Gravitational Waves from Phase Transitions

(an analytical study)

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Outline

- motivation
- sources of GW from phase transition: expected frequency and amplitude
- analytical model of the sources
- analytical model of the spectra

Why primordial sources?

Small perturbations in FRW metric: $(h_i{}^i = h_i{}^j{}_{|j} = 0)$

$$ds^2 = a^2(\eta)(d\eta^2 - (\delta_{ij} + 2h_{ij})dx^i dx^j) \qquad G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$\ddot{h}_{ij}(\mathbf{k},\eta) + \frac{2}{\eta}\dot{h}_{ij}(\mathbf{k},\eta) + k^2 h_{ij}(\mathbf{k},\eta) = 8\pi G a^2(\eta)\Pi_{ij}(\mathbf{k},\eta)$$

Source: $\Pi_{ij}(\mathbf{k},\eta)$ anisotropic stress

- Once emitted, propagate without interaction
- Direct probe of physical processes in the early universe (gravitons)
- Primordial source: stochastic background of GWs (example: inflation)

GW energy density: $$\Omega_G = \frac{\langle \dot{h}_{ij}\dot{h}^{ij}\rangle}{G\rho_c} = \int \frac{dk}{k}\frac{d\Omega_G(k)}{d\log(k)}$$

Characteristic frequency

- Characteristic frequency of GWs produced at time η_*

$$\mathcal{H}_*^{-1} = \frac{a}{\dot{a}}\Big|_* = \eta_* \quad \text{causality:} \quad k_* \geq \mathcal{H}_*$$

- Frequency window from a cosmological source: $\begin{cases} k_{\rm eq} \simeq 10^{-15}\text{Hz} \\ k_{\rm inf} \simeq 1\text{GHz} \end{cases}$

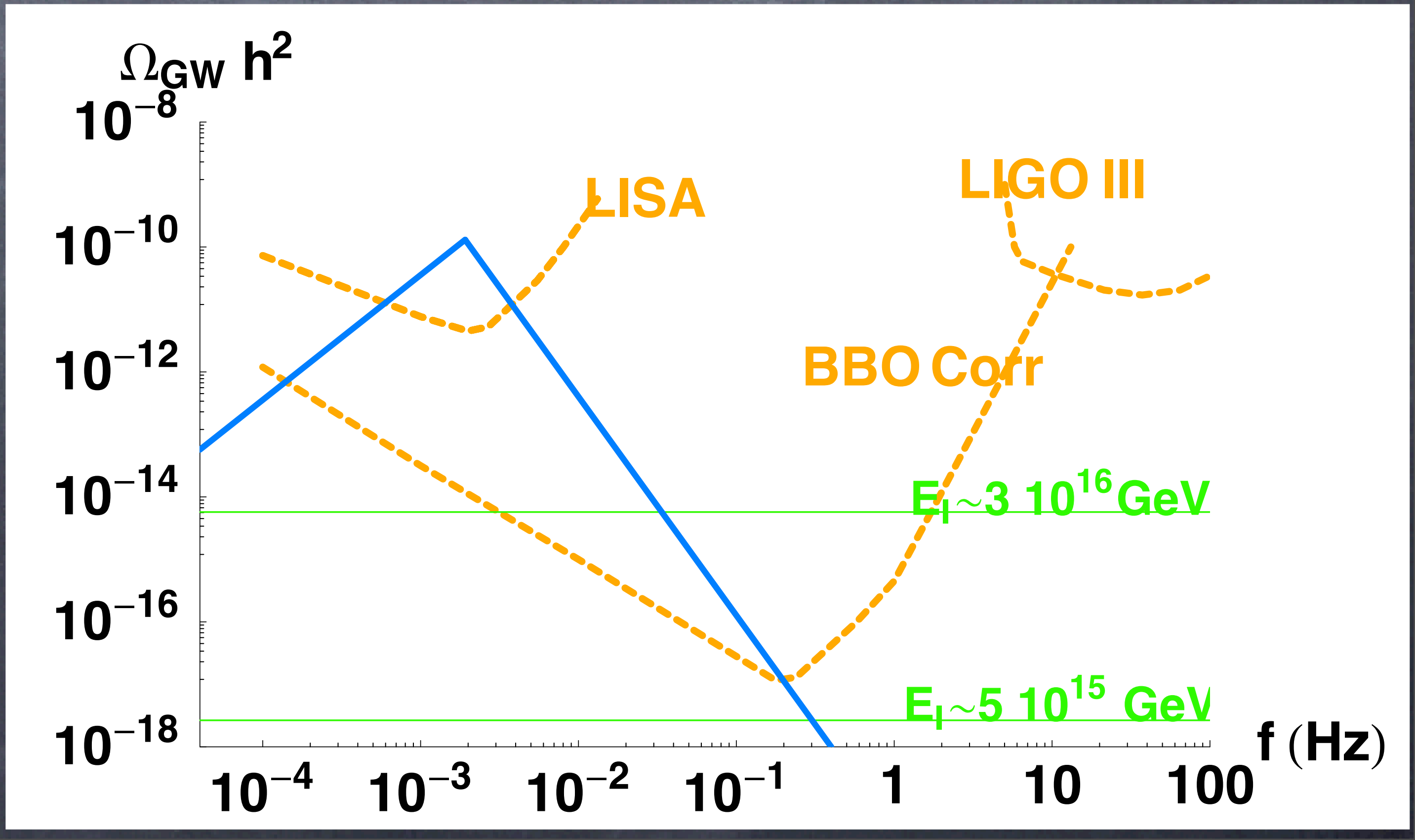
100 GeV (EW phase transition): $k_{100\text{GeV}} \geq 10^{-5}\text{Hz}$

