev-Wang quant-ph/0001071; Freedman-Larsen-Wang quant-ph/0001108

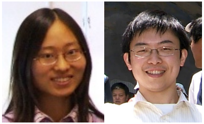
Topological order is the natural medium (the "silicon") to do topological quantum computation

Pattern of long-range entanglements = topo. order

For gapped systems with no symmetry:

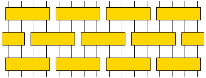
- According to Landau, no symmetry to break
  $\to$ all systems belong to one trivial phase

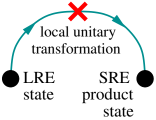
Pattern of long-range entanglements = topo. order



For gapped systems with no symmetry:

- According to Landau, no symmetry to break
  $\to$ all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - **long range entangled (LRE) states**
  - **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



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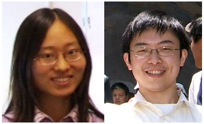
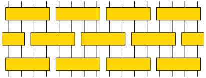
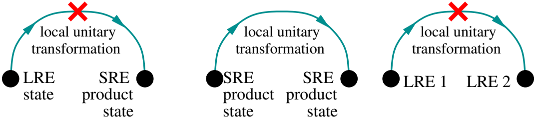
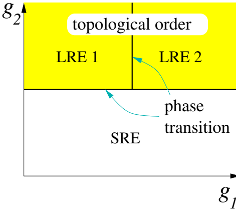


- According to Landau, no symmetry to break
  → all systems belong to one trivial phase
- Thinking about entanglement: Chen-Gu-Wen 2010
  - **long range entangled (LRE) states** → many phases
  - **short range entangled (SRE) states** → one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements**
  = different **topological orders** Wen 1989

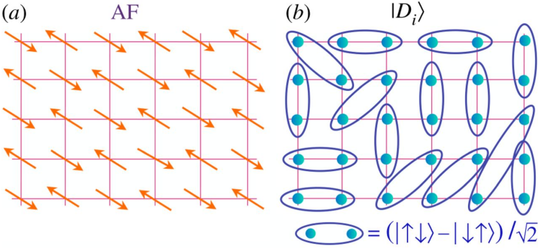
Lattice Hamiltonians to realize Z_2 topological order

- Frustrated spin-1/2 model on square lattice (slave-particle meanfield theory) Read Sachdev, PRL **66** 1773 (91); Wen, PRB **44** 2664 (91).

$$H = J \sum_{nn} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + J' \sum_{nnn} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

- Dimer model on triangular lattice (Mont Carlo numerics)

Moessner Sondhi, PRL **86** 1881 (01)



Why dimmer liquid has topological order

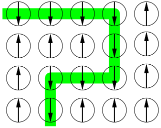
To make topological order, we need to sum over many different product states, but we should not sum over everything.

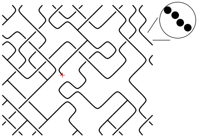
$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

Why dimmer liquid has topological order

To make topological order, we need to sum over many different product states, but we should not sum over everything.

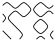
$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

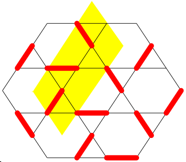


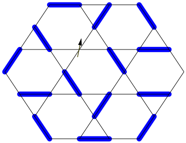


- sum over a subset of spin configurations:

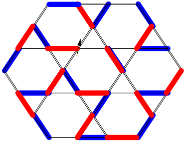
$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum$ |  $\rangle$

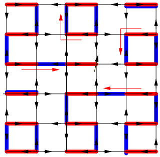
$|\Phi^{DS}_{\text{loops}}\rangle = \sum(-)^{\#\text{ of loops}}$ |  $\rangle$





- Dimmer liquid $\sim$ string liquid:
  Non-bipartite lattice: unoritaded string
  Bipartite lattice: oritaded string





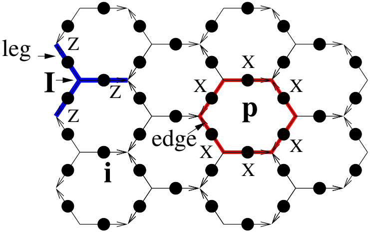
- Can the above wavefunction be the ground states of local Hamiltonians?

Toric-code model: Z_2 topo. order, Z_2 gauge theory

Local Hamiltonian enforces local rules: $\hat{P}\Phi_{\text{str}} = 0$

$\Phi_{\text{str}}(\;) - \Phi_{\text{str}}(\;) = \Phi_{\text{str}}(\;) - \Phi_{\text{str}}(\;) = 0$

- The Hamiltonian to enforce the local rules: Kitaev quant-ph/9707021



$$H = -U \sum_{I} \hat{Q}_I - g \sum_{p} \hat{F}_p, \quad \hat{Q}_I = \prod_{\text{legs of } I} \sigma_i^z, \quad \hat{F}_p = \prod_{\text{edges of } p} \sigma_i^x$$

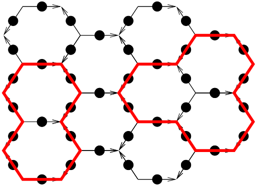
- The Hamiltonian is a sum of commuting operators

$[\hat{F}_p, \hat{F}_{p'}] = 0$, $[\hat{Q}_I, \hat{Q}_{I'}] = 0$, $[\hat{F}_p, \hat{Q}_I] = 0$. $\hat{F}_p^2 = \hat{Q}_I^2 = 1$

- Ground state $|\Psi_{\text{grnd}}\rangle$: $\hat{F}_p|\Psi_{\text{grnd}}\rangle = \hat{Q}_I|\Psi_{\text{grnd}}\rangle = |\Psi_{\text{grnd}}\rangle$

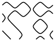
$\rightarrow (1 - \hat{Q}_I)\Phi_{\text{grnd}} = (1 - \hat{F}_p)\Phi_{\text{grnd}} = 0.$

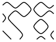
Physical properties of exactly soluble model



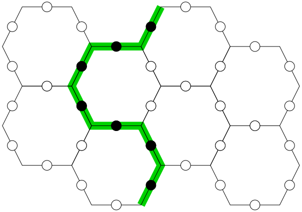
A string picture

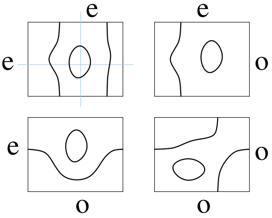
- The $-U\sum_I \hat{Q}_I$ term enforces closed-string ground state.
- $\hat{F}_p$ adds a small loop and deform the strings →

  permutes among the loop states $\Big|$  $\Big\rangle$ → Ground states

  $|\Psi_{\text{grnd}}\rangle = \sum_{\text{loops}} \Big|$  $\Big\rangle$ → highly entangled

- There are four degenerate ground states $\alpha = ee, eo, oe, oo$



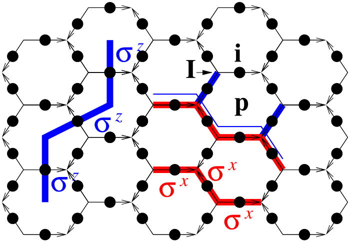


$D^{\text{tor}}=4$

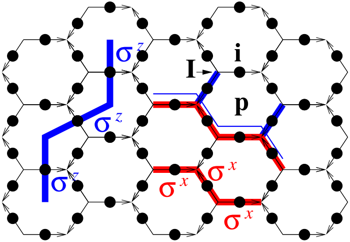
- On genus g surface, ground state degeneracy $D_g = 4^g$

The string operators and topological excitations

- Topological excitations:
  e-type: $\hat{Q}_I = 1 \to \hat{Q}_I = -1$
  m-type: $\hat{F}_p = 1 \to \hat{F}_p = -1$
- *e*-type and *m*-type excitations cannot be created alone due to identiy: $\prod_I \hat{Q}_I = \prod_p \hat{F}_p = 1$



