Supersymmetric models for lattice fermions: critical in 1D, frustrated in 2D

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Acknowledgements & references

K. Schoutens, UvA
P. Fendley, J. Halverson, UVA

P. Fendley, K. Schoutens, J. de Boer, PRL (2003)
L. Huijse, K. Schoutens, EPJ B (2008)
L. Huijse, J. Halverson, P. Fendley, K. Schoutens, PRL (2008)

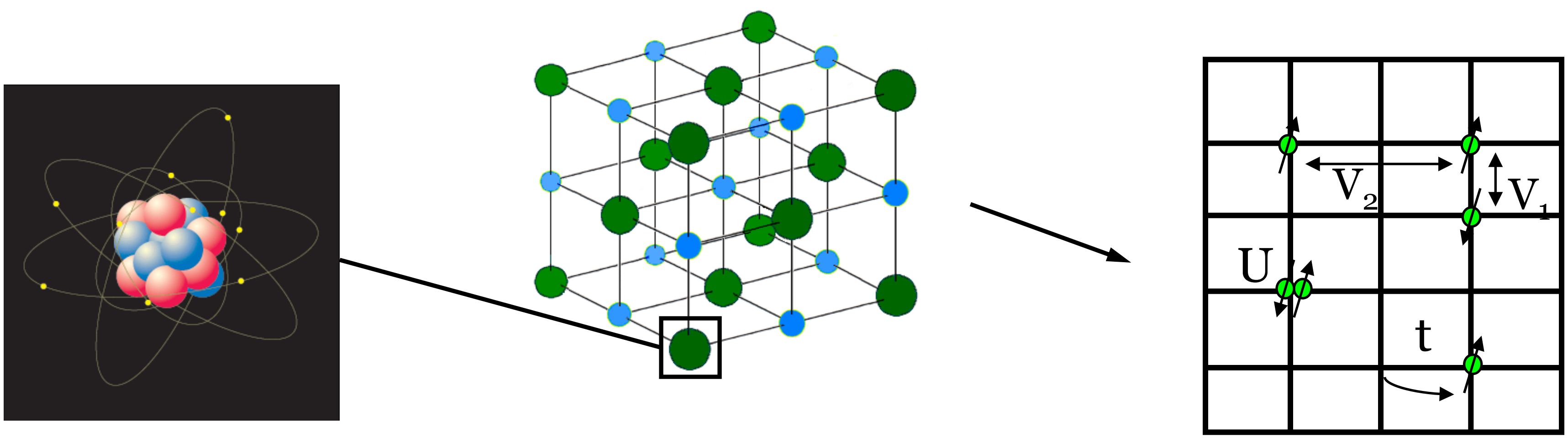
Introduction

condensed matter theory:

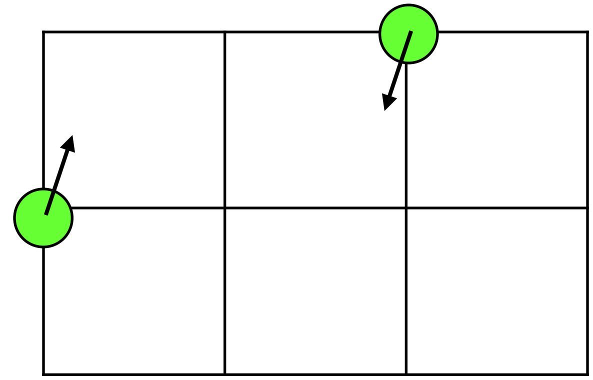
microscopic constituent
vs macroscopic behavior of solids

many-body quantum mechanics:

strip down the problem: study lattice models



Lattice models



configurations:

electrons located on the sites of an ionic lattice in a solid

Hamiltonian (energy operator):

typically a sum of kinetic (hopping) terms and short range repulsive interactions

$$\{c_i^+, c_j\} = \delta_{ij}$$

$$n_i = c_i^+ c_i$$

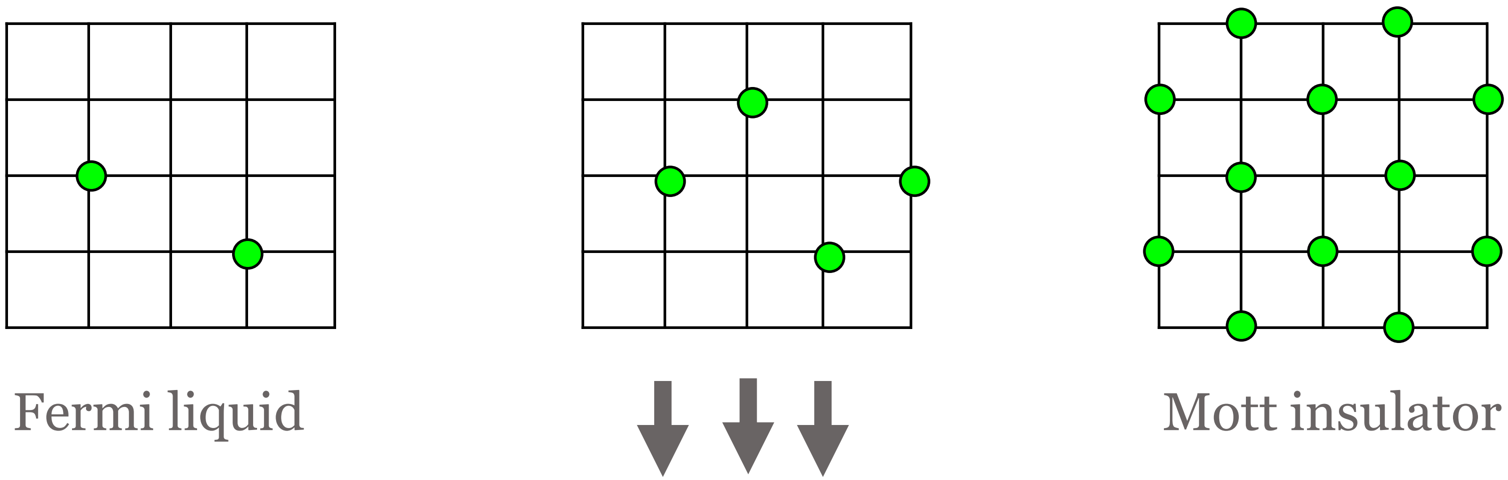
typical example (tV model):

$$N_f = \sum_i n_i$$

$$H_{tV} = t \sum_{<ij>} [c_i^+ c_j + c_j^+ c_i] + V \sum_{<ij>} n_i n_j - \mu N_f$$

Motivation

challenge: understand strongly repelling lattice fermions at densities intermediate between the low-density Fermi-liquid and a high-density Mott insulator.



hardly any analytical tools (non-perturbative regime)

hardly any numerical results (Fermi sign problem)

Supersymmetric model for lattice fermions

allows for various analytical results:

• quantum criticality in 1D
(CFT description of continuum limit)

• superfrustration in 2D
(extensive ground state entropy)

ground states sit at intermediate density:

degeneracy is due to subtle interplay between kinetic and potential terms.

Outline

- Supersymmetric QM
- The model
- Powerful tool: Witten index
- Superfrustration in 2D
- Quantum criticality in 1D
- Critical egde modes

Supersymmetric QM

Supersymmetric QM: algebraic structure

susy charges Q^+, $Q^-=(Q^+)^+$ and fermion number N_f:

$$(Q^+)^2 = 0, \quad (Q^-)^2 = 0, \quad [N_f, Q^\pm] = \pm Q^\pm$$

Hamiltonian defined as

$$H = \left\{Q^+, Q^-\right\}$$

satisfies

$$[H, Q^+] = [H, Q^-] = 0\,, \quad [H, N_f] = 0$$

Spectrum of supersymmetric QM

- $E \geq 0$ for all states

- $E > 0$ states are paired into doublets of the susy algebra

$$\{|\psi>, Q^{+}|\psi>\}, \quad Q^{-}|\psi>=0$$

- $E = 0$ iff a state is a singlet under the susy algebra

$$Q^{+}|\psi>=Q^{-}|\psi>=0$$

if $E = 0$ ground state exist, supersymmetry is unbroken.

Witten index

$$W = \mathrm{tr}(-1)^{N_f} e^{(-\beta H)}$$

Independent of β:

E>0 doublets: $\{|\psi\rangle, Q^+|\psi\rangle\}$

have f and f+1 particles and same energy

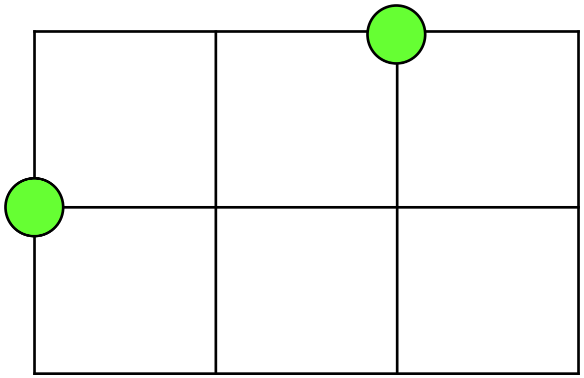
$$\rightarrow \quad W = \mathrm{tr}(-1)^{N_f}$$

|W| is lower bound on # of GS

The model

Basic susy lattice model

configurations:
lattice fermions with nearest neighbor exclusion



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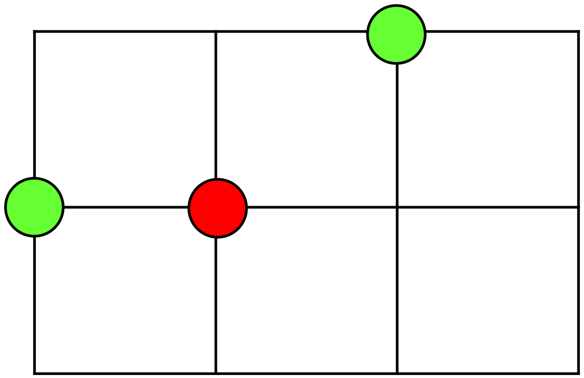

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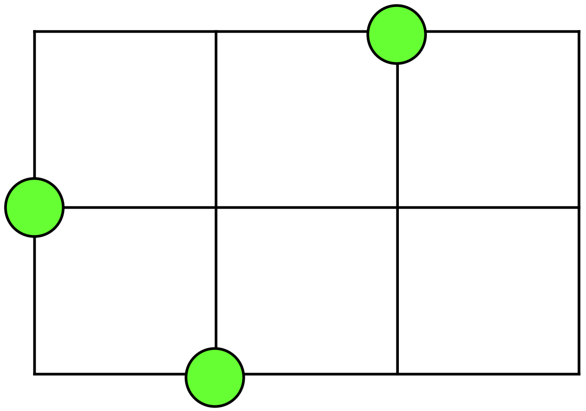

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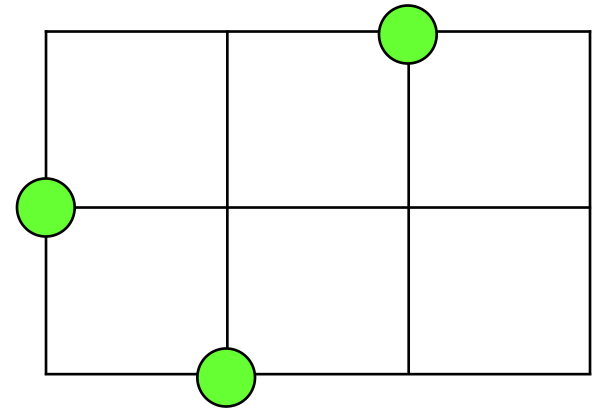