- LISA detection at low frequency:

$$10^{-4}\text{Hz} \leq k \leq 1\text{Hz}$$

$$\Omega_G \sim 10^{-12}$$

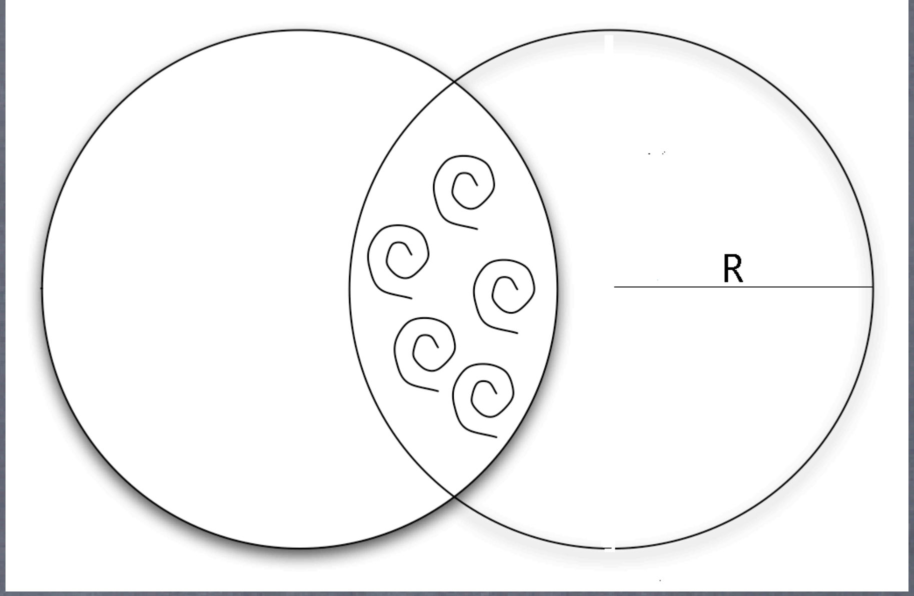
at about 1 mHz



GW from phase transitions

FIRST ORDER:

- Collision of bubbles walls
- Turbulent motions
- Magnetic fields



$\beta^{-1} \simeq 0.01\, \mathcal{H}_*^{-1}$ duration of the phase transition

$R \simeq v_b \beta^{-1}$ size of the bubbles at collision

$v_b \leq 1$ speed of the bubbles walls

$$f \simeq \beta \simeq 10^{-2} \frac{\beta}{\mathcal{H}_*} \frac{T_*}{100\,\mathrm{GeV}} \mathrm{mHz}$$

Scaling of the GW amplitude

- **Bubbles:** $T_{ij} = (\rho + p)\gamma^2 v_i v_j$ (phase transition in a thermal bath)

$\frac{4}{3}\rho_*$ enthalpy density $\qquad v_i$ velocity profile in the bubble

Energy density in bubble walls to radiation energy density: $\frac{\Omega^*_{\rm kin}}{\Omega^*_{\rm rad}} = \frac{4}{3}\frac{v^2}{1-v^2}$

- **Turbulence:** $T_{ij} = (\rho + p) v_i v_j$ $\qquad \frac{\Omega^*_T}{\Omega^*_{\rm rad}} = \frac{2}{3}\langle v^2 \rangle$ $\qquad \langle v^2 \rangle \le \frac{1}{3}$