The string operators and topological excitations



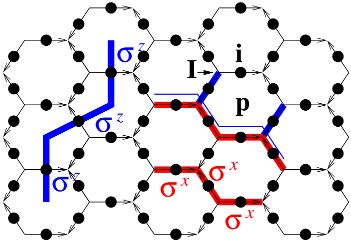
- Topological excitations:
  e-type: $\hat{Q}_I = 1 \to \hat{Q}_I = -1$
  m-type: $\hat{F}_p = 1 \to \hat{F}_p = -1$
- e-type and m-type excitations cannot be created alone due to identiy: $\prod_I \hat{Q}_I = \prod_p \hat{F}_p = 1$
- Type-e string operator: $W_e = \prod_{\text{string}} \sigma_i^x$
- Type-m string operator: $W_m = \prod_{\text{string*}} \sigma_i^z$
- Type-f string operator: $W_f = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$
- $[H, W_e^{\text{close}}] = [H, W_m^{\text{close}}] = [H, W_f^{\text{closed}}] = 0$.
  $\to$ Closed strings cost no energy
- $[\hat{Q}_I, W_e^{\text{open}}] \neq 0 \to W_e^{\text{open}}$ flip $\hat{Q}_I \to -\hat{Q}_I$,
  $[\hat{F}_p, W_m^{\text{open}}] \neq 0 \to W_m^{\text{open}}$ flip $\hat{F}_p \to -\hat{F}_p$
  An open-string creates a pair of topo. excitations at its ends

Three types of topological excitations and their fusion

- Type-e string operator $W_e = \prod_{\text{string}} \sigma_i^x$
- Type-m string operator $W_m = \prod_{\text{string}^*} \sigma_i^z$
- Type-f string operator $W_f = \prod_{\text{string}} \sigma_i^x \prod_{\text{legs}} \sigma_i^z$

- **Fusion algebra** of string operators
  $W_e^2 = W_m^2 = W_\epsilon^2 = W_e W_m W_\epsilon = 1$ when strings are parallel

- Fusion of topo. excitations:
  e-type. $e \times e = 1$
  m-type. $m \times m = 1$
  f-type $= e \times m$



- 4 types of excitations:
  $1, e, m, f$

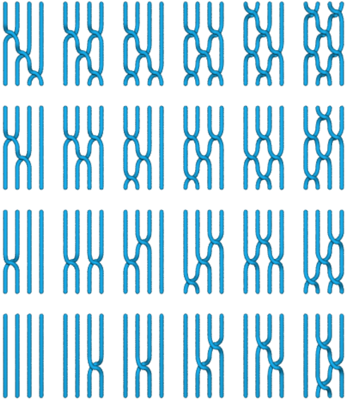
What are bosons? What are fermions?

- **Statistical distribution**
  Boson: $n_b = \frac{1}{e^{\epsilon/k_B T} - 1}$ Fermion: $n_f = \frac{1}{e^{\epsilon/k_B T} + 1}$
  They are just properties of non-interacting bosons or fermions
- **Pauli exclusion principle**
  Only works for non-interacting bosons or fermions
- **Symmtric/anti-symmetric wave function**.
  For identical particles $|x, y\rangle$ and $|y, x\rangle$ are just differnt names of same state. A generic state $\sum_{x,y} \psi(x, y)|x, y\rangle$ is always described symmetric wave function $\psi(x, y) = \psi(y, x)$ regardless the statistics of the identical particles.
- **Commuting/anti-commuting operators**
  Boson: $[a_x, a_y] = 0$ Fermiion: $\{c_x, c_y\} = 0$
- **C-number-field/Grassmann-field**
  Boson: $\phi(x)$ Fermion: $\psi(x)$

"Exchange" statistics and Braid group

- Quantum statistics is defined via phases induced by exchanging identical particles.
- *Quantum statistics is not defined via exchange, but via braiding.*

  Yong-Shi Wu, PRL **52** 2103 (84)



- **Braid group:**

“Exchange” statistics and Braid group

- Quantum statistics is defined via phases induced by exchanging identical particles.
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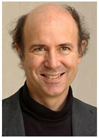
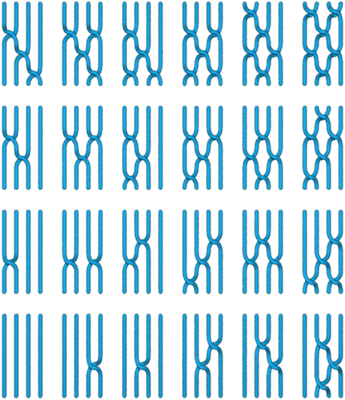
  Yong-Shi Wu, PRL **52** 2103 (84)





- **Braid group:**
- Representations of braid group (not permutation group) define quantum statistics:

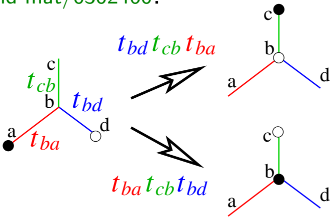
- Trivial representation of braid group → Bose statistics.
- 1-dimensional representation of braid group → Fermi/fractional statistics → **anyon**.

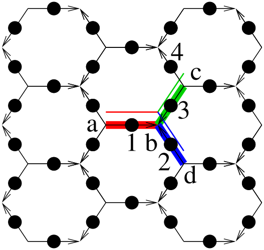
  Leinaas-Myrheim 77; Wilczek 82
- higher dimensional representation of braid group → non-Abelian statistics → **non-Abelian anyon**. Wen 91; More-Read 91

Statistics of ends of strings

- The statistics is determined by particle hopping operators

Levin-Wen cond-mat/0302460:

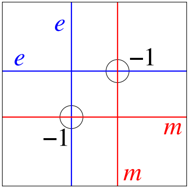




- An open string operator is a hopping operator of the 'ends'. The algebra of the open string op. determines the statistics.
- For type-*e* string: $t_{ba} = \sigma_1^x$, $t_{cb} = \sigma_3^x$, $t_{bd} = \sigma_2^x$
  We find $t_{bd}t_{cb}t_{ba} = t_{ba}t_{cb}t_{bd}$
  The ends of type-*e* string are bosons
- For type-*f* strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x}\sigma_4^z$, $t_{bd} = \sigma_2^x\underline{\sigma_3^z}$
  We find $t_{bd}t_{cb}t_{ba} = -t_{ba}t_{cb}t_{bd}$
  The ends of type-*f* strings are fermions

Topo. ground state degeneracy and code distance

- When strings cross,
  $W_e W_m = (-)^{\#\text{ of cross}} W_m W_e \to$
  4^g degeneracy on genus g surface
  $\to$ **Topological degneracy**
  Degeneracy remain exact for any perturbations localized in a finite region.



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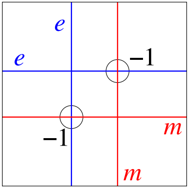
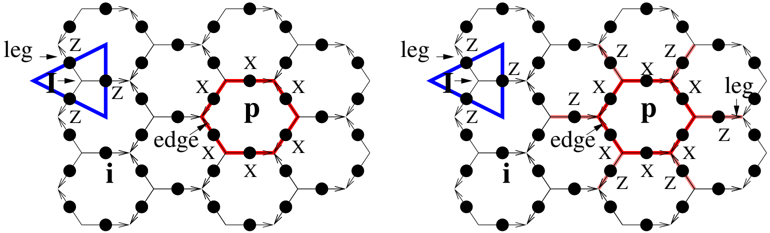


- The above degenerate ground states form a "code", which has a large **code distance** of order L (the size of the system).
- Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by first-order local perturbation δH: $\langle\psi'|\delta H|\psi\rangle > O(|\delta H|), \quad L \to \infty$
  $\to$ code distance $= 1$.
  Two states $|\psi\rangle$ and $|\psi'\rangle$ that can be connected by n^{th}-order local perturbation $\to$ code distance $= n$.
- Symm. breaking ground states in d-dim have code distance $\sim L^d$ respected to symm. preserving perturbation. code distance ~ 1 respected to symm. breaking perturbation.

Toric-code model and closed string operators



- Toric-code Hmailtonian

$$H = -U \sum_{\mathbf{I}} W_m^{\text{closed}} - g \sum_{\mathbf{p}} W_e^{\text{closed}}$$

- A new Hamitonian

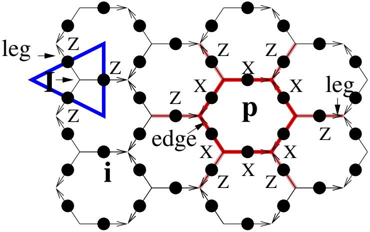
$$H = -U \sum_{\mathbf{I}} W_m^{\text{closed}} - g \sum_{\mathbf{p}} W_f^{\text{closed}}$$

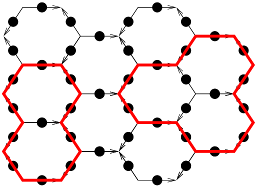
which realizes the same Z_2 topological order.