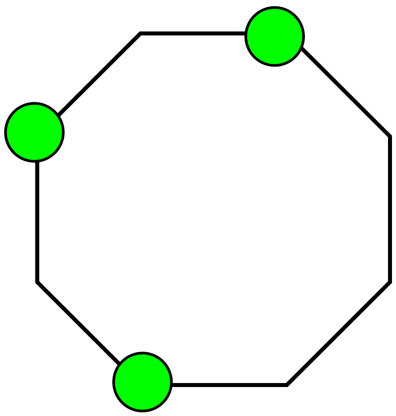
nilpotent supercharges, respecting exclusion rule:

$$Q^{+} = \sum_i c_i^{+} \prod_{\delta} (1 - n_{i+\delta}), \quad Q^{-} = (Q^{+})^{+} \qquad n_i = c_i^{+} c_i$$

Hamiltonian: kinetic (hopping) plus potential terms

$$H = \left\{ Q^{+}, Q^{-} \right\} = H_{kin} + H_{pot}$$

Basic susy model in 1D



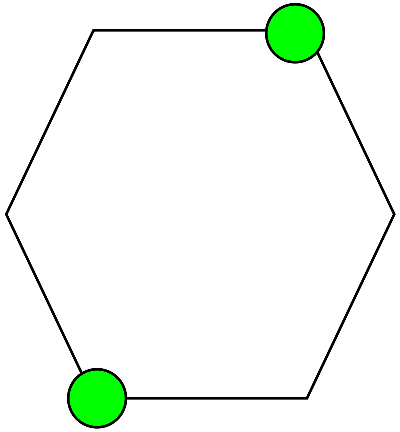
supercharges

$$Q^{+} = \sum_i (1-n_{i-1}) c_i^{+} (1-n_{i+1}), \quad Q^{-} = (Q^{+})^{+}$$

Hamiltonian:

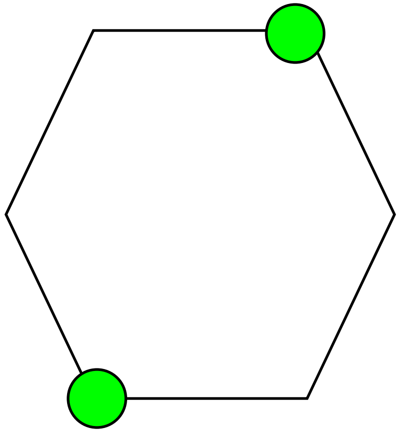
$$H = \sum_i [(1-n_{i-1}) c_i^{+} c_{i+1} (1-n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1} n_{i+1} - 2N_f + L$$

L=6 model: Witten index



$$W = \mathrm{Tr}\left((-1)^{N_f} e^{-\beta H} \right)$$

L=6 model: Witten index



$$W = \mathrm{Tr}\left((-1)^{N_f} e^{-\beta H}\right)$$

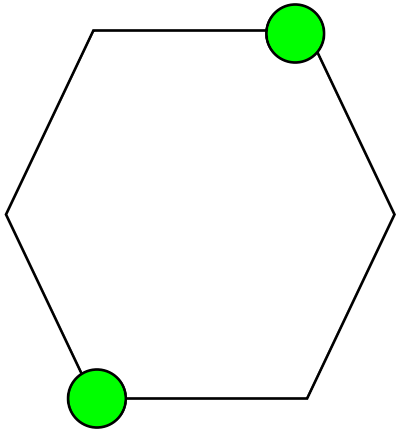
$N_f = 0$: 1 state

$N_f = 1$: 6 states

$N_f = 2$: 9 states

$N_f = 3$: 2 states

L=6 model: Witten index



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$N_f = 0$: 1 state

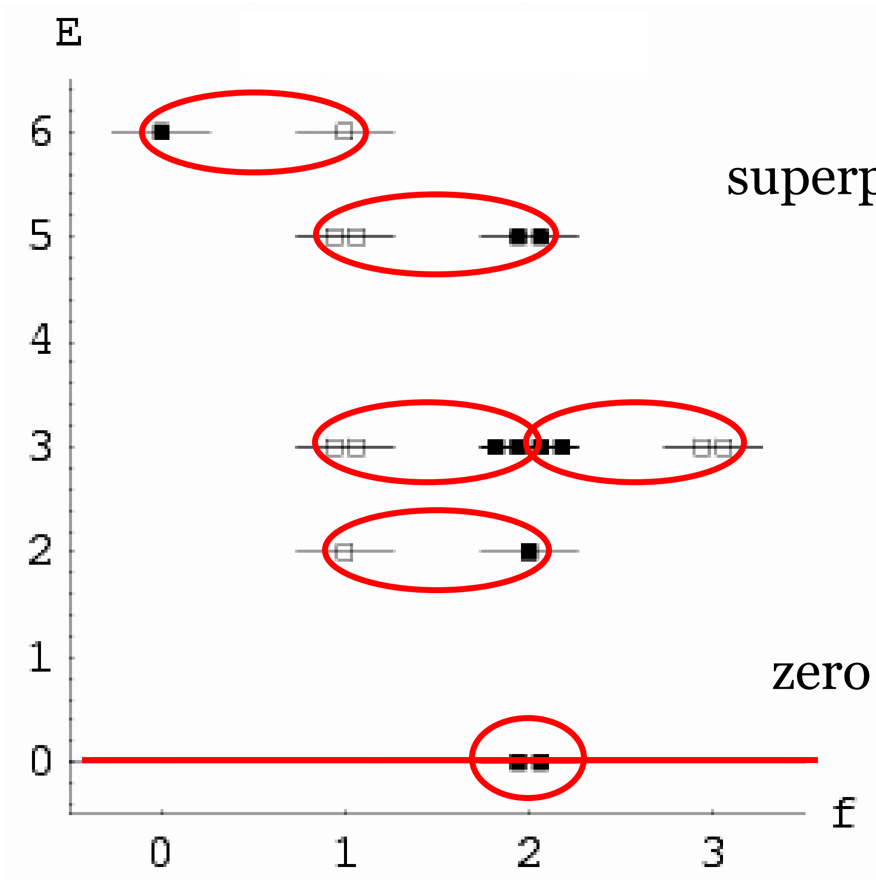
$N_f = 1$: 6 states

$N_f = 2$: 9 states

$N_f = 3$: 2 states

$$\Rightarrow W = 1 - 6 + 9 - 2 = 2$$

Spectrum for *L=6* sites



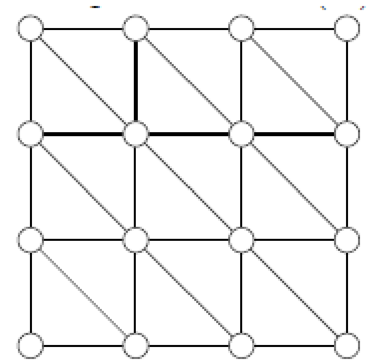
superpartners grouped in doublets

zero energy ground states, singlets

Superfrustration in 2D

Triangular lattice: Witten index

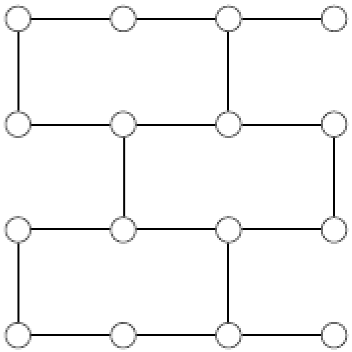
NB Lower bound to the number of ground states!



	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

[H. van Eerten]

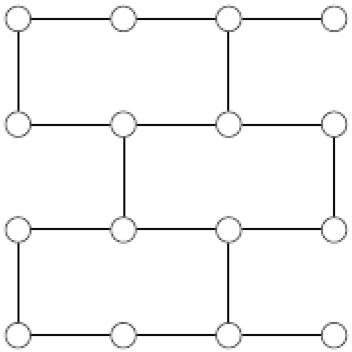
Hexagonal lattice: Witten index



	2	4	6	8	10	12	14	16	18
2	-1	-1	2	-1	-1	2	-1	-1	2
4	3	7	18	47	123	322	843	2207	5778
6	-1	-1	32	-73	44	356	-1387	2087	2435
8	3	7	18	55	123	322	843	2215	5778
10	-1	-1	152	-321	-171	7412	-26496	10079	393767
12	3	7	156	1511	6648	29224	150069	1039991	6208815
14	-1	-1	338	727	-5671	1850	183560	-279497	-4542907
16	3	7	1362	12183	31803	379810	5970107	55449303	327070578

[H. van Eerten]

Hexagonal lattice: Witten index



Superfrustration

	2	4	6	8	10	12	14	16	18
2	-1	-1	2	-1	-1	2	-1	-1	2
4	3	7	18	47	123	322	843	2207	5778
6	-1	-1	32	-73	44	356	-1387	2087	2435
8	3	7	18	55	123	322	843	2215	5778
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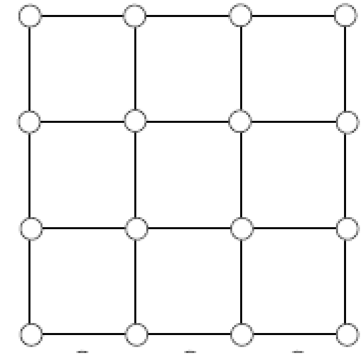
[H. van Eerten]

Square lattice: Witten index



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

[H. van Eerten]

Square lattice: Witten index

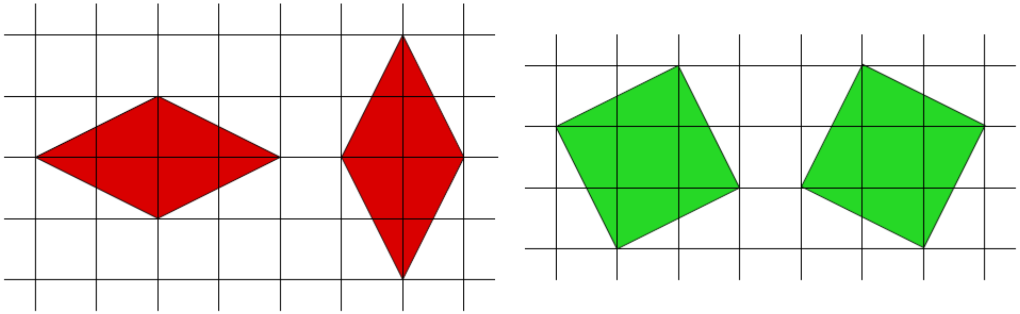
Witten index is related to tiling configurations

