LECTURE 2

The goal of this lecture is to understand in detail, what happens to the degrees of freedom of the massive graviton in the limit $m_g^2 \to 0$ and expose the vDVZ-discontinuity. This is straightforwardly addressed using Stückelberg fields which picks out the longitudinal modes, or Goldstone modes in a spontaneously broken theory. But before we get there, let us understand in detail the degrees of freedom of the massive graviton.

Particle representations

All particles must form representations of the Poincaré algebra ISO(3,1) including 4-translations and 4-rotations. This tells us how particles behaves under translations and rotations. In particular, particles also carry a representations under the rotation subalgebra.

For example a massive spin-1 particle $\vec{A}$ transforms as a 3 under SO(3). Lorentz invariance then demands that we place this 3 into a full SO(3,1) representation $A_\mu = (A_0, \vec{A})$ that is we have to embed the 3 in a $4 = 1 + 3$. To describe a 3 we then have to impose 1 constraint on the dynamics of the 4, which is exactly what we saw before.

Going on to the massive graviton, described by a 5 under $SO(3)$, we have to embed into a symmetric tensor $h_{\mu\nu}$, with 10 components. Actually h is a scalar and can be removed but let us keep it. Then we can decompose the 10 into $10 = 5+3+1+1$ under $SO(3)$. Indeed

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{0i} \\ h_{i0} & h_{ij} \end{pmatrix}, \quad h_{00} : 1, \quad h_{0i} : 3,$$

$$h_{ij} = \left(h_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{tr}\,h\right) + \frac{1}{3}\delta_{ij}\,\mathrm{tr}\,h \;:\; 5+1$$

As we saw in the FP case we could use the equations of motion to reduce the dofs from 10 to 5.

In the case of massless fields we had to introduce a gauge invariance to restrict the number of degrees of freedom further such that only the 2 dofs of both the photon and the graviton survives. We will see that this comes automatic from the analysis of massless representations of the Poincaré algebra.

Representations of the Poincaré Algebra

All particles must transform as representations under the Poincaré group $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$. To find the algebra of the generators we will now consider infinitesimal transformations

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu \,, \quad a^\mu = \varepsilon^\mu \quad \text{where } \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} .$$

Consider for concreteness a vector field A_α. Under the above transformations we have that

$$\tilde{A}^\alpha(\tilde{x}) = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} A^\beta(x) = \Lambda^\alpha{}_\beta A^\beta(x) .$$

The infinitesimal shift in the field is given by $(\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu)$

$$\begin{aligned}
\delta A^\alpha(x) \equiv \tilde{A}^\alpha(x) - A^\alpha(x) &\simeq \tilde{A}^\alpha(\tilde{x} - \delta x) - A^\alpha(x) \\
&\simeq \tilde{A}^\alpha(\tilde{x}) - \delta x^\mu \partial_\mu \tilde{A}^\alpha(\tilde{x}) - A^\alpha(x) \\
&\simeq A^\alpha(x) + \varepsilon^\alpha{}_\beta A^\beta(x) - \delta x^\mu \partial_\mu A^\alpha(x) - A^\alpha(x) \\
&= \varepsilon^\alpha{}_\beta A^\beta(x) - \delta x^\mu \partial_\mu A^\alpha(x) ,
\end{aligned}$$

up to small quantities squared.

$$\begin{aligned}
\delta A^\alpha = \varepsilon^\alpha{}_\beta A^\beta - \delta x^\mu \partial_\mu A^\alpha &= \varepsilon^\alpha{}_\beta A^\beta - (\varepsilon^\mu{}_\nu x^\nu + \varepsilon^\mu)\partial_\mu A^\alpha \\
&= \varepsilon^{\mu\nu}(\delta^\alpha_\mu \delta^\beta_\nu - \eta^{\alpha\beta} x_\nu \partial_\mu) A_\beta - \varepsilon^\mu \eta^{\alpha\beta} \partial_\mu A_\beta \\
&\equiv \frac{i}{2} \varepsilon^{\mu\nu} (J_{\mu\nu})^{\alpha\beta} A_\beta - i \varepsilon^\mu (P_\mu)^{\alpha\beta} A_\beta
\end{aligned}$$

where we have defined

$$(J_{\mu\nu})^{\alpha\beta} \equiv -2i\, \delta^\alpha_{[\mu} \delta^\beta_{\nu]} - 2i\, \eta^{\alpha\beta} x_{[\mu} \partial_{\nu]} \equiv (S_{\mu\nu})^{\alpha\beta} + (L_{\mu\nu})^{\alpha\beta}$$

$$(P_\mu)^{\alpha\beta} \equiv -i \eta^{\alpha\beta} \partial_\mu$$

Suppressing the spin-indices the P_μ, $J_{\mu\nu}$ satisfy the following algebra, known as the Poincaré algebra

$$[P_\mu, P_\nu] = 0$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho} J_{\sigma\nu} - \eta_{\nu\rho} J_{\sigma\mu} - \eta_{\mu\sigma} J_{\rho\nu} + \eta_{\nu\sigma} J_{\rho\mu})$$

$$[P_\mu, J_{\rho\sigma}] = -i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho),$$

see for example Weinberg vol 1.