Double-semion model

Local rules: Levin-Wen cond-mat/0404617

- The Hamiltonian to enforce the local rules:



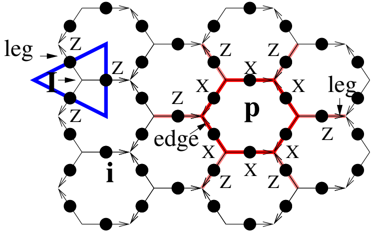


$$H = -U \sum_I \hat{Q}_I - \frac{g}{2} \sum_p (\hat{F}_p + h.c.), \quad \sigma^{z/2} \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \sim \sqrt{\sigma^z}$$

$$\hat{Q}_I = \prod_{\text{legs of } I} \sigma_i^z, \quad \hat{F}_p = \Big(\prod_{\text{edges of } p} \sigma_j^x\Big)\Big(-\prod_{\text{legs of } p} \sigma_i^{z/2}\Big)$$

Double-semion model

- The action of operator $\hat{F}_{\boldsymbol{p}} = (\prod_{\text{edges of } \boldsymbol{p}} \sigma^x_{\boldsymbol{j}})(-\prod_{\text{legs of } \boldsymbol{p}} \sigma^{z/2}_{\boldsymbol{i}})$:
  (1) flip string around the loop;
  (2) add a phase $-\mathrm{i}^{\#\text{ of strings attatched to the loop}}$.
  Combine the above two in the closed-string subspace: add a loop and a sign $(-)^{\text{change in } \# \text{ of loops}}$

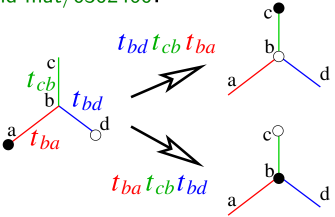


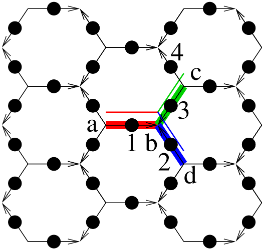
- $\hat{F}_{\boldsymbol{p}}$ is hermitian in the closed-string subspace.
- $\hat{F}_{\boldsymbol{p}}\hat{F}_{\boldsymbol{p}'} = \hat{F}_{\boldsymbol{p}'}\hat{F}_{\boldsymbol{p}}$ in the closed-string subspace.
- Ground state wave function $\Phi(X) = (-)^{\sigma^x_c}$, where σ^x_c is the number of loops in the string configuration X.

Statistics of ends of dressed strings

- The statistics is determined by particle hopping operators

Levin-Wen cond-mat/0302460:

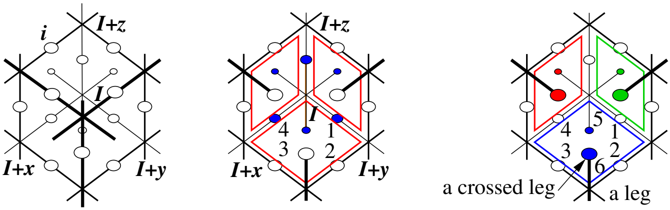




- For the dressed strings: $t_{ba} = \sigma_1^x$, $t_{cb} = \underline{\sigma_3^x}\sigma_4^{z/2}$, $t_{bd} = \sigma_2^x \underline{\sigma_3}^{z/2}$
  We find $t_{bd}t_{cb}t_{ba} = -\mathrm{i}\, t_{ba}t_{cb}t_{bd}$ via

  $\sigma_1^x\ \underline{\sigma_3^x}\sigma_4^{z/2}\ \ \sigma_2^x\underline{\sigma_3}^{z/2} = -\mathrm{i}\sigma_2^x\underline{\sigma_3}^{z/2}\ \ \underline{\sigma_3^x}\sigma_4^{z/2}\ \ \sigma_1^x$
  when acting on a state with two ends of strings at a, b
  → **The ends of dressed strings are semions**

3D Z_2 topological order on Cubic lattice



- Untwisted-string model: $H = -U\sum_I Q_I - g\sum_p F_p$

$$Q_I = \prod_{i \text{ next to } I} \sigma_i^z, \qquad F_p = \sigma_1^x\sigma_2^x\sigma_3^x\sigma_4^x$$

Can get 3D fermions for free (almost) **Levin & Wen 03**

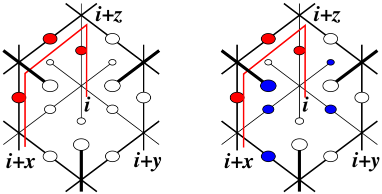
Just add a little twist

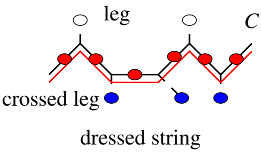
- Twisted-string model: $H = U\sum_I Q_I - g\sum_p F_p$

$$F_p = \sigma_1^x\sigma_2^x\sigma_3^x\sigma_4^x\sigma_5^z\sigma_6^z$$

String operators and Z_2 charges Levin & Wen 03

- A pair of Z_2 charges is created by an open string operator which commute with the Hamiltonian except at its two ends. Strings cost no energy and is unobservable.





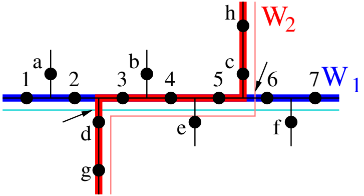
- In untwisted-string model – untwisted-string operator

$$\sigma^x_{i_1}\sigma^x_{i_2}\sigma^x_{i_3}\sigma^x_{i_4}\cdots$$

- In twisted-string model – twisted-string operator

$$(\sigma^x_{i_1}\sigma^x_{i_2}\sigma^x_{i_3}\sigma^x_{i_4}\dots)\prod_{i \text{ on crossed legs of } C}\sigma^z_i$$

Twisted string operators commute $[W_1, W_2] = 0$



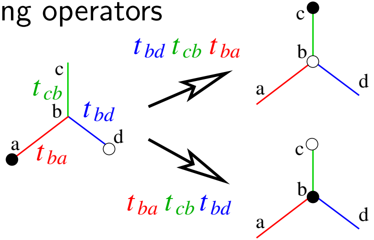
$$W_1 = (\sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4\sigma^x_5\boldsymbol{\sigma}^x_6\sigma^x_7)\ [\boldsymbol{\sigma}^z_d\sigma^z_e\sigma^z_f]$$
$$W_2 = (\sigma^x_h\sigma^x_c\sigma^x_5\sigma^x_4\sigma^x_3\boldsymbol{\sigma}^x_d\sigma^x_g)\ [\boldsymbol{\sigma}^z_6\sigma^z_e]$$

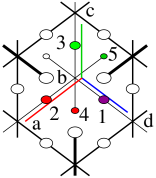
- We also have $[W, Q_I] = 0$ for closed string operators W, since W only create closed strings.

Statistics of ends of twisted strings

- The statistics is determined by particle hopping operators

Levin-Wen 03:



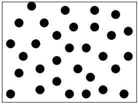


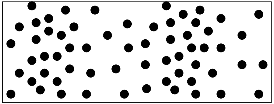
- An open string operator is a hopping operator of the 'ends'. The algebra of the open string op. determine the statistics.
- For untwisted-string model: $t_{ba} = \sigma^x_2$, $t_{cb} = \sigma^x_3$, $t_{bd} = \sigma^x_1$
  We find $t_{bd} t_{cb} t_{ba} = t_{ba} t_{cb} t_{bd}$
  The ends of untwisted-string are bosons
- For twisted-string model: $t_{ba} = \sigma^z_4 \sigma^z_1 \sigma^x_2$, $t_{cb} = \sigma^z_5 \sigma^x_3$, $t_{bd} = \sigma^x_1$
  We find $t_{bd} t_{cb} t_{ba} = -t_{ba} t_{cb} t_{bd}$
  The ends of twisted-string are fermions

Principle of emergence

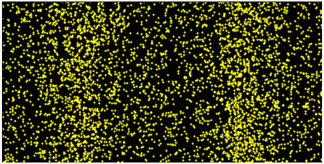
Different orders → different wave equations → different physical properties.