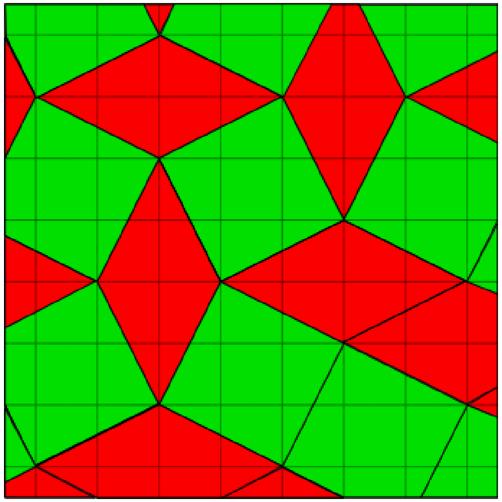
Theorem [Jonsson]:

$$W_{\vec{u},\vec{v}} = t_{even} - t_{odd} - (-1)^{d_-}\theta_{d_-}\theta_{d_+}$$

$$d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$$

$$\theta_{3p} = 2 \quad \theta_{3p\pm1} = -1$$





t is the number of tilings
u and v give the periodicities

Square lattice: cohomology

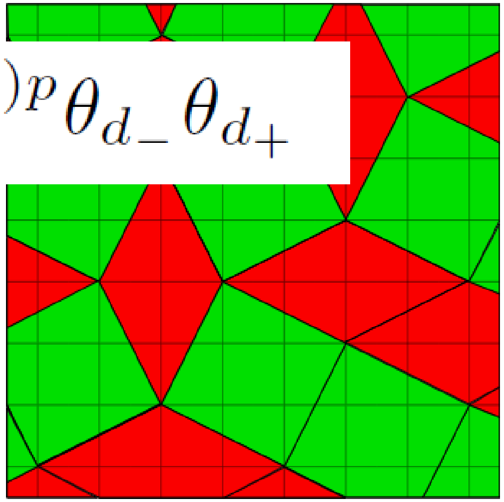
total number of GS is related to tiling configurations

Theorem [LH, Halverson, Fendley, Schoutens]:

$$\#\ \mathrm{GS} = t_{even} + t_{odd} - (-1)^{(\theta_m+1)p}\theta_{d_-}\theta_{d_+}$$

$$d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$$

$$\theta_{3p} = 2 \quad \theta_{3p\pm1} = -1$$



$$v_1 + v_2 = 3p$$

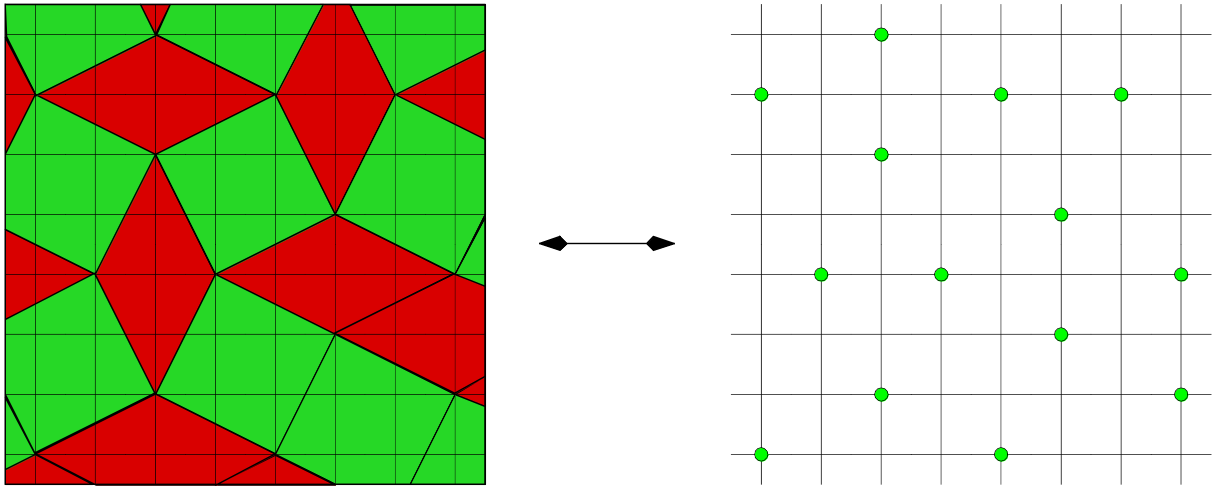
$$\vec{u} = (m, -m)$$

t is the number of tilings
u and v give the periodicities

Square lattice: physics

- square lattice: # GS grows exponentially with the linear dimensions of the system
- zero energy ground states at intermediate filling:

$$\frac{N_f}{L} \in [1/5, 1/4] \cap \mathbb{Q}$$

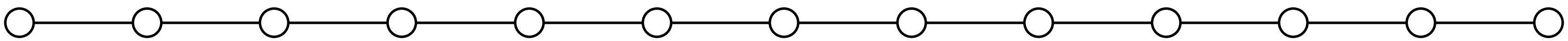


Quantum criticality in 1D

Quantum criticality in 1D lattice models

- correlation length diverges (powerlaw decay)
- in continuum limit: there is no scale (lattice spacing vanishes, correlation length diverges)
- massless/gapless system
- conformal invariance (angle preserving transformations)
- continuum limit of 1D critical lattice model is described by a 1+1D CFT
- direct relation between states in the lattice model and states in the CFT

Quantum criticality for SUSY model in 1D



- 2 gs for L multiple of 3, else 1 gs (periodic chain)
- exactly solvable via Bethe Ansatz

  → continuum limit:

  $\mathcal{N}$=2 SCFT with central charge c=1
- mapping to XXZ chain with Δ=1/2
- free boson:

$$S = \frac{2}{3\pi} \int dx\, dt \ \left[(\partial_t \Phi)^2 - (\partial_x \Phi)^2 \right]$$

$\mathcal{N}$=2 SCFT description for the chain

$$S = \frac{2}{3\pi}\int dx\,dt\ \left[(\partial_t\Phi)^2 - (\partial_x\Phi)^2\right]$$

vertex operators: $$V_{m,n} = \exp\left(\imath m\Phi + \imath n\tilde{\Phi}\right)$$

$$\Phi = \Phi_L + \Phi_R,\ \tilde{\Phi} = \frac{2}{3}(\Phi_L - \Phi_R)$$

conformal dimensions: $$h_{L,R} = \frac{3}{8}(m \pm \frac{2}{3}n)^2$$

Ramond sector: $$(-1)^{m+2n} = -1$$

$\mathcal{N}$=2 SCFT description for the chain

states in lattice model correspond to operators acting on the vacuum in the SCFT: $V_{m,n}|0\rangle$

the corresponding energy is: $(h_L + h_R - c/12)$

for finite size: $E_{\text{num}} = E_{\text{CFT}} v_F / L = (h_L + h_R - c/12) v_F / L$

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U(1) charges (Kac-Moody algebra): $q_{L,R} = n/3 \pm m/2$

fermion number: $N_f - N_{f_{GS}} = q_L - q_R = m$

momentum: $P = (Q_0 \pi + 2\pi (h_L - h_R)/L) \mod 2\pi$

$Q_0 = q_L + q_R = 2n/3$

Virasoro algebra: descendents

$$L_{-k,L}L_{-l,R}|h_L,h_R\rangle = |h_L+k,h_R+l\rangle$$

Virasoro algebra: descendents

$$L_{-k,L}L_{-l,R}|h_L,h_R\rangle = |h_L+k,h_R+l\rangle$$

$$\to E = h_L+k+h_R+l-c/12$$

$$P = (Q_0\pi + 2\pi(h_L+k-h_R-l)/L) \mod 2\pi$$

Virasoro algebra: descendents

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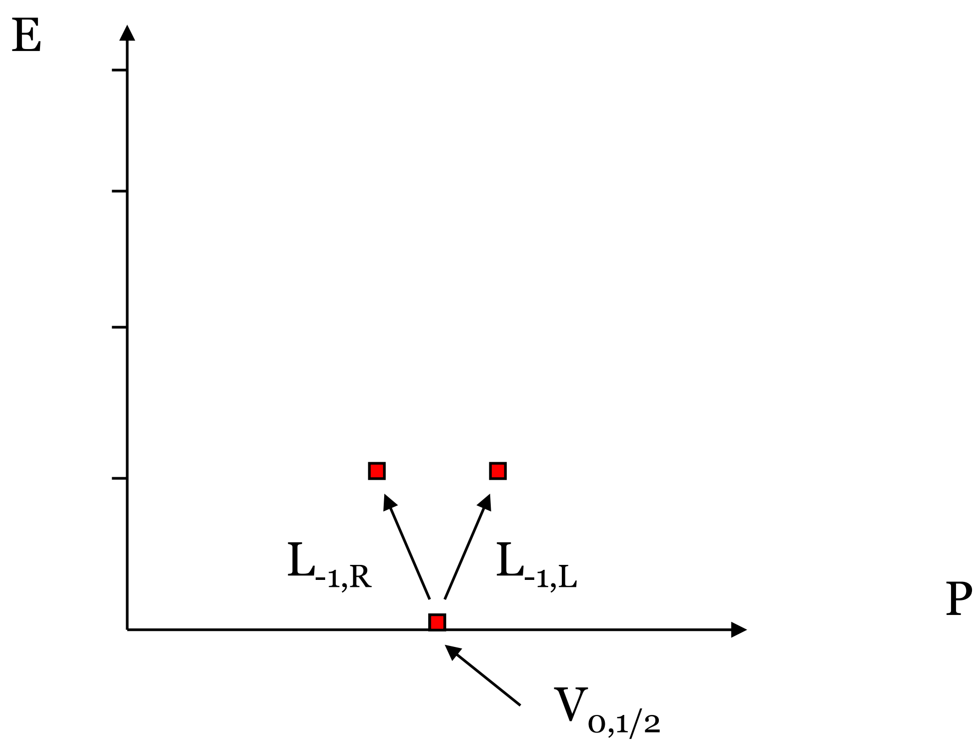

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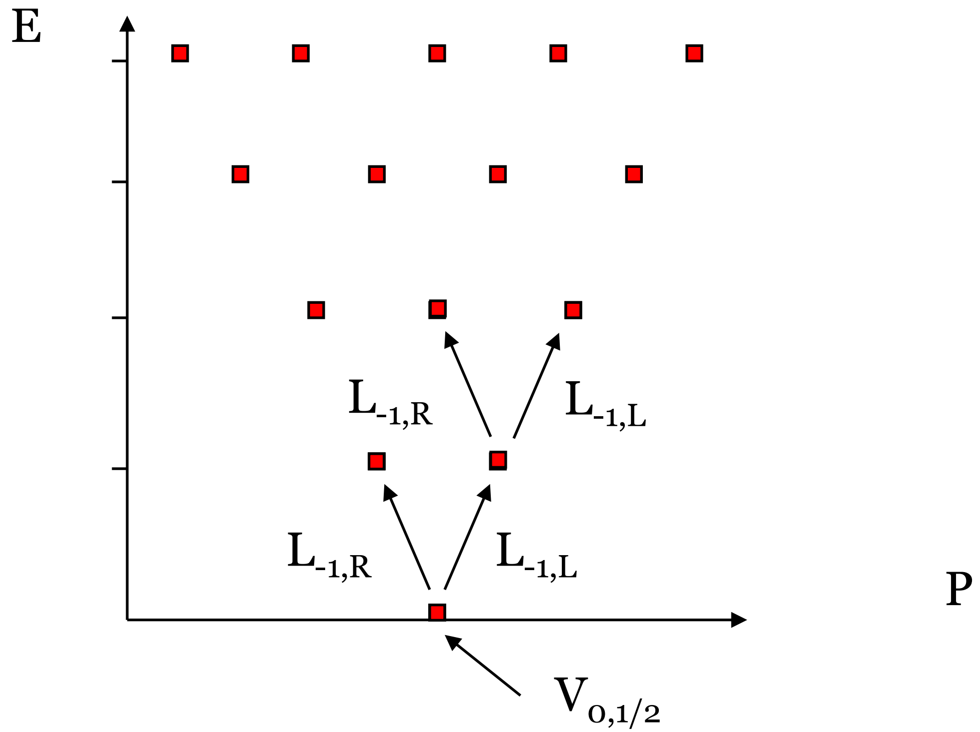