We now want to find the maximal number of commuting observables ang the operators above, ie find the "good quantum numbers". They are the energy and momentum together with the spin of the particle.

We can start by choosing eigenfunctions of P_μ:

$$P_\mu e^{ikx} = k_\mu e^{ikx},$$

with eigenvalues k_μ. Now $C_1 \equiv -P_\mu P^\mu \equiv m^2$ is a quadratic Casimir, ie C_1 commuts with all operators of the algebra above, and we will seperate the two cases $m^2 \neq 0$, $m^2 = 0$: massive and massless reps.

To find the other commuting observables we define the so called Pauli–Lubanski vector W_μ

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma .$$

Now since $L_{\mu\nu} = -2i x_{[\mu}\partial_{\nu]} = 2x_{[\mu}P_{\nu]}$, the $\varepsilon^{\mu\nu\rho\sigma}$ picks out the spin part of $J_{\mu\nu}$, that is

$$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho}P_\sigma = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(L_{\nu\rho}+S_{\nu\rho})P_\sigma$$
$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(2x_{[\mu}P_{\nu]}P_\sigma + S_{\nu\rho}P_\sigma)$$
$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} S_{\nu\rho}P_\sigma \quad \text{, due to antisymmetry in } \nu\sigma$$

From this it is also obvious that W^μ commutes with P_ν

$$[W_\mu, P_\nu] = 0 \; ,$$

thus we have found the observables commuting with P_μ describing the spin of the particle. We now want to find the representations of W_μ which satisfy the following algebra

$$[W^\mu, W^\nu] = i\varepsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma \; .$$

We will now characterize this algebra for the two cases $m^2 \neq 0$ and $m^2 = 0$. But first notice that since W_μ and P_μ commute we can specify a specific eigenvalue k_μ of P_μ and work with that wavefunction, that is we can write $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} S_{\nu\rho} k_\sigma$ and choose k_σ conviniently.

<u>Massive representations</u>

Now choose $k^\mu = (m; 0,0,0)$ then

$$W^0 = 0 \; , \; W^i = \frac{1}{2}\varepsilon^{ijk0} S_{jk}\, m \equiv m S^i \; ,$$

where we have defined $S^i \equiv \frac{1}{2}\varepsilon^{ijk} S_{jk}$.

Then the algebra $[W^i, W^j] = i\varepsilon^{ijk0} W_k m$ translates into

$$[S^i, S^j] = i\varepsilon^{ijk} S_k \,,$$

which is the familiar $SO(3)$ rotation algebra and the states are characterized by their eigenvalues of the operators

$$\vec{S}^2 = s(s+1) \quad , \quad s = 0, 1/2, 1, \ldots$$
$$S_z = m \quad , \quad m = -s, -s+1, \ldots, s-1, s$$

Note also that $C_2 = W_\mu W^\mu$ is another quadratic Casimir with value $C_2 = m^2 s(s+1)$.

Massive vector field

For the vector field

$$(\vec{S}^2)^\mu{}_\nu = \ldots = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad , \quad 2 = 1\cdot(1+1)$$

and there are three eigenvectors $\varepsilon^\mu(0,\lambda)$, $k^\mu = (m, \vec{0})$ with eigen values $\lambda = -1, 0, +1$ under

$$(S_z)^\mu{}_\nu = \ldots = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Explicitly we have that the eigenvectors are

$$\varepsilon^\mu(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \varepsilon^\mu(0,\pm 1) = \mp \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} ,$$

with $\vec{S}^2 \varepsilon(0,\lambda) = 2\varepsilon(0,\lambda)$ and $S_z \varepsilon(0,\lambda) = \lambda\, \varepsilon(0,\lambda)$.

Thus we conclude that the wave function of a massive vector boson takes the form

$$A_\alpha(x; k_\mu, \lambda) = \varepsilon_\alpha(\vec{k}, \lambda)\, e^{ikx}.$$

Notice that $k \cdot \varepsilon = 0$ which translates into $\partial \cdot A = 0$ which we found before.

<u>Propagator</u>

The quantum field takes the form

$$A_\alpha(x) = \sum_{\lambda=1}^{3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_\lambda(\vec{k})\, \varepsilon_\alpha(\vec{k}, \lambda)\, e^{ikx} + a^\dagger_\lambda(\vec{k})\, \varepsilon^*_\alpha(\vec{k}, \lambda)\, e^{-ikx} \right)$$

where $a, a^\dagger$ are the relativistically normalized creation and annihilation operators.