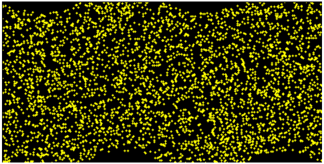
- Atoms in fluid have a random distribution
  → cannot resist shear deformations (do nothing)
  → liquids do not have shapes





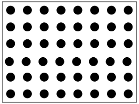
Wave Eq. → Euler Eq.
$\partial_t^2 \rho - \partial_x^2 \rho = 0$ One longitudinal mode



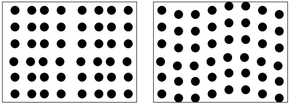


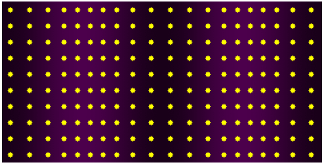
Principle of emergence

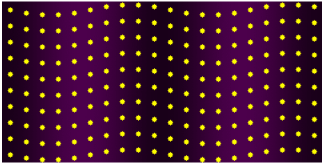
- Atoms in solid have a ordered lattice distribution
  $\to$ can resist shear deformations
  $\to$ solids have shapes



Wave Eq. $\to$ elastic Eq. $\partial_t^2 u^i - C^{ijkl}\partial_{x^j}\partial_{x^k} u^l = 0$
One longitudinal mode and
two transverse modes





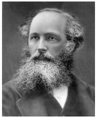


Origin of photons, gluons, electrons, quarks, etc

- Do all waves and wave equations emerge from some orders?

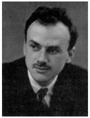
Wave equations for elementary particles

- Maxwell equation → Photons
$\boldsymbol{\partial} \times \boldsymbol{E} + \partial_t \boldsymbol{B} = \boldsymbol{\partial} \times \boldsymbol{B} - \partial_t \boldsymbol{E} = \boldsymbol{\partial} \cdot \boldsymbol{E} = \boldsymbol{\partial} \cdot \boldsymbol{B} = 0$



- Yang-Mills equation → Gluons
$\partial^\mu F^a_{\mu\nu} + f^{abc} A^{\mu b} F^c_{\mu\nu} = 0$

- Dirac equation → Electrons/quarks (spin-$\frac{1}{2}$ fermions!)
$[\partial_\mu \gamma^\mu + m]\psi = 0$



What orders produce the above waves?
What are the origins of light (gauge bosons) and electrons (fermions)?

Elementary or emergent?

- We used to think all orders are described by symmetry breaking, and different symmetry breaking orders indeed leads to different wave equations.
  - We just pick a particular symmetry breaking to produce the Maxwell equation.

Elementary or emergent?

- We used to think all orders are described by symmetry breaking, and different symmetry breaking orders indeed leads to different wave equations.
  - We just pick a particular symmetry breaking to produce the Maxwell equation.
- But none of the symmetry breaking orders can produce:
  - electromagnetic wave satisfying the Maxwell equation
  - gluon wave satisfying the Yang-Mills equation
  - electron wave satisfying the Dirac equation.

Elementary or emergent?

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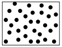
  - We just pick a particular symmetry breaking to produce the Maxwell equation.

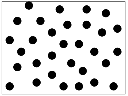
- But none of the symmetry breaking orders can produce:
  - electromagnetic wave satisfying the Maxwell equation
  - gluon wave satisfying the Yang-Mills equation
  - electron wave satisfying the Dirac equation.

Two choices:

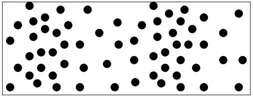
- Declare that photons, gluons, and electrons are elementary, and do not ask where do they come from.
- Declare that the symmetry breaking theory is incomplete. Maybe new orders beyond symmetry breaking can produce the Maxwell, Yang-Mills, and the Dirac equations.

Long range entanglements (closed strings) → emergence of electromagnetic waves (photons)

- Wave in superfluid state $|\Phi_{\text{SF}}\rangle = \sum_{\text{all position conf.}} \Big|$  $\Big\rangle$:

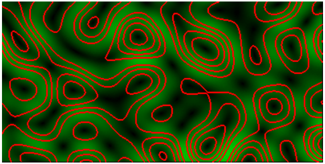


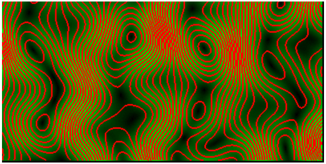
density fluctuations:
Euler eq.: $\partial_t^2 \rho - \partial_x^2 \rho = 0$
→ Longitudinal wave



- Wave in closed-string liquid $|\Phi_{\text{string}}\rangle = \sum_{\text{closed strings}} \Big|$  $\Big\rangle$:

Wen 03, Levin-Wen 05

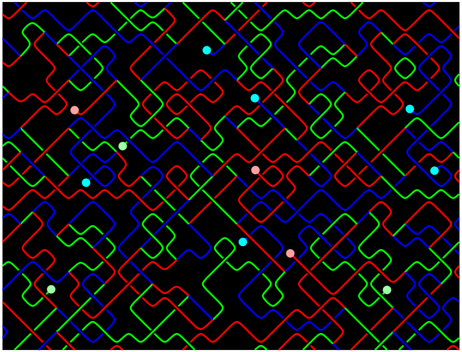




String density $\boldsymbol{E}(x)$ fluctuations → waves in string liquid.
Closed string → $\boldsymbol{\partial} \cdot \boldsymbol{E} = 0$ → **only two transverse modes**.
Equation of motion for string density → Maxwell equation:
$\dot{\boldsymbol{E}} - \boldsymbol{\partial} \times \boldsymbol{B} = \dot{\boldsymbol{B}} + \boldsymbol{\partial} \times \boldsymbol{E} = \boldsymbol{\partial} \cdot \boldsymbol{B} = \boldsymbol{\partial} \cdot \boldsymbol{E} = 0.$

Long range entanglements (string nets)$\to$ Emergence of Yang-Mills theory (gluons)

- If string has different types and can branch
  $\to$ string-net liquid
  $\to$ Yang-Mills theory
- Different ways that strings join $\to$ different gauge groups



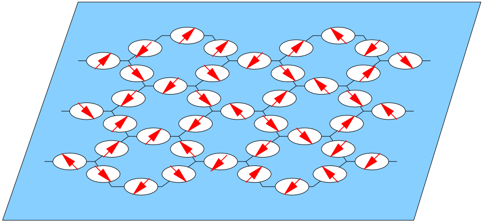
A picture of our vacuum   A string−net theory of light and electrons

Closed strings $\to$ $U(1)$ gauge theory
String-nets $\to$ Yang-Mills gauge theory   Levin-Wen 05

XXZ Spin-1 model on 2D Kagome lattice

- Only has nearest-neighbor and two-spin interactions:

$$H = J_1 \sum (S_i^z)^2 + J_2 \sum S_i^z S_j^z - J_{xy} \sum (S_i^x S_j^x + S_i^y S_j^y)$$



Eigenstates of S^z:

$$S^z|\uparrow\rangle = |\uparrow\rangle \qquad S^z|0\rangle = 0 \qquad S^z|\downarrow\rangle = -|\downarrow\rangle$$

A spin state with spin pointing in x-direction:

$$|\rightarrow\rangle = |\uparrow\rangle + |0\rangle + |\downarrow\rangle$$

Pictures of a few ground states of the spin system

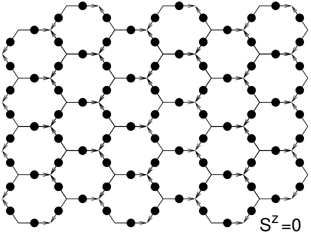
- $J_1 > 0$, $J_2 = g = 0$:
  All spins in the $|0\rangle$ state:
  $$|\Phi_0\rangle = |00...0\rangle = |0\rangle \otimes |0\rangle \otimes ... \otimes |0\rangle$$
  Excitations above the ground state:
  spin flips with finite gap.
- $J_{xy} > 0$, $J_1 = J_2 = 0$:
  All spins in the $|\rightarrow\rangle$ state:
  $$\begin{aligned} |\Phi_0\rangle &= |\rightarrow\rangle \otimes |\rightarrow\rangle \otimes ... \otimes |\rightarrow\rangle \\ &= (|\uparrow\rangle + |0\rangle + |\downarrow\rangle) \otimes (|\uparrow\rangle + |0\rangle + |\downarrow\rangle) \otimes ... \\ &= |\uparrow 00...\rangle + |0 \uparrow\downarrow ...\rangle + |\downarrow\uparrow\uparrow ...\rangle + ... \\ &= \text{a superposition of all } S^z\text{-spin configurations} \end{aligned}$$
  Excitations above the ground state:
  spin waves with no energy gap.