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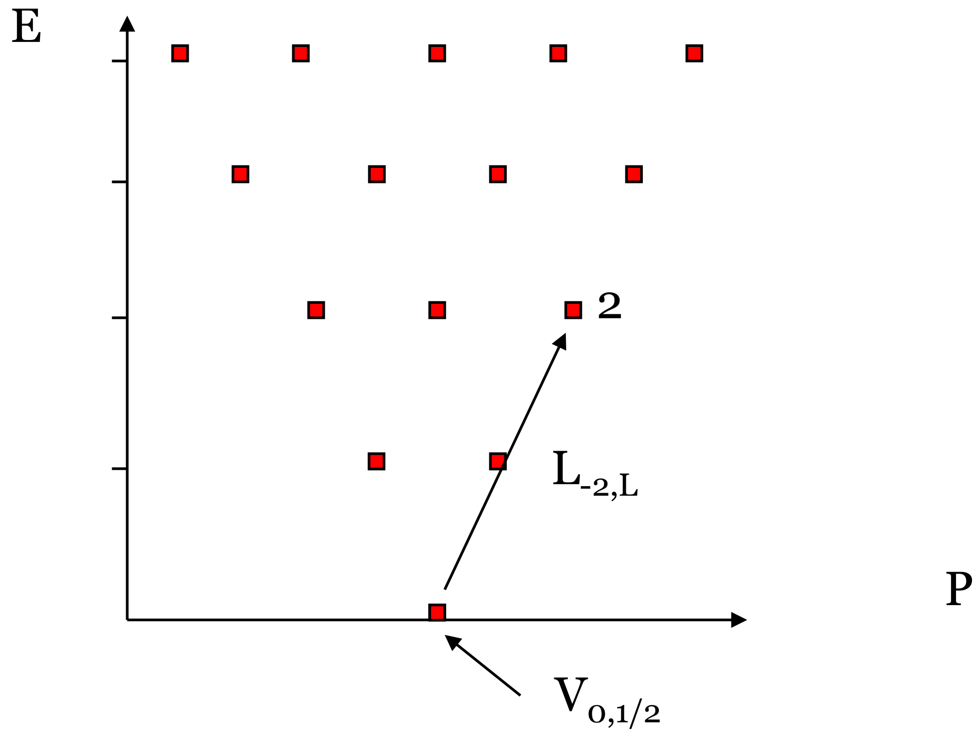
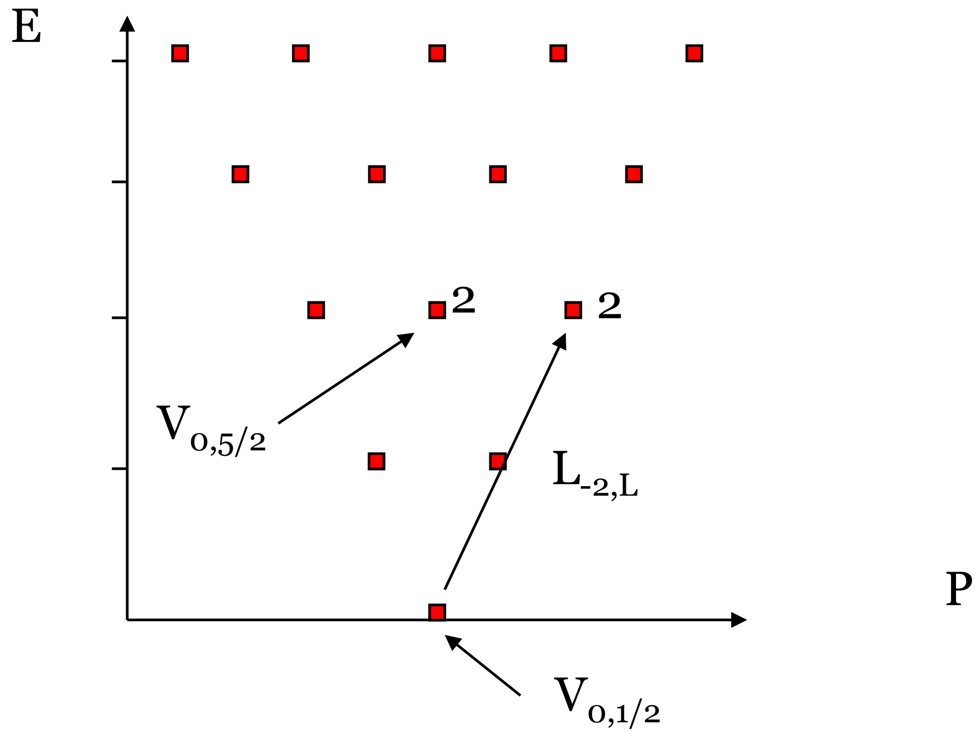


Virasoro algebra: descendents

$$L_{-k,L}L_{-l,R}|h_L,h_R\rangle = |h_L+k,h_R+l\rangle$$

$$\rightarrow E = h_L + k + h_R + l - c/12$$

$$P = (Q_0\pi + 2\pi(h_L+k-h_R-l)/L) \quad \mathrm{mod}\, 2\pi$$



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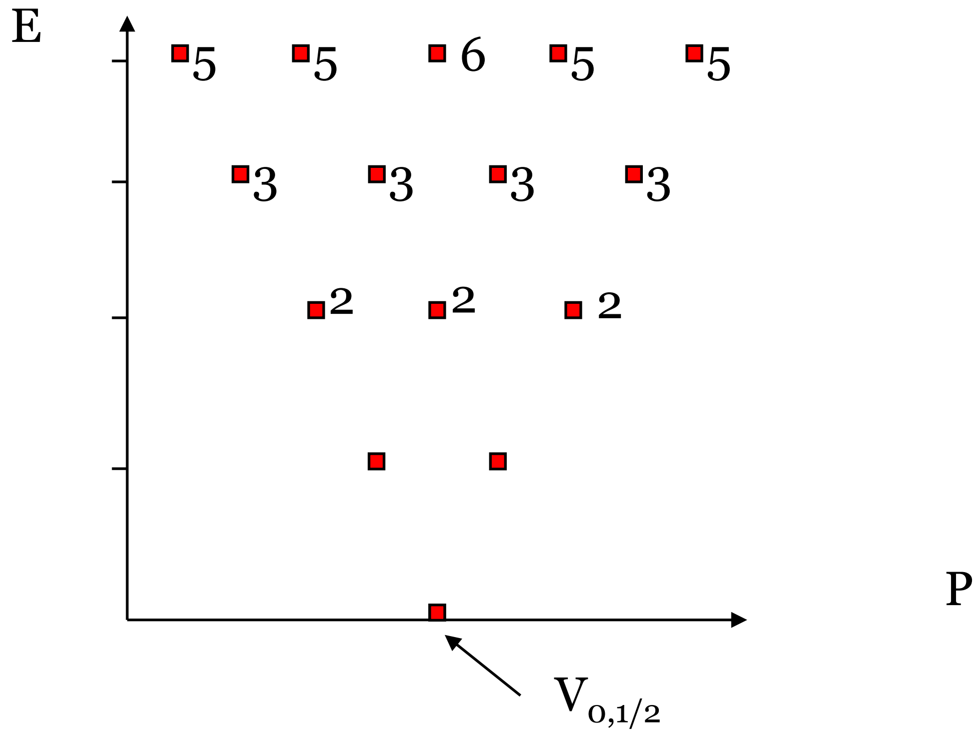
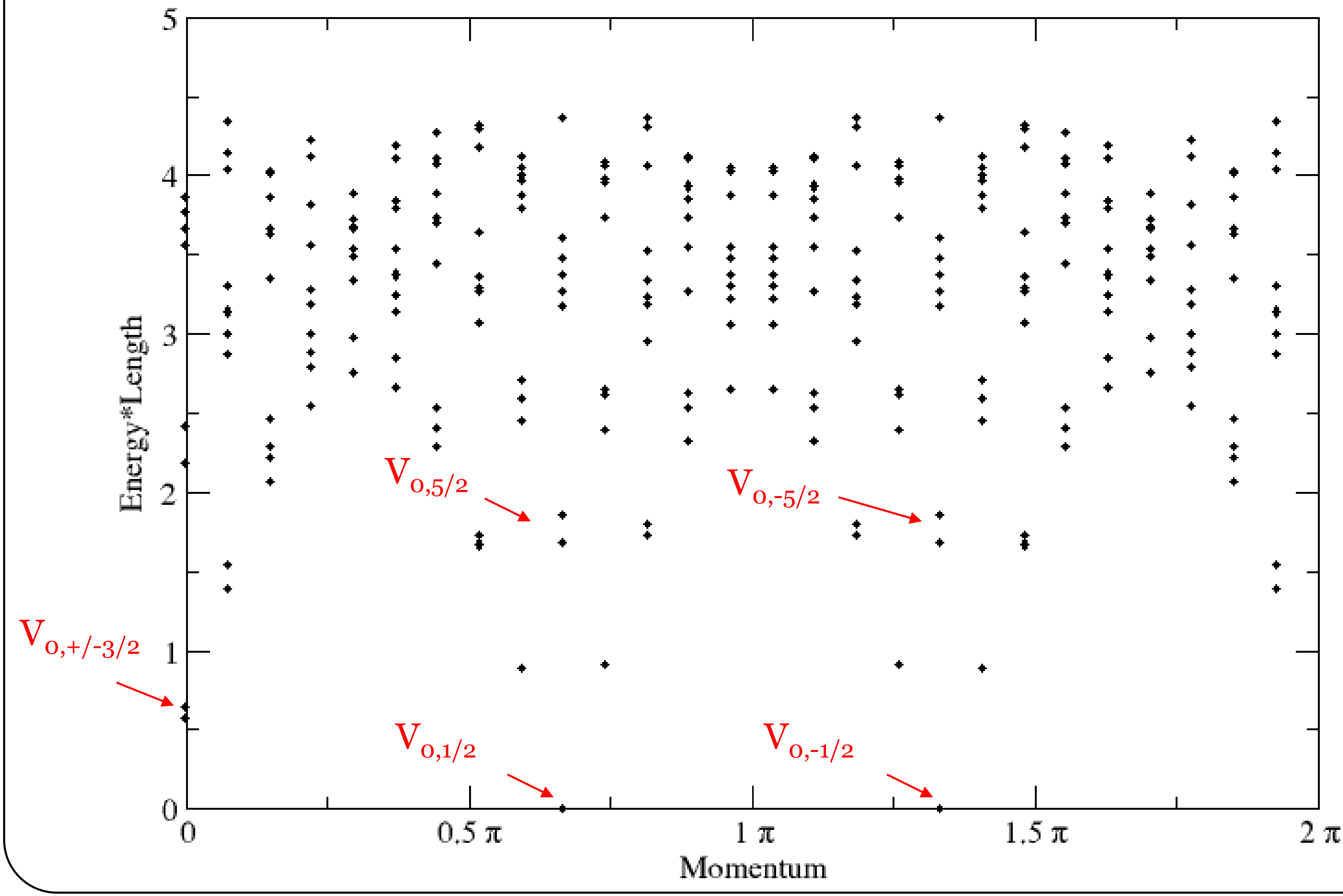