- **Magnetic fields:** $T_{ij} = \frac{1}{8\pi} B_i B_j$ $\qquad \frac{\Omega^*_B}{\Omega^*_{\rm rad}} = \frac{\langle B^2 \rangle}{8\pi \rho_{\rm rad}}$

Scaling of the GW amplitude

$$\delta G_{ij} = 8\pi G T_{ij}$$

$$\beta^2 h \sim 8\pi G\, T$$

characteristic time of evolution (β)

tensor perturbation (h)

energy density: $\rho_G \sim \frac{\dot{h}^2}{8\pi G}$

$$\dot{h} \sim \frac{8\pi G\, T}{\beta}$$

$$T \sim \rho_{\rm rad} \frac{\Omega^*_{\rm kin}}{\Omega^*_{\rm rad}}$$

$$\Longrightarrow \quad \Omega_G \sim \Omega_{\rm rad} \left(\frac{\mathcal{H}_*}{\beta}\right)^2 \left(\frac{\Omega^*_{\rm kin}}{\Omega^*_{\rm rad}}\right)^2$$

$\Omega_{\rm rad}$: 10^{-5}

$\mathcal{H}_*/\beta$: 10^{-4}

$\Omega^*_{\rm kin}/\Omega^*_{\rm rad}$: 10^{-2}

Analytic study of the GW signal

GW power spectrum:

$$\frac{d\Omega_G}{d\ln k} = \frac{k^3|\dot{h}|^2}{G\rho_c} \qquad \langle \dot{h}_{ij}(\mathbf{k},\eta)\dot{h}^*_{ij}(\mathbf{q},\eta)\rangle = \delta(\mathbf{k}-\mathbf{q})|\dot{h}|^2(k,\eta)$$

Wave equation: $$h_{ij}(\mathbf{k},\eta) = \int_{\eta_{\rm in}}^{\eta} d\tau \mathcal{G}(\tau,\eta)\Pi_{ij}(\mathbf{k},\tau)$$

Anisotropic stress power spectrum: $$\langle \Pi_{ij}(\mathbf{k},\tau_1)\Pi^*_{ij}(\mathbf{q},\tau_2)\rangle = \delta(\mathbf{k}-\mathbf{q})\Pi(k,\tau_1,\tau_2)$$

Energy momentum tensor: $$\Pi_{ij} = \mathcal{P}_{ij}^{lm} T_{lm}$$

$$T_{ij}(\mathbf{k},\tau) = \frac{w(\tau)}{1-v^2(\tau)} \int d^3p \, v_i(\mathbf{k}-\mathbf{p},\tau) v_j(\mathbf{p},\tau)$$

4 point correlation function ⟹ Wick theorem ⟹ Power spectrum of the source

Bubble walls power spectrum

Hydrodynamics of bubble growth at late times:

broken phase	v2 ← | ← v1	symmetric phase
$\rho_2 = aT_2^4$		$\rho_1 = aT_1^4 + \rho_{\rm vac}$
$p_2 = aT_2^4/3$		$p_1 = aT_1^4/3 - \rho_{\rm vac}$

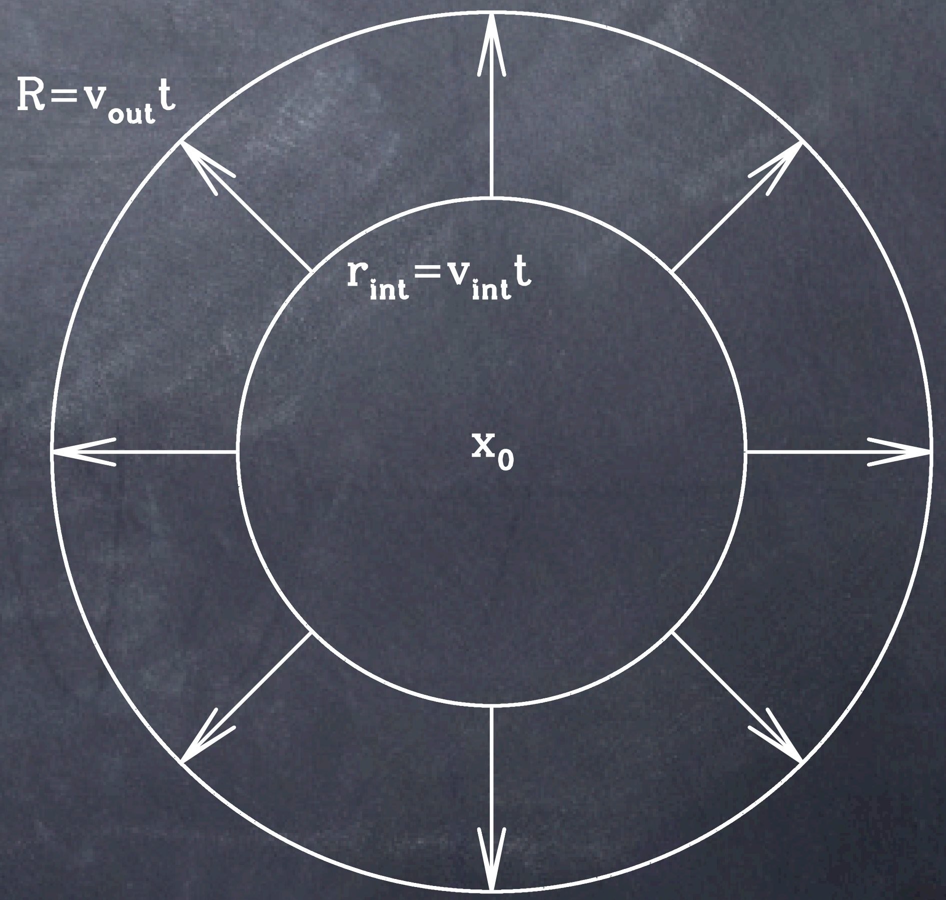
combustion front
(conservation of energy and momentum)