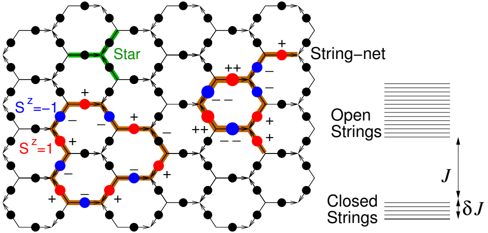
String liquid ground state

Introduce $\Delta J = J_1 - J_2$ and rewrite

$$H = \frac{J_2}{2}\sum(S_1^z + S_2^z + S_3^z)^2 + \Delta J \sum (S_i^z)^2 - g\sum(S_1^+ S_2^- S_3^+ S_4^- S_5^+ S_6^- + h.c.)$$

When $\Delta J = g = 0$, the no string state and closed string states all have zero energy:

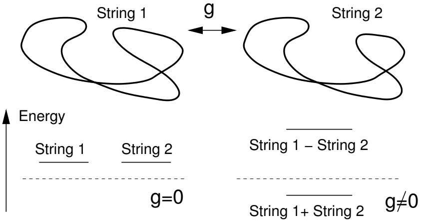




No string state: $|000...\rangle$ Closed-string state

- The strings are oriented.

- The effect of ΔJ term: String tension
- The effect of g term: String hopping

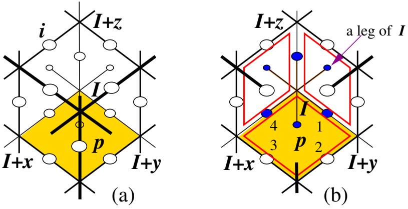


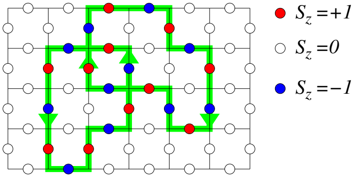
When $\Delta J \ll g \ll J_2$, the ground state is a superposition of all closed-string states. Such a state is called *string-net condensed state* – a new state of matter that breaks no symmetries.

Compare with some well known states

- Crystal: Particles have a fixed regular positions.
- Superfluid (liquid): Particles have uncertain positions. Ground state = superposition of all particle positions.
- Plastic: Polymers have a fixed random configuration.
- String liquid: Strings have uncertain configurations. Ground state = superposition of all string-net configurations.

3D String-net condensation in cubic lattice





$$H = U \sum_{\mathbf{I}} Q_{\mathbf{I}}^2 + J \sum (S_{\mathbf{i}}^z)^2 - g \sum_{\mathbf{p}} (B_{\mathbf{p}} + h.c.)$$

$$Q_{\mathbf{I}} = \sum_{\mathbf{i} \text{ next to } \mathbf{I}} S_{\mathbf{i}}^z, \qquad gB_{\mathbf{p}} = gS_1^+ S_2^- S_3^+ S_4^-$$

Here S^z is the angular momentun of a rotor.
$S^{\pm}$ is raising/lowering operator of S^z.

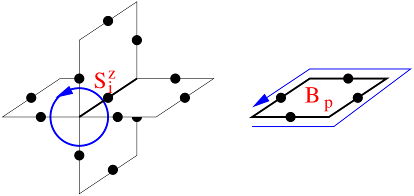
$U \sum_{\mathbf{I}} Q_{\mathbf{I}}^2$:	only closed string states have low energies
$J \sum (S_{\mathbf{i}}^z)^2$:	string tension
$g \sum_{\mathbf{p}} (B_{\mathbf{p}} + h.c.)$:	string hopping

Equation of motion approach → Maxwell equation

To understand the dynamics of $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K}{2}\hat{x}^2$:

$$\frac{d}{dt}\langle \hat{x}\rangle = \langle i[\hat{H},\hat{x}]\rangle = \langle \hat{p}/m\rangle, \quad \frac{d}{dt}\langle \hat{p}\rangle = \langle i[\hat{H},\hat{p}]\rangle = -\langle K\hat{x}\rangle$$

Equation of motion of an oscillator.



Emergence of Maxwell equation

$$B_{\boldsymbol{p}} = e^{i\phi_{\boldsymbol{p}}}, \qquad S^z_{\boldsymbol{i}} = E_{\boldsymbol{i}}$$

$$\partial_t \langle S^z_{\boldsymbol{i}}\rangle = \langle i[H, S^z_{\boldsymbol{i}}]\rangle \sim i\langle \sum_{a=1,..,4} B_{\boldsymbol{p}_a} - h.c.\rangle \sim \sum_{a=1,..,4} \phi_{\boldsymbol{p}_a}$$

$$\to \dot{\boldsymbol{E}} = \boldsymbol{\partial} \times \boldsymbol{B}$$

$$\mathrm{i}\partial_t \langle \phi_{\boldsymbol{p}}\rangle = \partial_t \langle B_{\boldsymbol{p}}\rangle = \langle i[H, B_{\boldsymbol{p}}]\rangle \sim i\langle \sum_{a=1,..,4} S^z_{\boldsymbol{i}_a} B_{\boldsymbol{p}}\rangle \sim i \sum_{a=1,..,4} S^z_{\boldsymbol{i}_a}$$

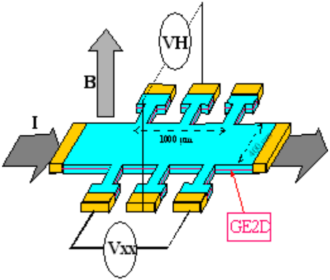
$$\to \dot{\boldsymbol{B}} = \boldsymbol{\partial} \times \boldsymbol{E}$$

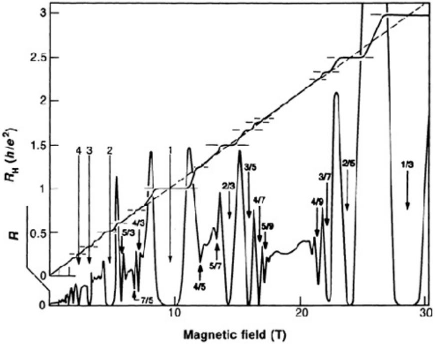
The experimental discovery of FQH effect

- Quantum Hall states (1980's)
  Quantized Hall conductance:

$$\sigma_{xy} = \frac{I}{V_H} = \frac{m}{n}\frac{e^2}{h} = \frac{1}{R_H}$$

$$\frac{m}{n} = \nu = \frac{\#\text{ of electrons}}{\#\text{ of flux quanta}}$$

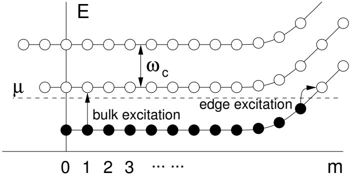


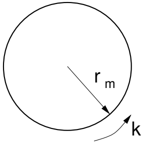


Introduction of IQH states

- One-particle in magnatic field (choose $B = 1$ and $z = x + \mathrm{i}y$): $\quad H_0 = -\sum(\partial_z - \frac{B}{4}z^*)(\partial_{z^*} + \frac{B}{4}z)$
- First Landau level state: $\Psi(z) = z^m \mathrm{e}^{-\frac{1}{4}|z|^2}$, since $\mathrm{e}^{\frac{1}{4}zz^*}(\mathrm{i}\partial_z - \mathrm{i}\frac{1}{4}z^*)(\mathrm{i}\partial_{z^*} + \mathrm{i}\frac{1}{4}z)\mathrm{e}^{-\frac{1}{4}zz^*} = (\mathrm{i}\partial_z - \mathrm{i}\frac{1}{2}z^*)\mathrm{i}\partial_{z^*}$