Spectrum for 1D chain, L=27, N_f=9



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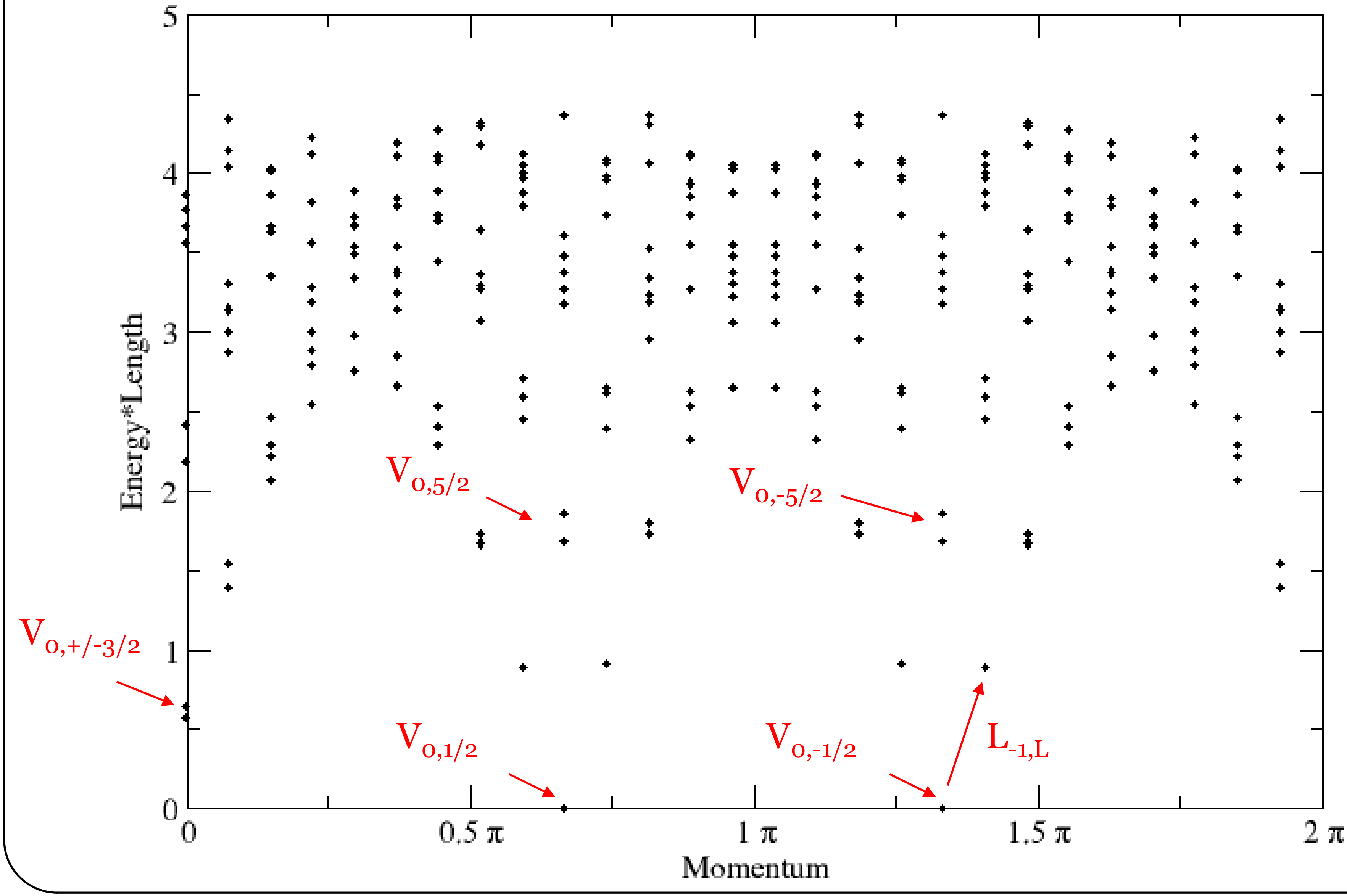
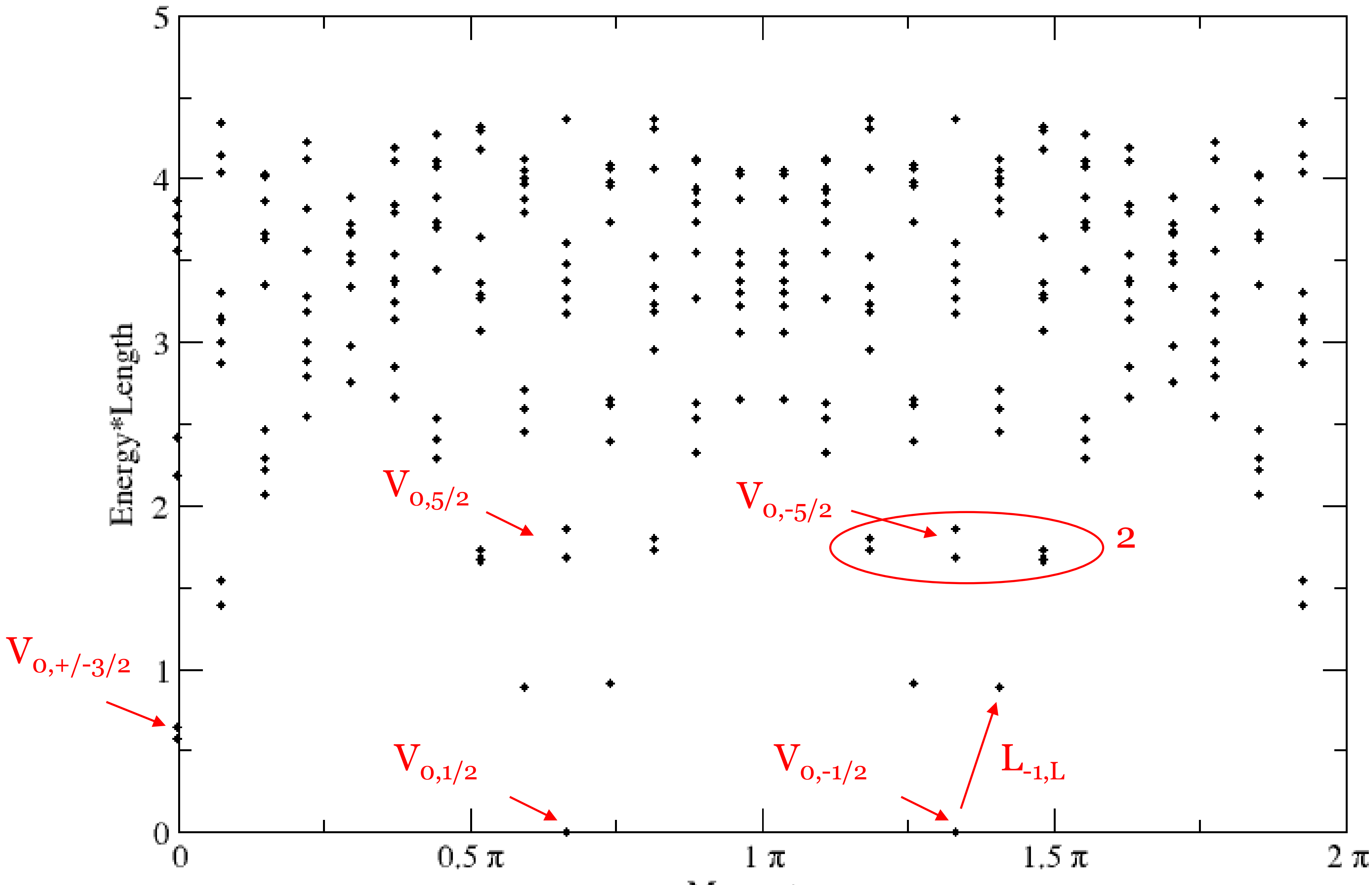


Spectrum for 1D chain, L=27, N_f=9



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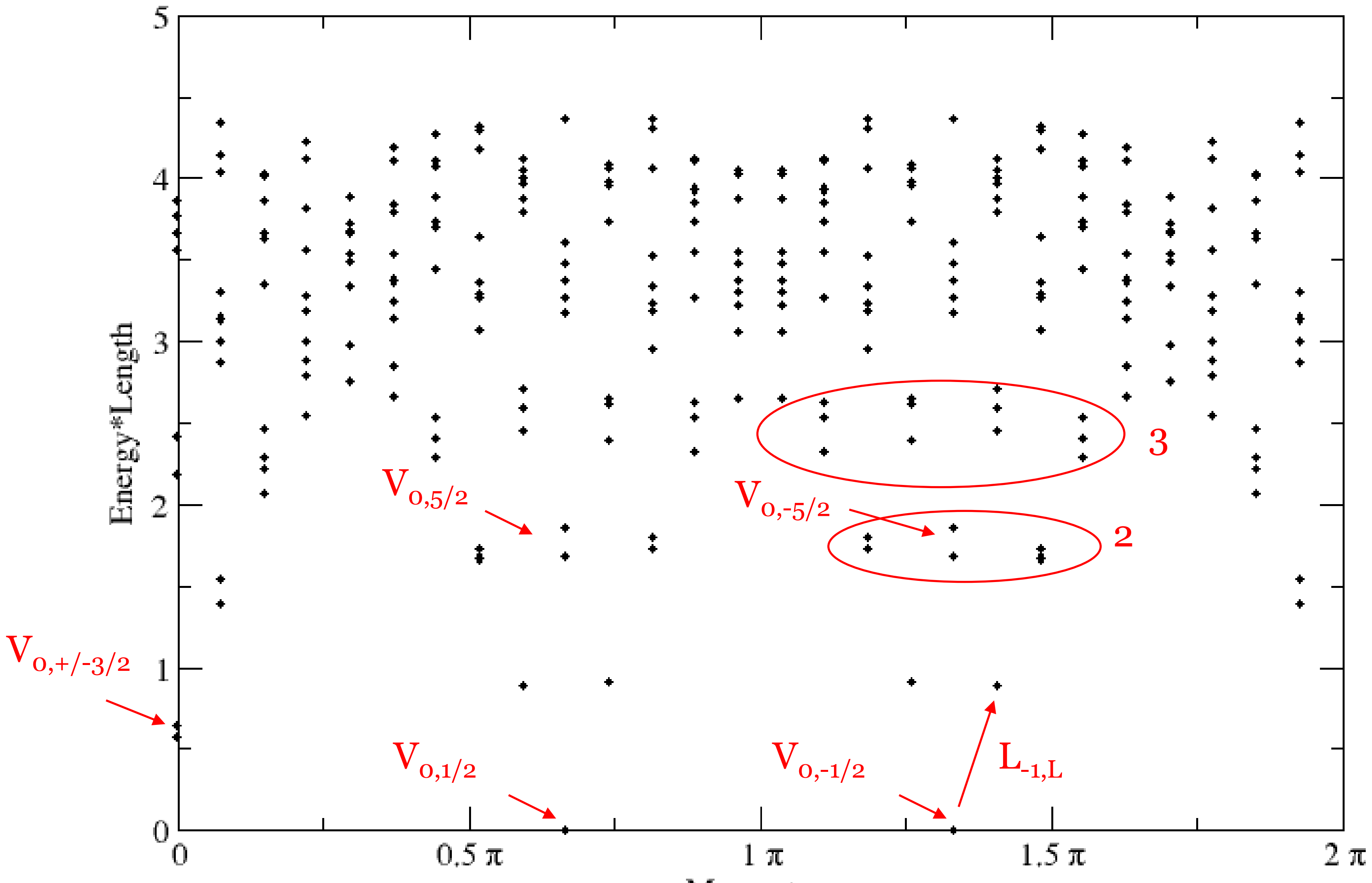