- DETONATIONS: $v_1 > c_s\,, v_2 = c_s$

  symmetric phase at rest
  (Steinhardt 82)

- DEFLAGRATIONS: $v_1 < c_s\,, v_2 < c_s$

  broken phase at rest
  shock wave in the symmetric phase



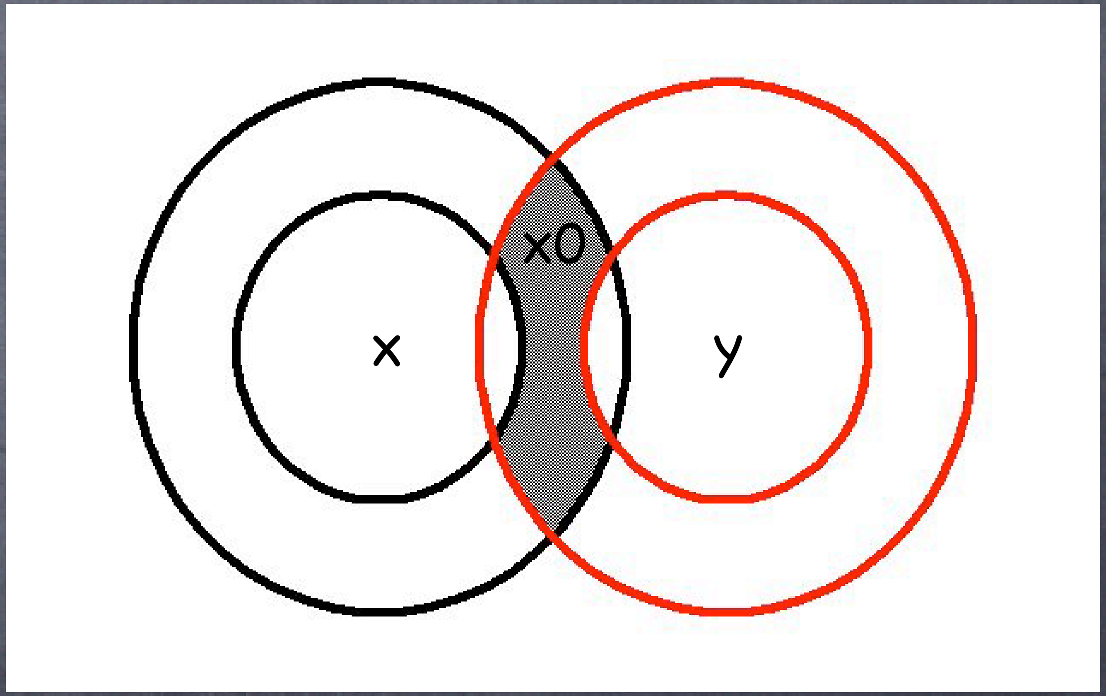
Bubble walls power spectrum

Velocity profile of a spherical bubble:

$$v_i(\mathbf{x},t) = \begin{cases} (v_f/R)\,(\mathbf{x}-\mathbf{x}_0)_i & \text{for} \quad r_{\rm int} < |\mathbf{x}-\mathbf{x}_0| < R \\ 0 & \text{otherwise} \end{cases}$$