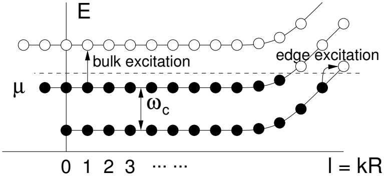
  $\nu = 1$ IQH state:





- Higher Landau levels:

  $\nu = 2$ IQH state:



Introduction of FQH states

- N-electrons (fermionic or bosonic) in a magnetic field:

$$H = \sum_{i=1}^{N} (\mathrm{i}\partial_{z_i} - \mathrm{i}\frac{B}{4} z_i^*)(\mathrm{i}\partial_{z_i^*} + \mathrm{i}\frac{B}{4} z_i) + \sum_{i<j} V(x_i - x_j, y_i - y_j)$$

- When $V = 0$, there are many minimal energy wave functions

$$\Psi = P(z_1, \cdots, z_N)\mathrm{e}^{-\frac{1}{4}\sum_{i=1}^{N} z_i z_i^*}, \quad P = \text{ a (anti-)symm. polynomial}$$

all have zero energy (for any P):

$$\Big[\sum_{i=1}^{N} (\mathrm{i}\partial_{z_i} - \mathrm{i}\frac{B}{4} z_i^*)(i\partial_{z_i^*} + \mathrm{i}\frac{B}{4} z_i)\Big] P(z_1, \cdots, z_N)\mathrm{e}^{-\frac{1}{4}\sum_{i=1}^{N} z_i z_i^*} = 0$$

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- For small non-zero V, there is only one minimal energy wave function P whose form is determined by V.

3 ideal FQH states: the exact ground states

- $\nu = 1/2$ bosonic Laughlin state: $z_1 \approx z_2$, second order zero

$$P_{1/2} = \prod_{i<j}(z_i - z_j)^2, \quad V_{1/2}(z_1, z_2) = \delta(z_1 - z_2),$$

$$[\sum_{i<j} V_{1/2}(z_i - z_j)]P_{1/2} = 0.$$

All other states have finite energies in $N \to \infty$ limit (gapped).

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All other states have finite energies in $N \to \infty$ limit (gapped).

- $\nu = 1/4$ bosonic Laughlin state: $z_1 \approx z_2$, fourth-order zero

$$P_{1/4} = \prod_{i<j} (z_i - z_j)^4$$

$$V_{1/4}(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \partial^2_{z_1^*} \delta(z_1 - z_2) \partial^2_{z_1}$$

3 ideal FQH states: the exact ground states

- $\nu = 1$ Pfaffian state: $z_1 \approx z_2$, no zero; $z_1 \approx z_2 \approx z_3$, second-order zero:

$$P_{\text{Pf}} = \mathcal{A}\Big(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\Big)\prod_{i<j}(z_i - z_j)$$

$$= \text{Pf}(\frac{1}{z_i - z_j})\prod_{i<j}(z_i - z_j)$$

$$V_{\text{Pf}}(z_1, z_2, z_3)$$
$$= \mathcal{S}[v_0\delta(z_1 - z_2)\delta(z_2 - z_3) - v_1\delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}]$$

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- $\nu = 1$ fermionic IQH state: $z_1 \approx z_2$, first-order zero:

$$P_1 = \prod_{i<j}(z_i - z_j); \qquad V_1(z_1, z_2) = 0$$

Non-Abelian topo. order in quantum Hall systems

Abelian topological order $\to$ fractional statistics

- IQH and Laughlin many-body state   Laughlin PRL **50** 1395 (1983)

$\chi_1 = \prod_{1\leq i<j\leq N}(z_i - z_j)\mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \quad \Psi_{\nu=1/n} = \prod(z_i - z_j)^3\mathrm{e}^{-\frac{n}{4}\sum |z_i|^2}$
$= (\chi_1)^3$

where $z_i = x_i + \mathrm{i}\, y_i$ and $\chi_m = m$ filled Landau levels.

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Non-abelian topological order $\to$ non-abelian statistics

- $SU(m)_2$ state via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi_{SU(2)_2} = \chi_1(\chi_2)^2, \ \nu = \frac{1}{2}; \qquad \Psi_{SU(3)_2} = (\chi_2)^3, \ \nu = \frac{2}{3};$$

$\to$ $SU(m)_2$ Chern-Simons effective theory $\to$ non-abelian statistics

- Pfaffien state via CFT correlation — Moore-Read NPB **360** 362 (1991)

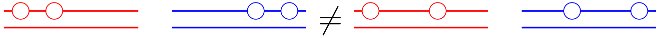
$$\Psi_{Pfa} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)^2\mathrm{e}^{-\frac{1}{4}\sum|z_i|^2}, \qquad \nu = \frac{1}{2}$$

- *The $\Psi_{SU(2)_2}$ and Ψ_{Pfa} have the same Ising non-abelian statistics*
- *The $\Psi_{SU(3)_2}$ state has the Fibonacci non-abelian statistics.*

Non-Abelian statistics

Non-Abelian statistics = presence of topo. degeneracy even when all the quasiparticles are fully trapped.

- The ground state $\chi_1(\chi_2)^2 = \chi_1\chi_2\chi_2$ is non-degenerate.
- Degeneracy D_{deg} of 4 trapped quasiparticles at x_1, x_2, x_3, x_4: many different wave functions: $\chi_1\chi_2^{x_1x_2}\chi_2^{x_3x_4} \neq \chi_1\chi_2^{x_1x_3}\chi_2^{x_2x_4}$



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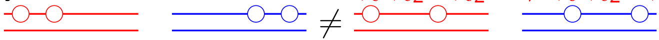
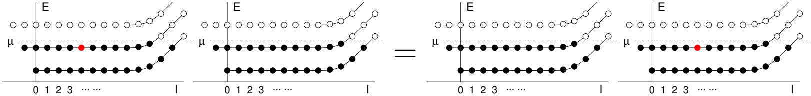


- The above represent a topological degeneracy, since locally the two wave functions $\chi_1\chi_2^{x_1}\chi_2$ and $\chi_1\chi_2\chi_2^{x_1}$ are identical.



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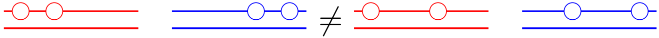

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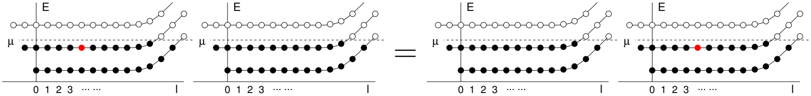

- The presence of the topological degeneracy → The braiding is described by unitary matrix $U(D_{\text{deg}})$ → non-Abelian statistics.

Fractionalized degrees of freedom

- N trapped quasiparticle $\to$ degeneracy $D_{\text{deg}}(N)$. Each particle carries degrees of freedom $d = \lim_{N\to\infty}[D_{\text{deg}}(N)]^{\frac{1}{N}}$ (the quantum dimension of the particle).

- $d = 2$ from spin-1/2 particles
  $d = 3$ from spin-1 particles.
- For $\chi_1(\chi_2)^2$ state $d = \sqrt{2}$
  (half qubit) – Ising anyon.
- For $(\chi_2)^3$ state $d = \frac{\sqrt{5}+1}{2}$
  (0.69 qubits) – Fibonacci anyon.

How to known $[\chi_m(z_1,...,z_N)]^n$
is a non-Abelian QH state?
What kind of non-Abelian state?

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Split an electron into partons

Projective construction for Laughlin states

Assume the bosonic electrons have an interaction to have a gaped ground wavefunction:

$$\Psi(z_1, ..., z_N) = [\chi_1(z_1, ..., z_N)]^2 = P[\chi_1(z_1^{(1)}, ...)\chi_1(z_1^{(2)}, ...)]$$

- Electron $\to$ 2 kinds of partons, each kind $\to$ $\nu = 1$ IQH χ_1
- The projection P binds 2-partons into an electron $z_i^{(1)} = z_i^{(2)} = z_i$

- Effective theory of independent partons ($I = 1, 2$)

$$L = \psi_I^\dagger[\partial_t - \mathrm{i}\frac{1}{2}(\bar{A}_0 + \delta A_0)]^2\psi_I + \frac{1}{2m}\psi_I^\dagger[\partial_i - \mathrm{i}\frac{1}{2}(\bar{\boldsymbol{A}} + \delta \boldsymbol{A})]^2\psi_I$$

The electron density (and the parton density) is such that each parton form a $\nu = 1$ IQH state χ_1.