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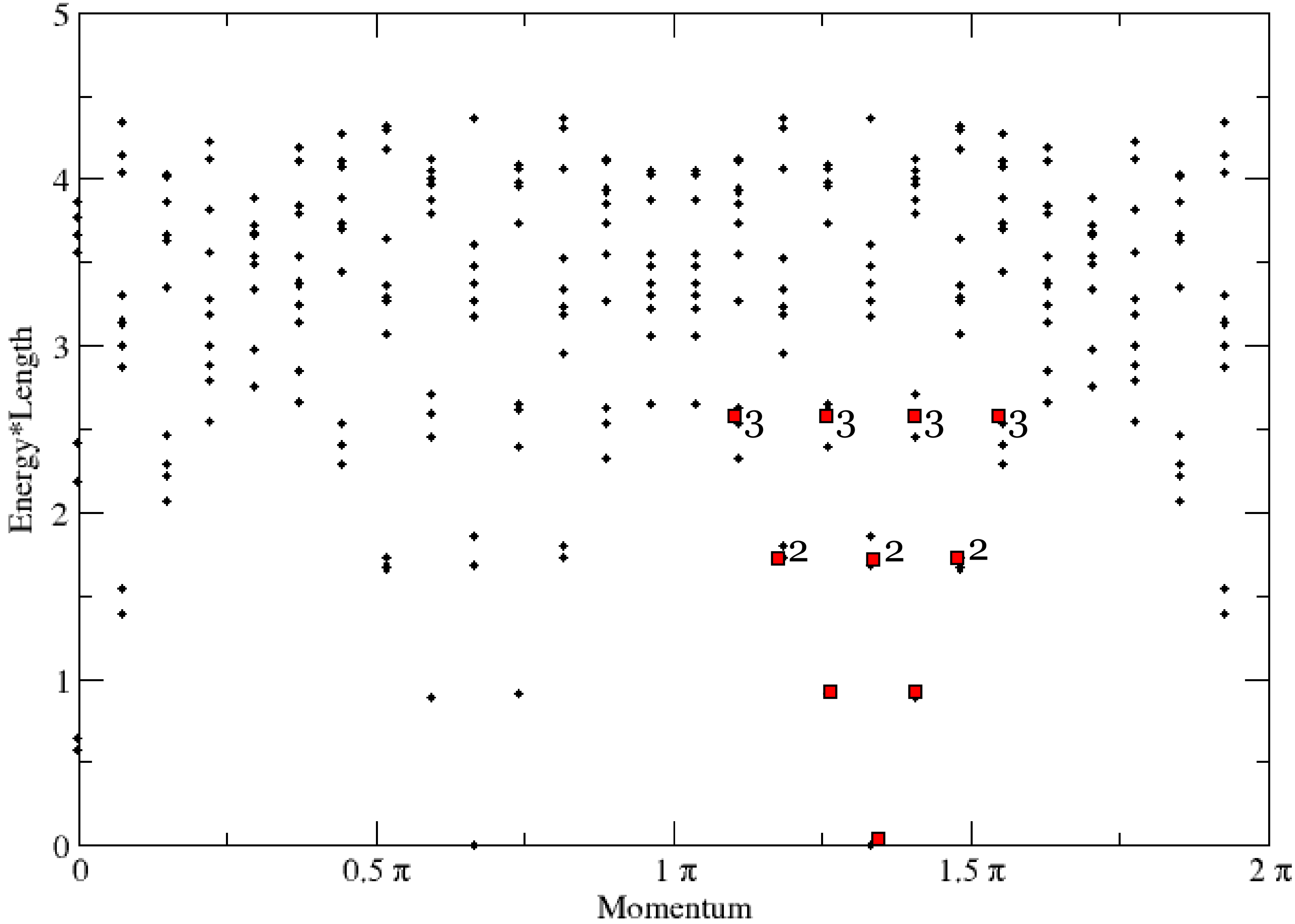
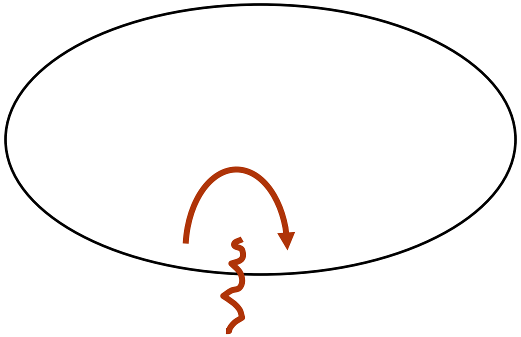


Spectral flow

Boundary twist: spectral flow

wave function picks up a phase exp($2\pi\iota\alpha$)
as a particle hops over a "boundary"



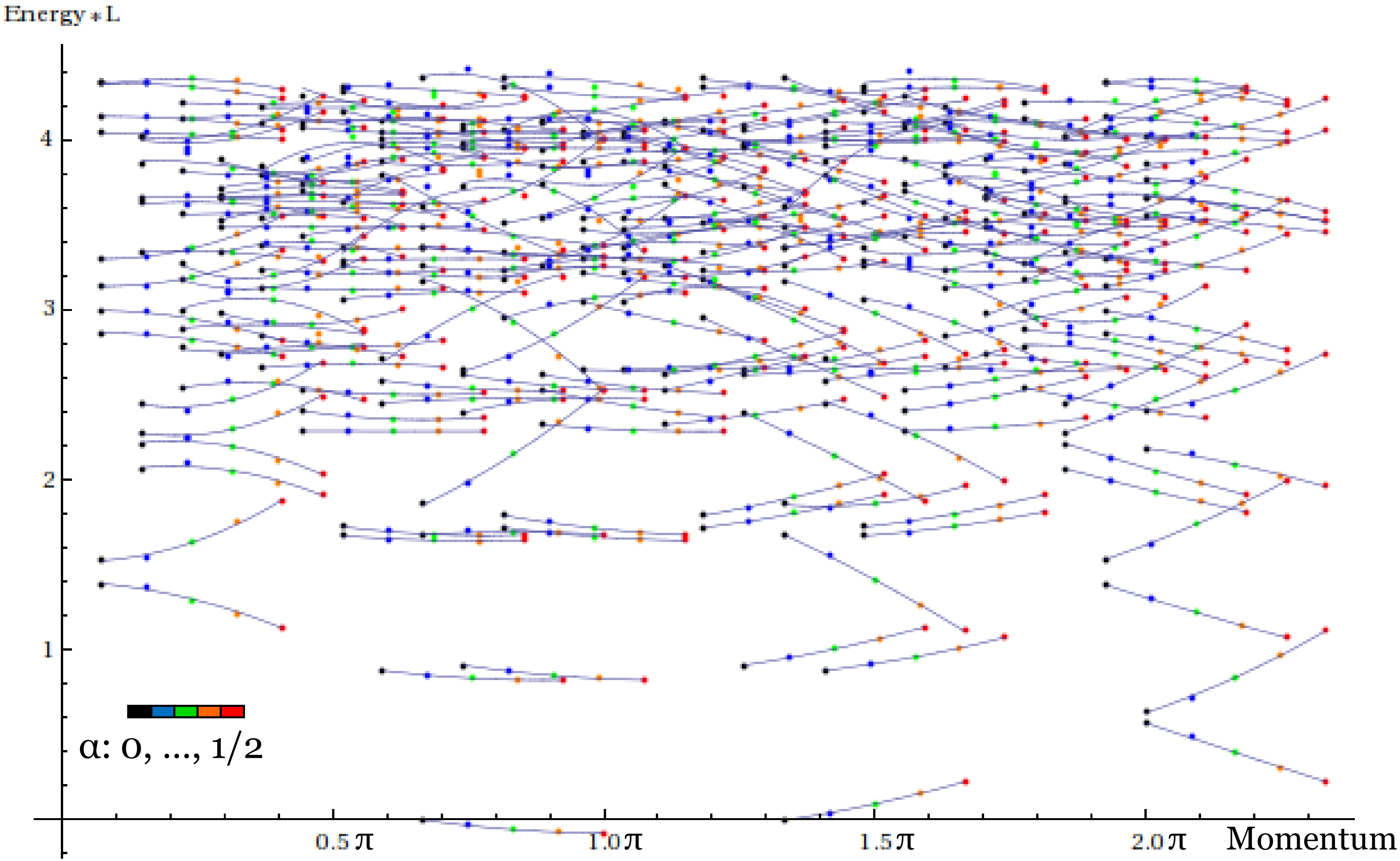
twist: α: $0 \leftrightarrow 1/2$
"pbc $\leftrightarrow$ apbc" = "R $\leftrightarrow$ NS sector"

in SCFT: twist operator: $V_{0,\alpha}$
⟶ energy is parabolic function of twist parameter

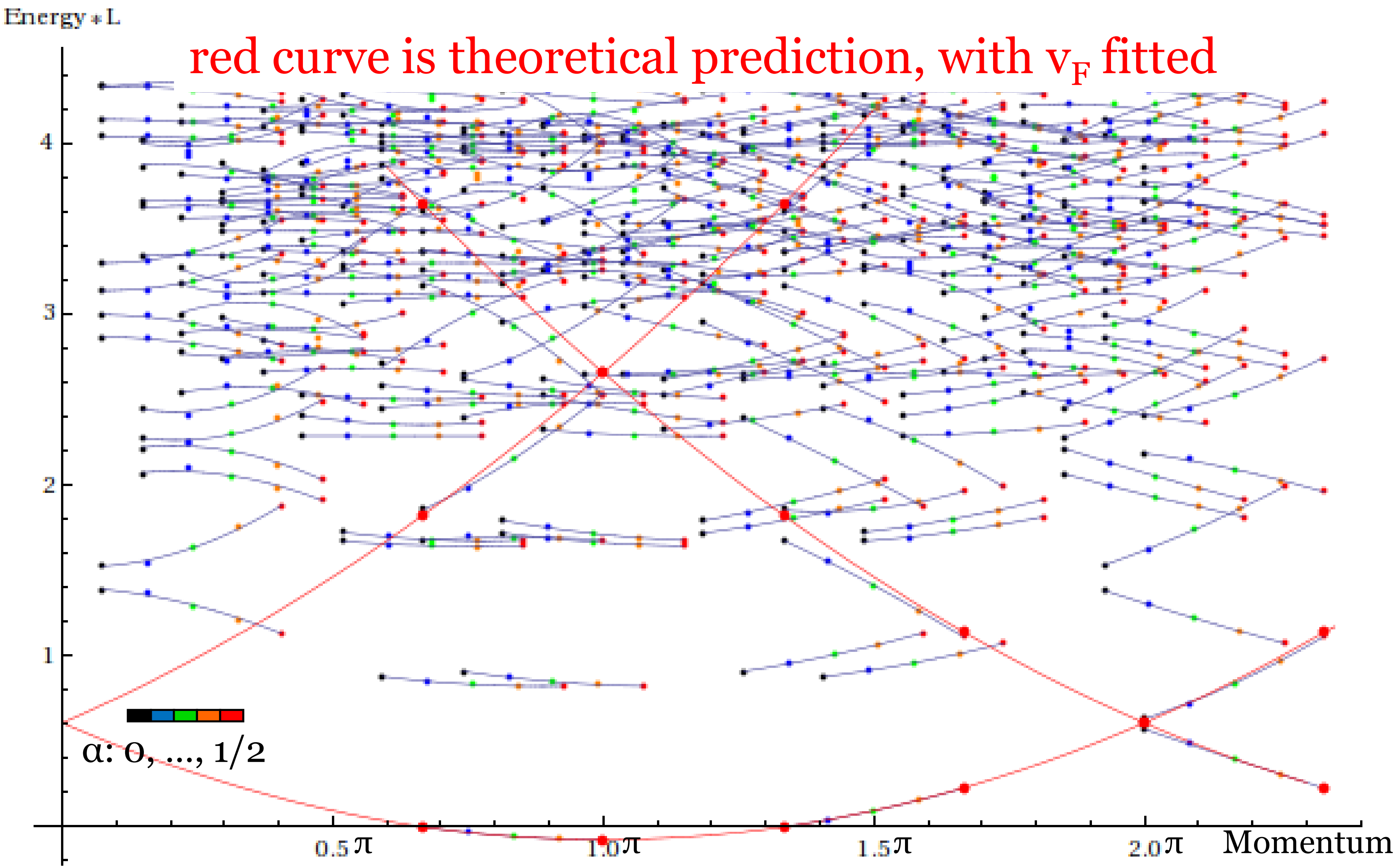
$$E_\alpha = E_0 - Q_0\alpha + \alpha^2 c/3$$