Two point correlation function $\langle v_i(\mathbf{x},t) v_j(\mathbf{y},t)\rangle$ non-zero only if

- $|\mathbf{x}-\mathbf{y}| \leq 2R(t)$
- x and y in the same bubble
- the velocity is not zero



$$\langle v_i(\mathbf{x},t) v_j(\mathbf{y},t)\rangle = \frac{v_f^2}{R^2}\,\frac{p}{V}\int_V d^3x_0 (\mathbf{x}-\mathbf{x}_0)_i (\mathbf{y}-\mathbf{x}_0)_j$$

p: probability that there is a centre in the intersection region

Bubble walls power spectrum

Power spectrum: Fourier transform of 2 p. correlation function

$$\langle v_i(\mathbf{k},t) v_j^*(\mathbf{q},t)\rangle = \delta(\mathbf{k}-\mathbf{q}) v_f^2 R(t)^3 P(t) [A(kR)\delta_{ij} + B(kR)\hat{k}_i\hat{k}_j]$$

- $P(t)$ probability that a point is in the broken phase at time t

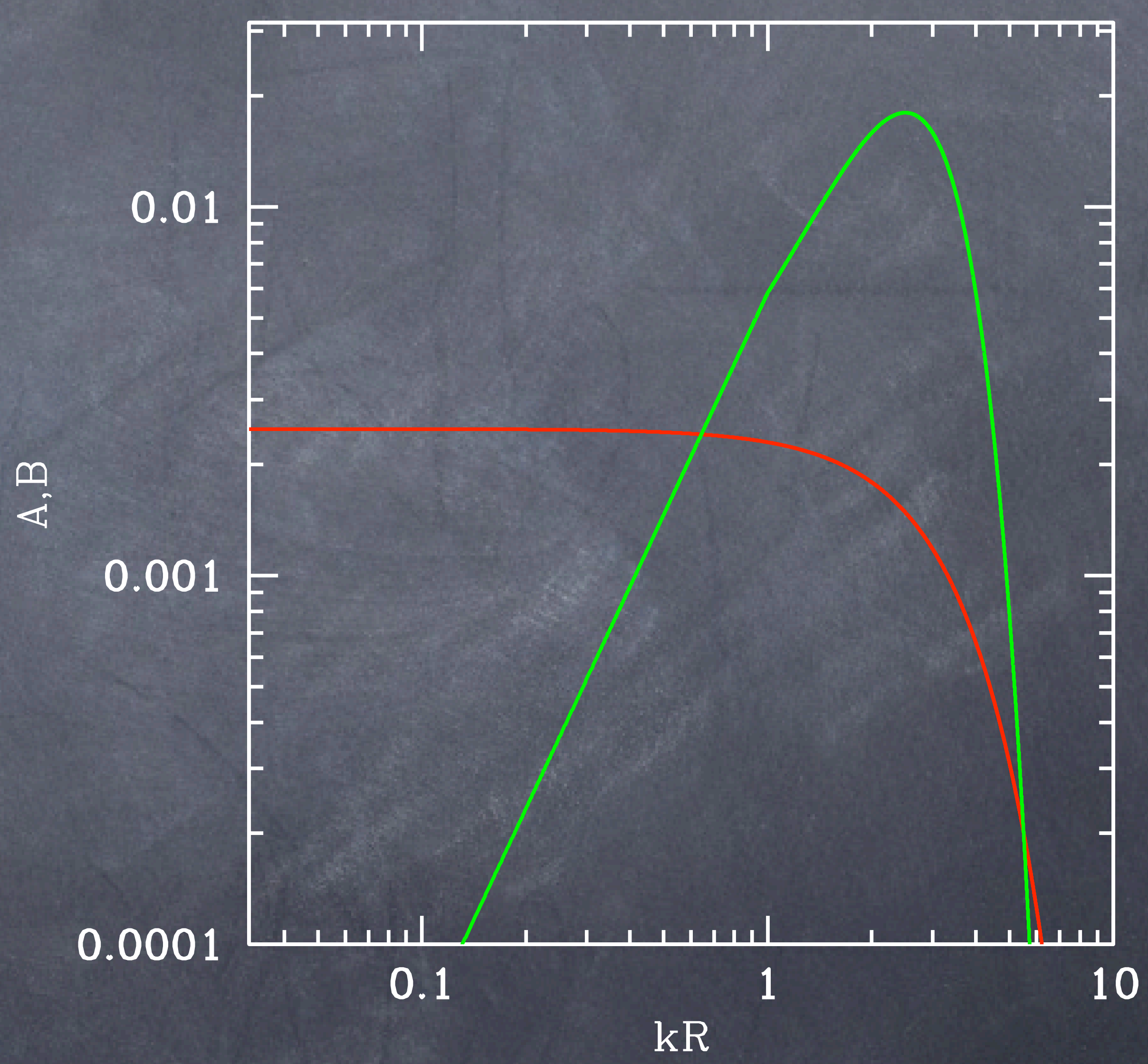
$$P(t) \simeq 1 - \exp(-\Gamma(t))$$

- correlation function with compact support $|\mathbf{x}-\mathbf{y}| \leq 2R(t)$

analytic power spectrum

$$A(k \to 0) \propto k^0$$

$$B(k \to 0) \propto k^2$$



- small scale part $A(k \gg \pi/R) \sim B(k \gg \pi/R) \propto k^{-4}$

Turbulence and magnetic field power spectra

- Correlated over the scale $2R$ ⟹ peak at $k \sim \pi/R$

- Divergence-free vector fields

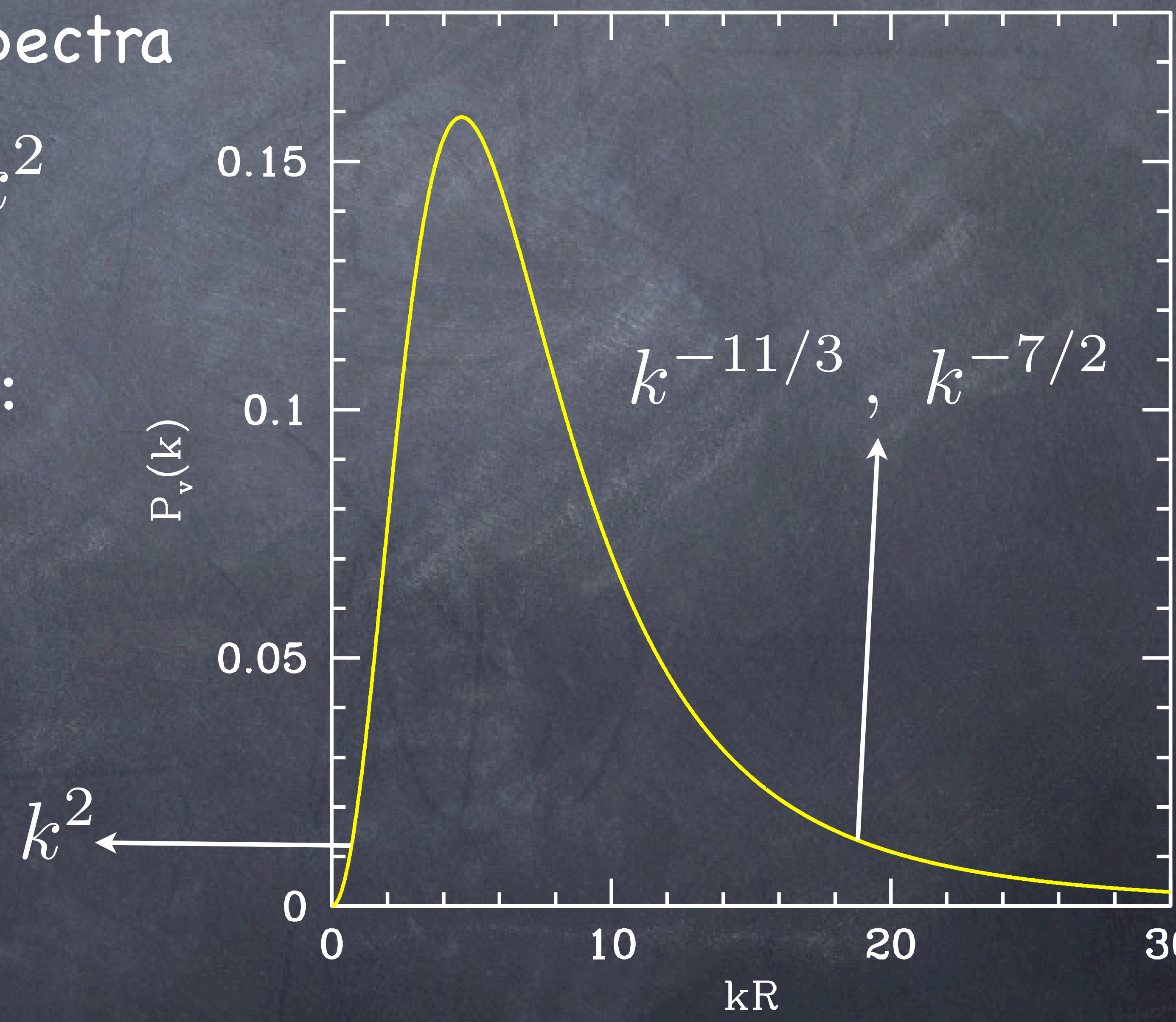
$$\langle v_i(\mathbf{k}) v_j^*(\mathbf{q}) \rangle = \delta(\mathbf{k} - \mathbf{q}) \, (\delta_{ij} - \hat{k}_i \hat{k}_j) P_v(kR)$$