- Integrating out ψ_I in path integral $\to$ effective Lagrangian

$$L(\delta A_\mu) = \frac{1}{4\pi}\delta A_\mu \partial_\nu \delta A_\lambda \epsilon^{\mu\nu\lambda}[(\frac{1}{2})^2 + (\frac{1}{2})^2]$$

$\to$ $U(1)$ Chern-Simons gauge theory.

Hall conductance $\sigma_{xy} = \frac{e^2}{h}[(\frac{1}{2})^2 + (\frac{1}{2})^2]$

The low energy effective theory

- Introduce dynamical $U(1)$ gauge field to do projection (glue partons back to electrons):

$$L = \psi_I^\dagger[\partial_t - \mathrm{i}\frac{1}{2}\bar{A}_0 - \mathrm{i}\, a_0]^2\psi_I + \frac{1}{2m}\psi_I^\dagger[\partial_i - \mathrm{i}\frac{1}{2}\bar{\boldsymbol{A}} - \mathrm{i}\,\boldsymbol{a}]^2\psi_I$$

- Integrating out ψ_I in path integral $\to$ effective Lagrangian

$$L(a_\mu) = \frac{1+1}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{1}{g}(f_{\mu\nu})^2$$

$\to$ $U(1)_2$ Chern-Simons gauge theory at level 2.

- $U(1)_m$-Chern-Simons theory at level m have fractional statistics $\theta = \pi/m$.
  $U(1)_2$ Chern-Simons gauge theory has semions $\theta = \pi/2$.

Projective construction for non-Abelain FQH states

Wen PRL **66** 802 (1991)

Assume electrons have an interaction such that the following many-body wave function is a gaped ground state:

$$\Psi(z_1,...,z_N) = [\chi_m(z_1,...,z_N)]^n = P[\chi_m(z_1^{(1)},...)\chi_m(z_1^{(2)},...)\cdots]$$

- Electron $\to$ n kinds of partons, each kind $\to$ $\nu = m$ IQH χ_m
- We then bind n-partons into an electron $z_i^{(I)} = z_i^{(J)} = z_i$

• Effective theory of independent partons

$$L = \psi_I^\dagger[\partial_t - \mathrm{i}\frac{1}{n}\bar{A}_0]\psi_I + \frac{1}{2m}\psi_I^\dagger[\partial_i - \mathrm{i}\frac{1}{n}\bar{\boldsymbol{A}}]^2\psi_I, \qquad I = 1,\cdots,n$$

The electron density (and the parton density) is such that each parton form a $\nu = m$ IQH state χ_m.

Projective construction for non-Abelain FQH states

- Introduce dynamical $SU(n)$ gauge field to do projection (glue partons back to electrons):

$$\psi_I^\dagger[\partial_t - \mathrm{i}\frac{1}{n}\bar{A}_0\delta_{IJ} - \mathrm{i}(a_0)_{IJ}]^2\psi_J + \frac{1}{2m}\psi_I^\dagger[\partial_i - \mathrm{i}\frac{1}{n}\bar{\boldsymbol{A}}\delta_{IJ} - \mathrm{i}\boldsymbol{a}_{IJ}]^2\psi_J$$

- Integrating out ψ_I in path integral $\to$ effective Lagrangian

$$L(a_\mu) = \frac{m}{4\pi}\mathrm{Tr}(a_\mu\partial_\nu a_\lambda + \frac{\mathrm{i}}{3}a_\mu a_\nu a_\lambda)\epsilon^{\mu\nu\lambda} - \frac{1}{g}(f_{\mu\nu})^2$$

$\to$ $SU(n)_m$ Chern-Simons gauge theory at level m.

- $SU(n)_m$-CS theory have non-Abelian statistics if $m > 1$.

- $SU(2)_2$ CS gauge theory has Ising non-Abelian anyon.
- $SU(2)_3$ CS gauge theory has Fibonacci non-Abelian anyon.
- $SU(3)_2$ CS gauge theory has Fibonacci non-Abelian anyon.

How to realize non-Abelian FQH states

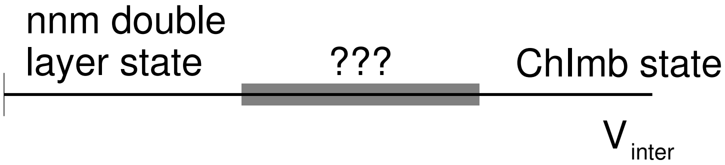
- $\Psi_{\nu=2/5} = (\chi_1)^2\chi_2$ can be realized if 2 LLs are degenerate
  $\Psi_{SU(2)_2} = \chi_1(\chi_2)^2$ can be realized if 3 LLs are degenerate
  $\Psi_{SU(3)_2} = (\chi_2)^3$ can be realized if 4 LLs are degenerate

- **Realizing non-Abelian FQH state in bi-layer systems**
  Starting with (nnm) state

$$\Phi_{nnm} = \prod (z_i - z_j)^n (w_i - w_j)^n (z_i - w_i)^m e^{-\frac{1}{4}\sum |z_i|^2 + |w_i|^2}$$

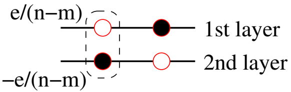
where $n = odd$ for fermionic electron.

- Phase diagram for increasing interlayer repulsion

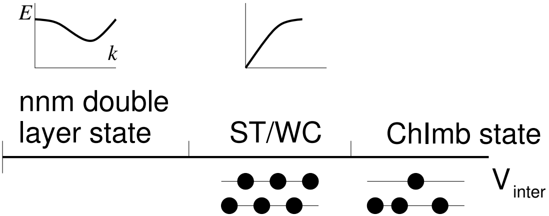


Two possibilities from exciton condensation

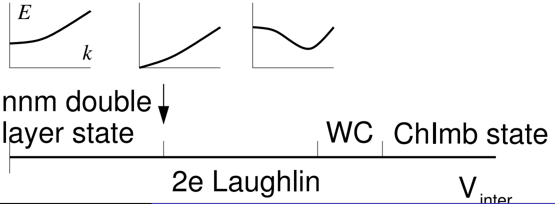
- Fractionalized exciton in (nnm) state has fractional statistics $\theta = \frac{2\pi}{n-m}$



- If the exciton has $k \neq 0$ → Wigner crystal:



- If the exciton has $k = 0$ → charge-$2e$ Laughlin state ($K = (8)$)



Critical theory for quantum phase transition

- Start with GL theory for bosonic excitons and anti-excitons:

$$\mathcal{L} = |\partial_\mu \phi|^2 + M|\phi|^2 + U|\phi|^4$$

  $M = 0$ at the transition.

- GL-CS theory to reproduce statistics $\theta = \frac{2\pi}{n-m}$

  $|(\partial - ia_1 + ia_2)\phi|^2 + M|\phi|^2 + U|\phi|^4 + \frac{1}{4\pi} a_I \partial a_J K^{IJ}, \quad K = \begin{pmatrix} n & m \\ m & n \end{pmatrix}$

- CS term does not destroy the critical point of GL theory, but changes the critical exponents

  $(nnm) \to$ *2e-Laughlin is a continuous transition between two states with the SAME symmetry*

- When $n - m = 2$, critical theory is a massless Dirac fermion theory

$$\mathcal{L} = \bar{\psi}\gamma^\mu \partial_\mu \psi + M\bar{\psi}\psi$$

  The mass $M = 0$ at the transition.