$$R \leftrightarrow \alpha = 0,\ NS \leftrightarrow \alpha = 1/2$$

Spectral flow for 1D chain, L=27 , N_f=9



Spectral flow for 1D chain, L=27 , N_f=9



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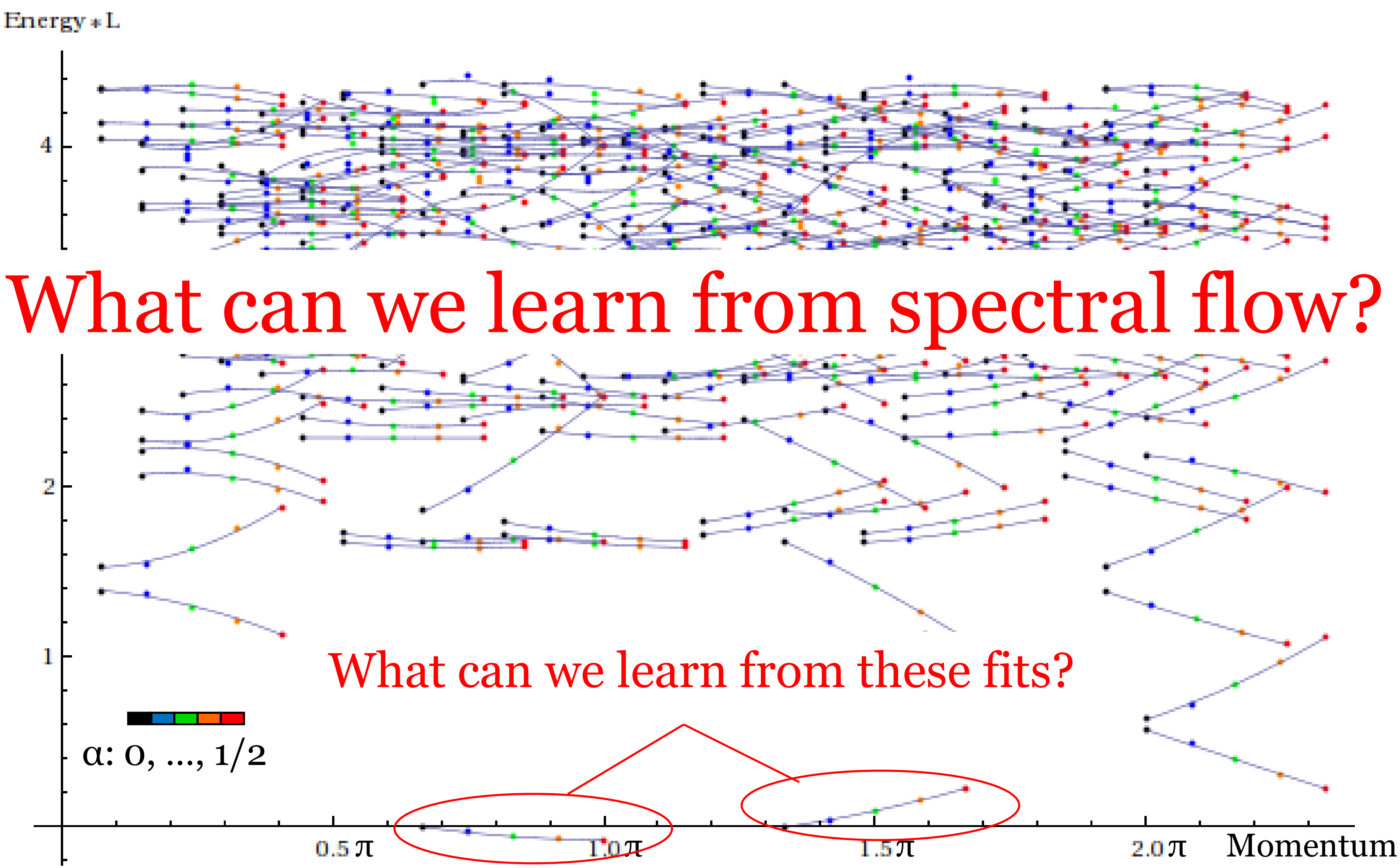
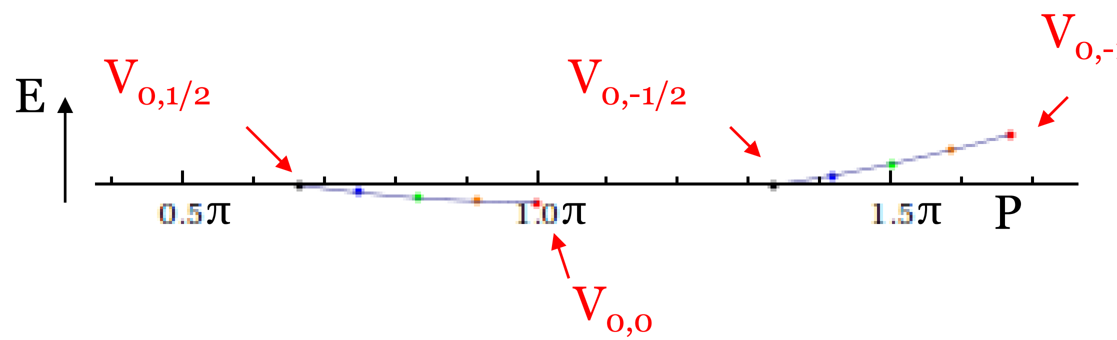


What can we learn from spectral flow?

What can we learn from spectral flow?

- 3 fit parameters
- 4 unknowns:
  $E,\ Q_0,\ c$ and v_F
- $\rightarrow$ ratios
- for 1D chain we extract:



numerics

sector	E/c	Q_0/c	c*v_F
R	0	-0.334	3.92
NS	-0.083	0	3.92
R	0	0.342	3.89
NS	0.254	0.675	3.89

SCFT

state	E	Q_0
$V_{0,1/2}$	0	-1/3
$V_{0,0}$	-1/12	0
$V_{0,-1/2}$	0	1/3
$V_{0,-1}$	1/4	2/3

What can we learn from spectral flow?

→ very accurate, also for very small system sizes:
L=6: NS: $(E/c,Q_0/c)=(-0.085,0)$ and R: $(0,-0.337)$!
NB: no extrapolation for L to infinity necessary!

numerics

sector	E/c	Q_0/c	$c*v_F$
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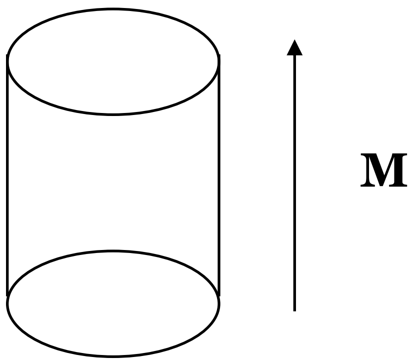
Square lattice: critical edge modes

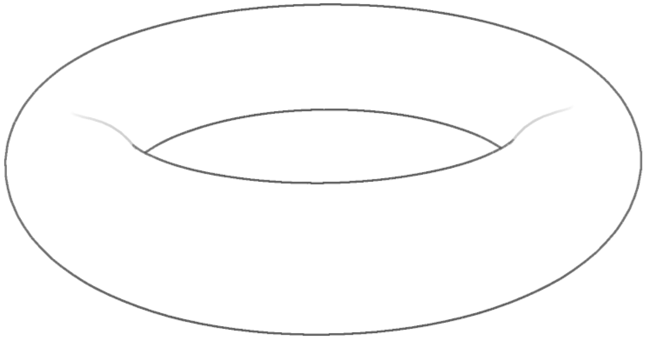
Spectral flow for the square lattice

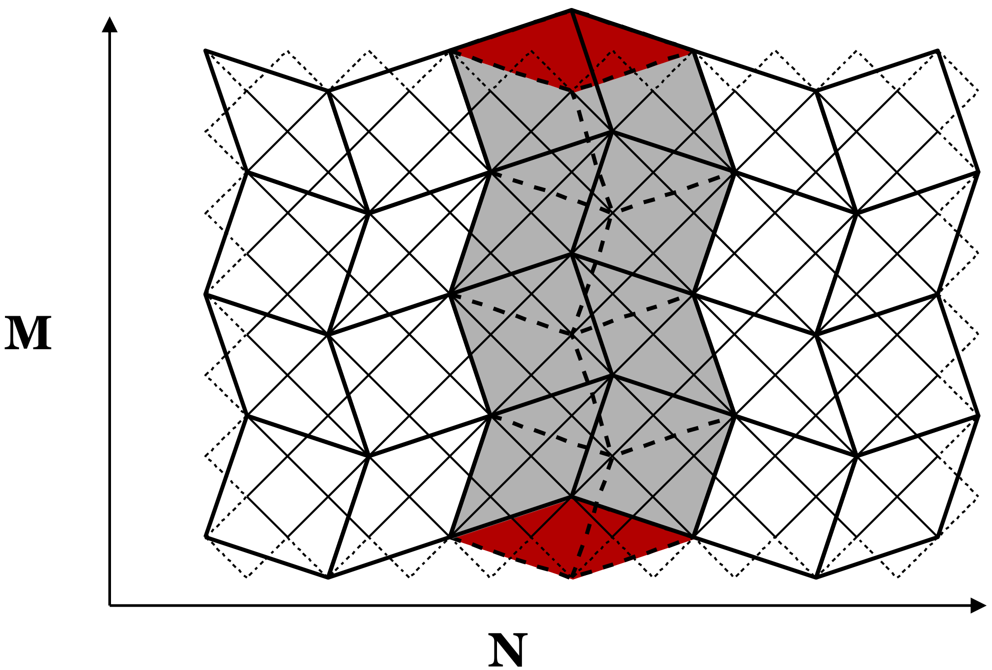
in combination with tiling correspondence we argue that the square lattice on the cylinder has critical edge modes

Edge modes (heuristic argument)

- plane: #gs = 1
- cylinder: #gs ~ 2^{M}
- torus : #gs ~ 2^{M+N}







Spectral flow for the square lattice

- square ladder

  (2,0)x(0,L)