⟹ large scale part of the spectra

$$P_v(k \ll \pi/R) \propto k^2$$

- Small scale part of the spectra:

Kolmogorov (turbulence)
Iroshnikov Kraichnan
(magnetic field)



Large scale part of the GW spectrum: k^3

Generic CAUSAL power spectrum:

$$P(k) \propto v^2 R^3 \begin{cases} (Rk)^n & \text{for } Rk < \pi \\ (Rk)^m & \text{for } Rk > \pi \end{cases}$$

The anisotropic stress power spectrum is the CONVOLUTION:

$$\Pi(k) \propto \int_0^\infty dq\, q^2\, P(|\mathbf{k}-\mathbf{q}|)P(q) \to v^4 R^3 \left(\frac{1}{2n+3} - \frac{1}{2m+3} \right)$$

$$\downarrow$$

$$k \to 0$$
$$n > -3/2$$
$$m < -3/2$$

White noise for the anisotropic stress $\to k^3$ for the GW energy density

$$\frac{d\Omega_G}{d\ln k} = \frac{k^3 |\dot{h}|^2}{G\rho_c}$$

Small scale part of the GW spectrum?

Time dependence of anisotropic stress power spectrum

$$\langle \Pi_{ij}(\mathbf{k},\tau)\Pi^*_{ij}(\mathbf{q},\zeta)\rangle = \delta(\mathbf{k}-\mathbf{q})\Pi(k,\tau,\zeta)$$

1) totally incoherent:

$$\langle \Pi_{ij}(\mathbf{k},\tau)\Pi^*_{ij}(\mathbf{q},\zeta)\rangle = \delta(\mathbf{k}-\mathbf{q})\Pi(k,\tau,\tau)\frac{\delta(\tau-\zeta)}{\beta}$$

2) totally coherent: correct one according to SIMULATIONS

$$\langle \Pi_{ij}(\mathbf{k},\tau)\Pi^*_{ij}(\mathbf{q},\zeta)\rangle = \delta(\mathbf{k}-\mathbf{q})\sqrt{\Pi(k,\tau)}\sqrt{\Pi(k,\zeta)}$$

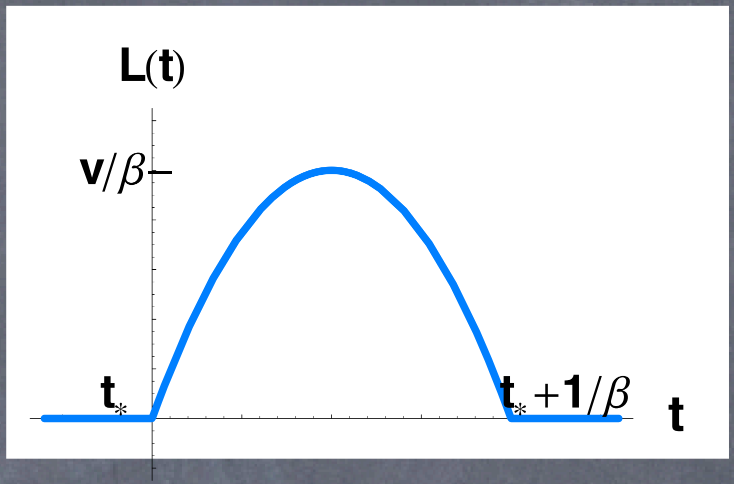
3) top hat in wavenumbers: $|\zeta-\tau| < \frac{x_c}{k}$

$$\begin{aligned}\langle \Pi_{ij}(\mathbf{k},\tau)\Pi^*_{ij}(\mathbf{q},\zeta)\rangle &= \delta(\mathbf{k}-\mathbf{q})[\Pi(k,\tau)\Theta(k\zeta-k\tau)\Theta(x_c-(k\zeta-k\tau)) \\ &+ \Pi(k,\zeta)\Theta(k\tau-k\zeta)\Theta(x_c-(k\tau-k\zeta))]\end{aligned}$$