Wen cond-mat/9908394

Phase diagram with interlayer tunneling

- **Without interlayer tunneling**: Effective theory near transition

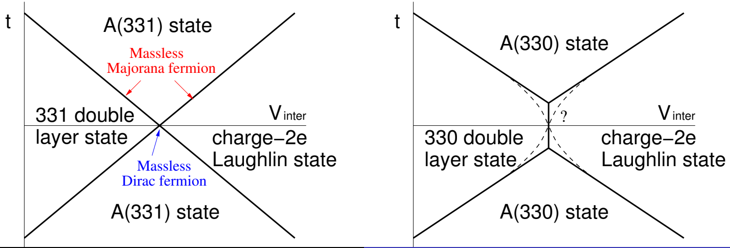
$$\mathcal{L} = |(\partial - ia_1 + ia_2)\phi|^2 + M|\phi|^2 + U|\phi|^4 + \frac{K^{IJ}}{4\pi} a_I \partial a_J .$$

$$\mathcal{L} = \bar\psi \gamma^\mu \partial_\mu \psi + M\bar\psi\psi, \qquad \text{for } n - m = 2$$

- **With interlayer tunneling**: Effective theory near transition ($n - m$ excitons = interlayer particle-hole)

$$\mathcal{L} = |(\partial - ia_1 + ia_2)\phi|^2 + M|\phi|^2 + U|\phi|^4 + (t\phi^{n-m}\hat{M} + h.c) + \frac{K^{IJ}}{4\pi} a_I \partial a_J .$$

$$\mathcal{L} = \bar\psi \gamma^\mu \partial_\mu \psi + m\bar\psi\psi + (t\psi^T\psi + h.c.), \qquad \text{for } n - m = 2$$



States from interlayer tunneling: $\mathcal{A}(331)$, $\mathcal{A}(330)$

- Two-layer state to one-layer state via anti-symmetrization:
$$\Psi_{\mathcal{A}(nnm)}(x_i) = \mathcal{A}[\prod(z_i - z_j)^n (w_i - w_j)^n (z_i - w_j)^m].$$

Wen-Wang arXiv:0801.3291

- **Characterize them with pattern-of-zeros**:
(similar to s-wave, p-wave, etc of superconducting states)

	S_2	S_3	S_4	S_5	$\cdots$
$\Psi_{\mathcal{A}(331)}$	1	5	10	18	$\cdots$
$\Psi_{\mathcal{A}(330)}$	1	3	6	12	$\cdots$
$\prod(z_i - z_j)^n$	n	$3n$	$6n$	$10n$	$\cdots$

$S_a =$ total relative angular momentum of a electrons.

POZ theory of FQG states

- **Obtain their properties using POZ** $\to$ Spectrum of gapless edge excitations. The ground state has a total angular momentum M_0. The chiral edge excitations have higher angluar mementa $M_0 + m$. $D_{\text{edge}}(m) =$ number of edge excitations at $M_0 + m$.
- How to compute $D_{\text{edge}}(m)$?
  $D_{\text{edge}}(m) =$ number of anti-symmetric holomorphic functions $\Psi(z_i)$ whose n-electron relative angular momentum $\tilde{S}_n \geq S_n$.

The edge spectrum $D_{\text{edge}}(m)$

$m:$	0	1	2	3	4	$\cdots$	c	remark
$\Psi_{\mathcal{A}(331)}$	1	1	3	5	10	$\cdots$	$\frac{3}{2}$	Z_2 parafermion
$\Psi_{\mathcal{A}(330)}$	1	1	3	6	13	$\cdots$	2	Z_4 parafermion
$\prod(z_i - z_j)^n$	1	1	2	3	5	P_m	1	Abelian Laughlin state

Central charge for the edge states

- The edge spectrum $D_{\text{edge}}(m)$ is described by central charge c.
  For $\prod(z_i - z_j)^m$: $P_m \sim \frac{1}{4m\sqrt{3}} e^{\pi\sqrt{\frac{2M}{3}}} \sim e^{c\pi\sqrt{\frac{2M}{3}}}$ with $c = 1$.
  In general $D_{\text{edge}}(m) \sim e^{c\pi\sqrt{\frac{2M}{3}}}$

- The central charge can be measured by specific heat
  $C = c\frac{\pi}{6}\frac{k_B^2 T}{v\hbar}$ or thermal Hall conductivity $\kappa_{xy} = c\frac{\pi}{6}\frac{k_B^2 T}{\hbar}$

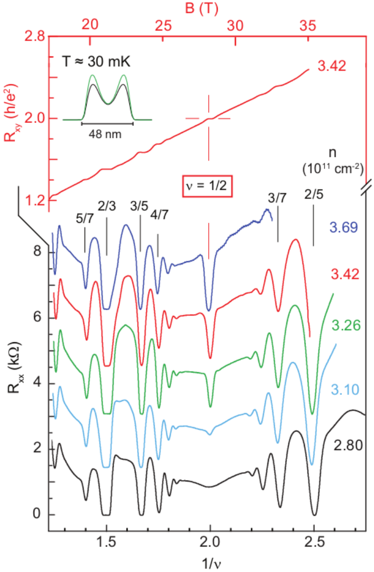
- The edge spectrum $D_{\text{edge}}(m)$ = finger print for FQH states:
- $D_{\text{edge}}(m)$ = partition number $\to \Psi_{\nu=1/m}$ is an Abelian state.
- $\Psi_{\mathcal{A}(331)}$ is a Z_2 parafermion state.
- $\Psi_{\mathcal{A}(330)}$ is a Z_4 parafermion state. (Related to $\chi_1(\chi_4)^2$ state $SU(2)_4$.) Blok-Wen Nucl. Phys. B374, 615 (92); Read-Rezayi cond-mat/9809384

- Interlayer tunneling can induce the above non-Abelian states.

Bilayer FQH in a quantum well (width = 48nm)

- For very large interlayer tunnelin
  we get a single-layer compressibl
  state at $\nu = 1/2$.
- For very small interlayer tunnelin
  we get a bi-layer (331) state.
- In between, we may get the Z_2
  parafermion non-Abelian state.

- To get (331) state from
  $\nu = 1/2$ FL state, we need
  a *d*-wave pairing $\to$ impossible.
- *p*-wave pairing on
  $\nu = 1/2$ FL state gives us Z_2
  parafermion non-Abelian state.
- *With less interlayer tunneling, can we see Z_2 parafermion $\to$ (331) transition?*



Shabani, Shayegan, etal arXiv:1306.5290

Two-component states in bi-layer systems

We have discussed one-component states (ie single-layer states) in bi-layers: $\Psi(\{x_i\})$.

- Now we consider two-component states in bi-layers, such as $\Psi(z_i, w_i) = \prod(z_i - z_j)^n (w_i - w_j)^n (z_i - w_j)^m$
- The pattern-of-zeros description of two-component states: S_{ab} = the total relative angular momentum for a cluster of a electron in layer-1 and b electron in layer-2.
- For the (nnm) state $S_{ab} = n\frac{a(a-1)}{2} + n\frac{b(b-1)}{2} + mab$:

$\Psi^{\nu=1/2}_{(331)}$, $c = 2$

S_{ab}	0	1	2	3
0	0	0	3	9
1	0	1	5	12
2	3	5	10	18
3	9	12	18	27

$\Psi^{\nu=1}_{(111)}$, gapless "superfluid"

S_{ab}	0	1	2	3
0	0	0	1	3
1	0	1	3	6
2	1	3	6	10
3	3	6	10	15

(331) state has a stronger intralayer avoidance than (111) state

Fibonacci non-Abelian statistics in bi-layer systems

- There are other more interesting FQH states described by different POZs, such as $\nu = \frac{4}{5}, \frac{4}{7}$ bi-layer states: Barkeshli-Wen arXiv:0906.0341

$\Psi^{\nu=4/5}_{SU(3)_2/U^2(1)}$, $c = 3\frac{1}{5}$

S_{ab}	0	1	2	3
0	0	0	1	5
1	0	1	2	7
2	1	2	4	9
3	5	7	9	15

$\Psi^{\nu=4/7}_{SU(3)_2/U^2(1)}$, $c = 3\frac{1}{5}$

S_{ab}	0	1	2	3
0	0	0	1	5
1	0	1	4	9
2	1	4	8	15
3	5	9	15	23

- Compare to the (111) state, the $\nu = 4/5$ state has a stronger intralayer avoidance and a weaker interlayer avoidance.
- Compare to the $\nu = \frac{2}{5} + \frac{2}{5}$ state, the $\nu = 4/5$ state has the same intralayer avoidance and a stronger interlayer avoidance.
- Appear in weak interlayer tunneling limit.

- Just like $(\chi_2)^3$ state, those $\Psi_{SU(3)_2/U^2(1)}$ states also have Fibonacci non-Abelian anyon with quantum dimension

Fibonacci non-Abelian statistics in wide quantum wells ?

- $\nu = 4/5$ FQH state was observed in bi-layer systems (wide quantum wells).
  Is it a Fibonacci FQH state that can do universal topological quantum computation?