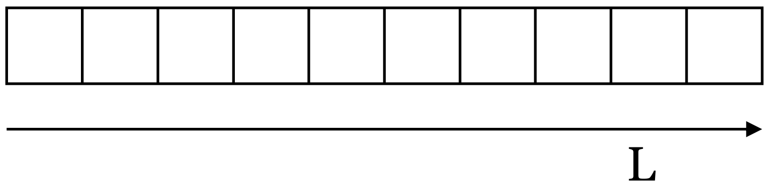
- zigzag ladder

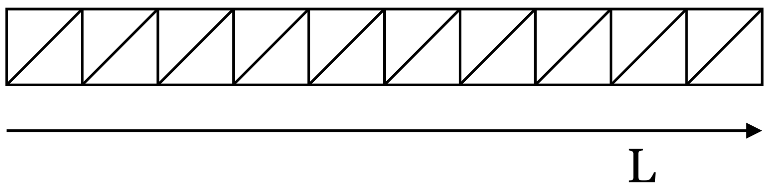
  (2,1)x(0,L)

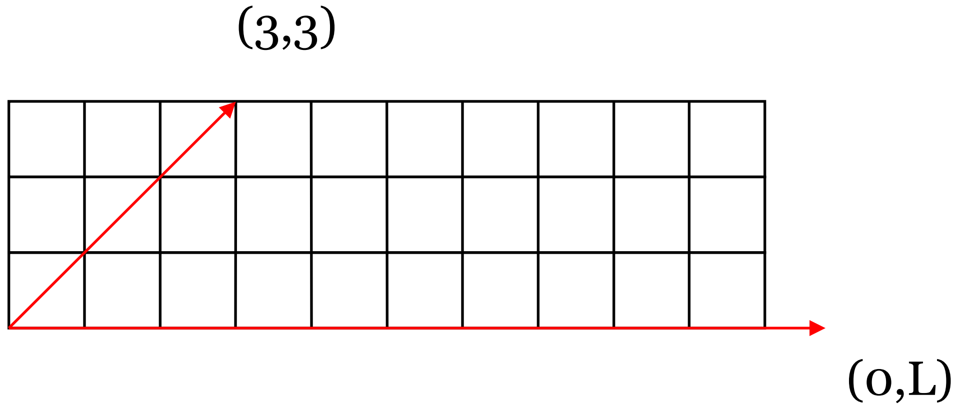
  GS for $\nu \in [1/5, 1/4]$

- (3,3)x(0,L)

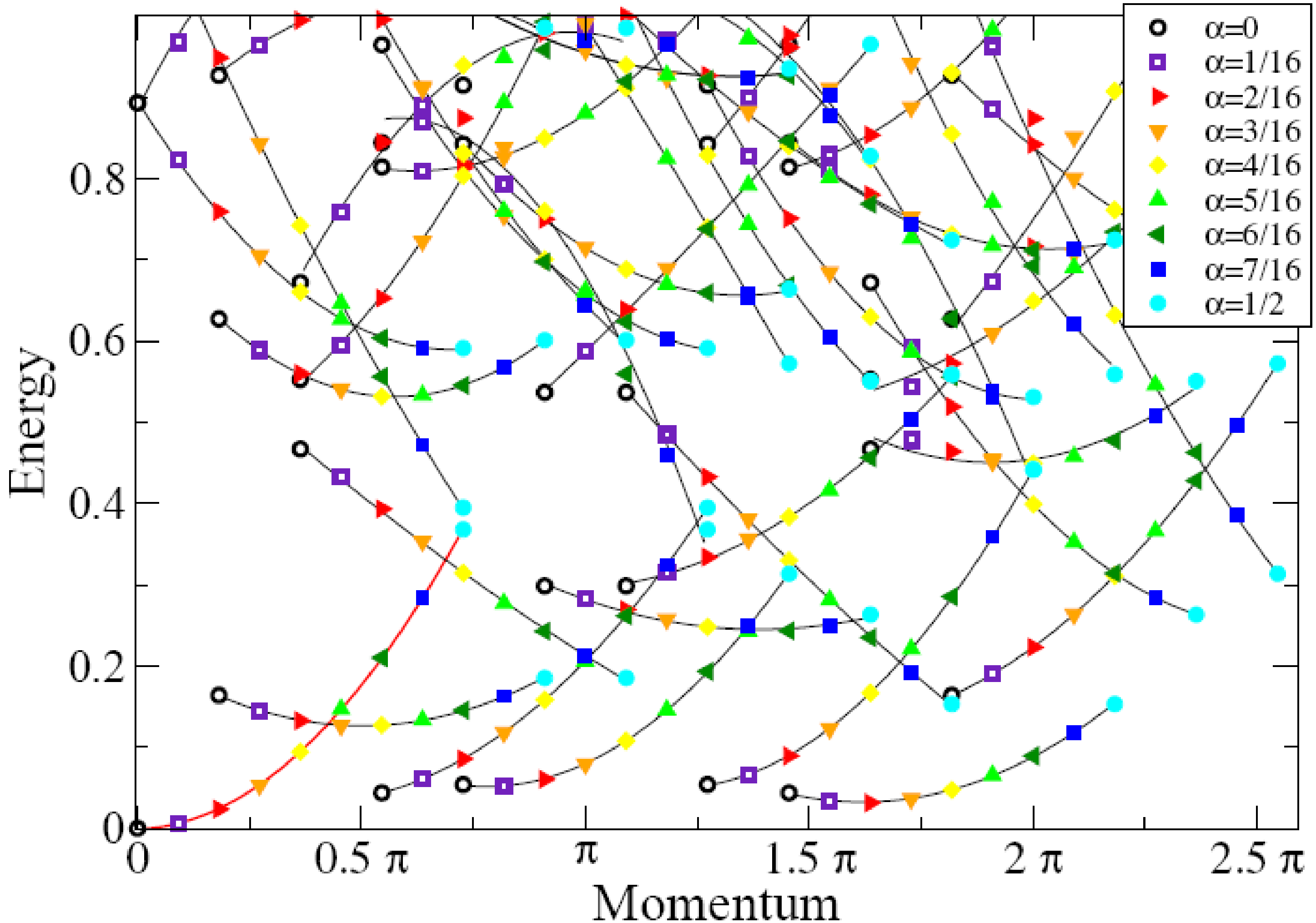
  fermions can hop past each other







Spectral flow results (3,3)x(0,11), N_f=8



Spectral flow results

$(L,0) \times (3,3)$

N	f	E/c	Q/c
18	4	-0.0851	0.004
36	8	-0.0841	-0.002
15	4	0.0898	0.349
21	4	0.0850	0.337
24	5	0.0850	0.337
30	7	0.0853	0.338
33	**8**	**0.0855**	**0.338**

$(L,0) \times (1,2)$

N	f	E/c	Q/c
9	2	-0.0858	-0.005
18	4	-0.0842	-0.002
27	6	-0.0839	-0.001
17	4	0.0844	0.336
26	6	0.0840	0.335
35	8	0.0839	0.335
14	3	0.2666	0.701
23	5	0.2458	0.657
32	7	0.2432	0.652

$(L,0) \times (0,2)$

N	f	E/c	Q/c
16	4	-0.0897	-0.014
24	6	-0.0889	-0.012
32	8	-0.0885	-0.011
12	3	0.0911	0.350
20	5	0.0900	0.348
28	7	0.0894	0.347
14	4	0.0855	0.338
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minimal models in SCFT: $c = \frac{3k}{k+2}$

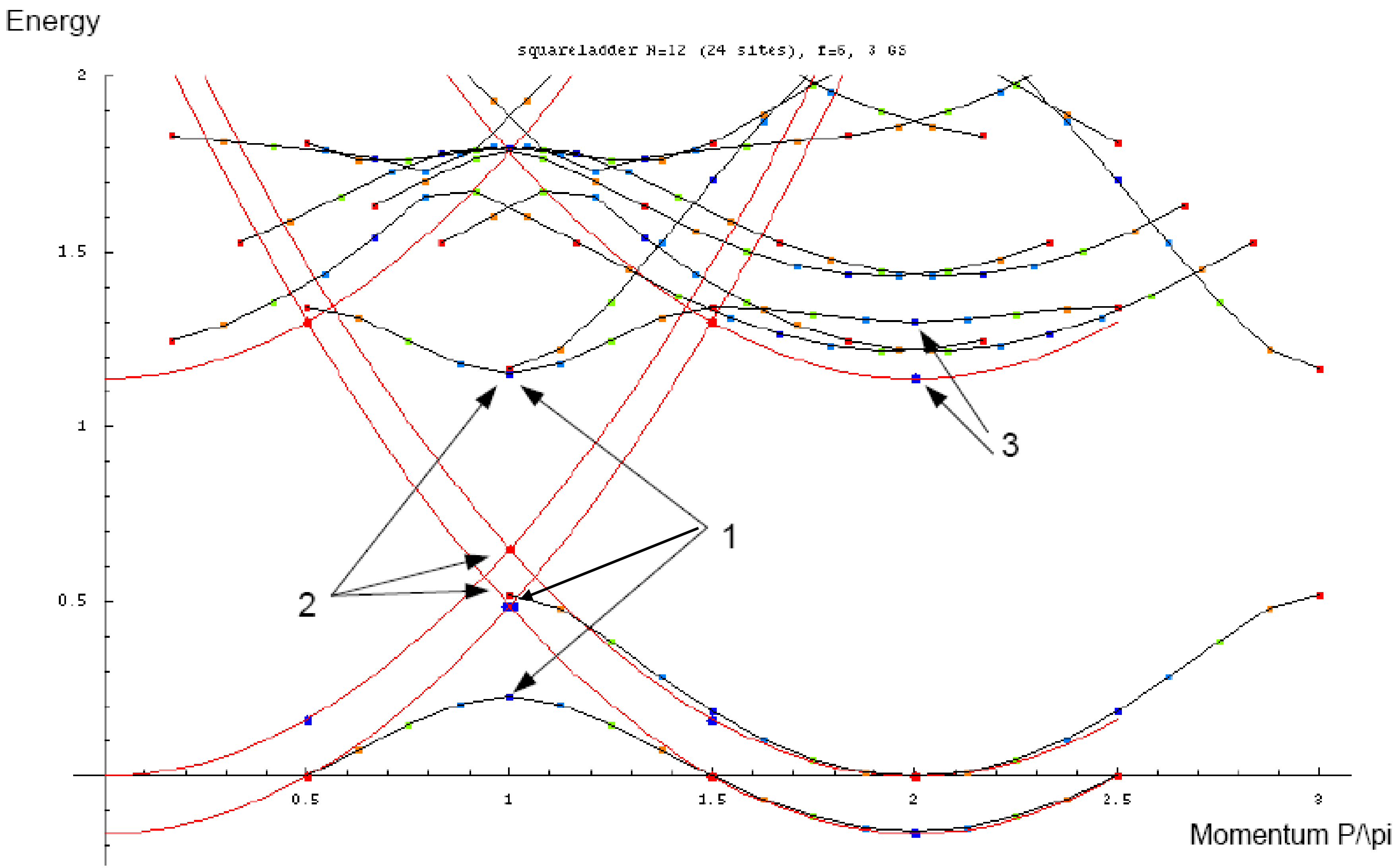
$$E/c = \frac{4l-k}{12k} \text{ and } Q_0/c = \frac{2l}{3k}$$

$$l=0:\ (-1/12,0),\ l=k/2:\ (1/12,1/3),\ l=k:\ (1/4,2/3)$$

Square ladder – the mystery

- DMRG $\rightarrow$ c=3/2
- #gs fits c=3/2
- Spectra do not fit:
  - For closed bc there is an avoided crossing as a function of the twist. Does it persist in continuum limit?
  - DMRG results for open bc also do not fit c=3/2
- Extremely slow convergence??

Square ladder – avoided crossing



Conclusions

supersymmetric lattice model exhibits novel features at intermediate densities:

- superfrustration
- quantum critical modes

exploited tools:
Witten index, cohomology, spectral flow

Thank you