Totally coherent

time Fourier transform, differentiability of time dependence $\mathcal{C}^m \rightarrow k^{-(m+2)}$

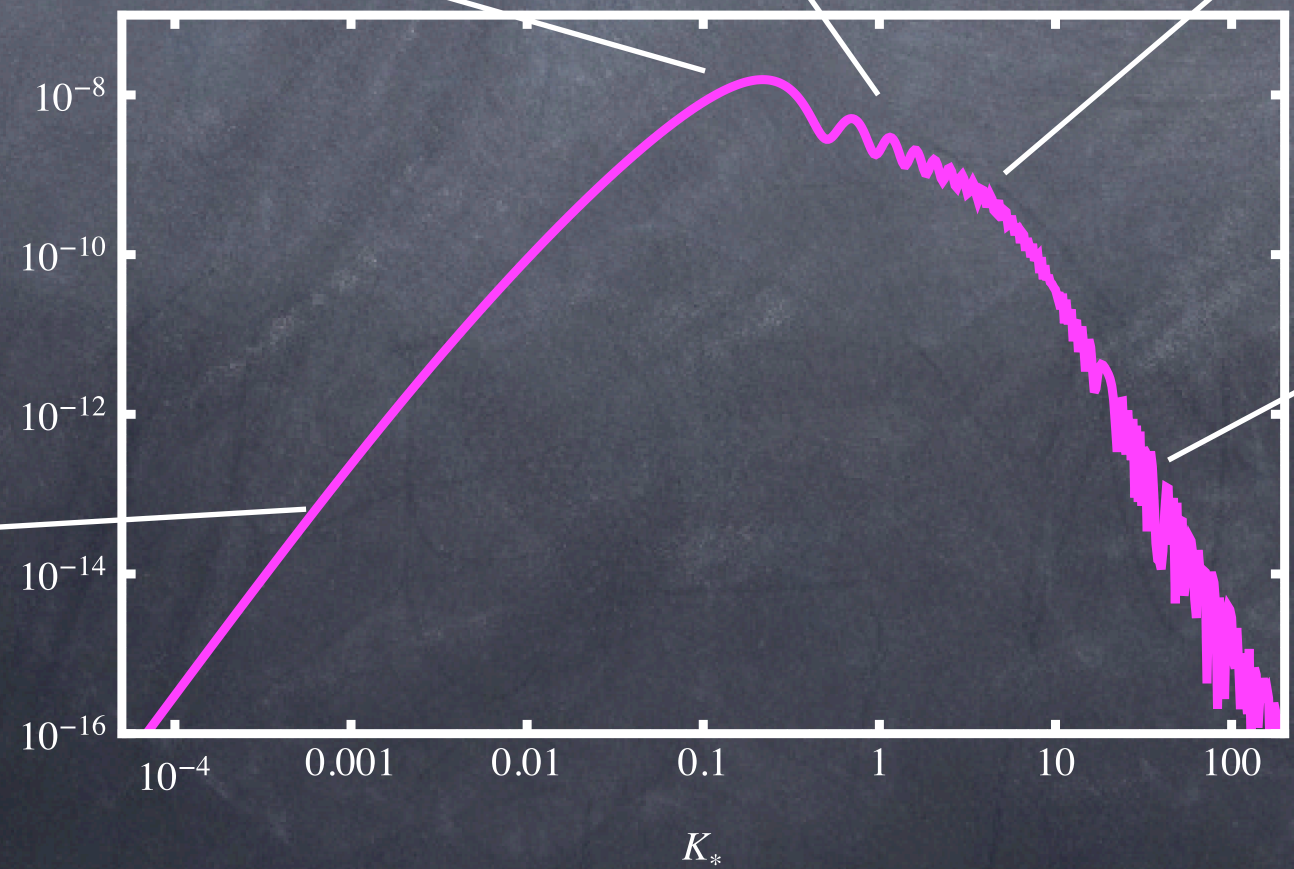
spectrum for a continuous source $\mathcal{C}^0$



k^{-1} slope due to continuity

feature at characteristic lengthscale $k \simeq \pi/R$

Peak at characteristic timescale $k \simeq \beta$



slope due to continuity and source

$k^{-1}k^{-4}$

causal k^3

not in simulations !

Conclusions

- GW production at EW symmetry breaking interesting for LISA
- Analytical model of the sources: bubbles, turbulence and magnetic fields
- Large scale part of the spectrum rises as k^3 (causality)
- Peak frequency correspond to the typical time/length scale of the source
- Small scale part of the spectrum very model dependent
- GW from bubbles observable for high fluid velocity or long lasting phase transition