In momentum space the propagator takes the form

$$i\tilde{G}_{\alpha\beta}(k, m^2) \equiv \int d^4(x-y)\, e^{-ik(x-y)} \langle 0 | T\left(A_\alpha(x) A_\beta(y)\right) | 0 \rangle$$

$$= i\tilde{G}(k, m) \sum_{\lambda=1}^{3} \varepsilon_\alpha(\vec{k}, \lambda)\, \varepsilon^*_\beta(\vec{k}, \lambda),$$

where $i\tilde{G}(k; m) = \dfrac{-i}{k^2 + m^2 - i\varepsilon}$ is the scalar propagator.

We now have to evaluate the spin sum.

In the rest frame $k^\mu = (m; 0, 0, 0)$ we have that

$$\sum_{\lambda=1}^{3} \varepsilon_\alpha(0, \lambda)\, \varepsilon^*_\beta(0, \lambda) = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} = \eta_{\alpha\beta} + \frac{k_\alpha k_\beta}{m^2}$$

thus for general k we get

$$\sum_{\lambda=1}^{3} \varepsilon_\alpha(k, \lambda)\, \varepsilon^*_\beta(k, \lambda) = \eta_{\alpha\beta} + \frac{k_\alpha k_\beta}{m^2}.$$

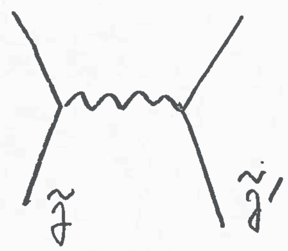
So the massive propagator is given by:

$$i\tilde{G}_{\alpha\beta}(k,m^2) = \frac{-i}{k^2+m^2-i\varepsilon}\left(\eta_{\alpha\beta} + \frac{k_\alpha k_\beta}{m^2}\right)$$

Notice that we can drop the $k_\alpha k_\beta/m^2$ term since when we calculate for example the current-current interaction this term drops out from current conservation $k_\mu \tilde{j}^\mu(k) = 0$ (from $\partial_\mu j^\mu(x) = 0$), that is

$$S_{int} = \int \frac{d^4k}{(2\pi)^4} \tilde{j}^\alpha(-k)\,\tilde{G}_{\alpha\beta}(k;m^2)\,\tilde{j}^\beta(k)'$$

$$= \int \frac{d^4k}{(2\pi)^4} \tilde{j}^\alpha(-k)\,\eta_{\alpha\beta}\,\tilde{G}(k;m^2)\,\tilde{j}^\beta(k)',$$



so $\tilde{G}(k;m^2)$ has a smooth limit when $m^2 \to 0$. When analyzing the massless propagator we will find that

$$i\tilde{G}_{\alpha\beta}(k) = \frac{-i}{k^2-i\varepsilon}\,\eta_{\alpha\beta},$$

so there is no discontinuity when the photon mass goes to zero.

Massive graviton

For the graviton we should analyze the eigenvectors of
$(\vec{S}^2)_{\alpha\beta}{}^{\gamma\delta}$, $(S_z)_{\alpha\beta}{}^{\gamma\delta}$
corresponding to spin-2. The matrix $(\vec{S}^2)_{\alpha\beta}{}^{\gamma\delta}$ has 10 eigenvalues; there are 10 independent symmetric pairs $(\alpha\beta)$:

5 eigenvalues	$6 = 2\cdot(2+1)$	corresponding to spin-2
3	$2 = 1\cdot(1+1)$	– 1
2	$0 = 0\cdot(0+1)$	– 0

The corresponding eigentensors are easy to construct using Clebsch-Gordan coefficients. The eigentensors corresponding to spin-2 are:

$$\varepsilon_{\alpha\beta}(\pm 2) = \varepsilon_\alpha(\pm 1)\varepsilon_\beta(\pm 1)$$
$$\varepsilon_{\alpha\beta}(\pm 1) = \frac{1}{\sqrt{2}}\left(\varepsilon_\alpha(\pm 1)\varepsilon_\beta(0) + \varepsilon_\alpha(0)\varepsilon_\beta(\pm 1)\right)$$
$$\varepsilon_{\alpha\beta}(0) = \frac{1}{\sqrt{6}}\left(\varepsilon_\alpha(+1)\varepsilon_\beta(-1) + 2\varepsilon_\alpha(0)\varepsilon_\beta(0) + \varepsilon_\alpha(-1)\varepsilon_\beta(+1)\right),$$

where $\varepsilon_\alpha(\lambda)$ denotes the vector polarization vectors.

In the rest frame $k^\mu = (m;0,0,0)$ they span the space of symmetric traceless spatial tensors, that is

$$\sum_{\lambda=1}^{5} \varepsilon^{ij}(\lambda)\varepsilon^*_{kl}(\lambda) = \delta^i_{(k}\delta^j_{l)} - \frac{1}{3}\delta_{kl}\delta^{ij},$$

$i,j,k,l = 1,2,3$.

As for the vectors, we can introduce momentum vectors such that

$$\sum_{\lambda=1}^{5} \varepsilon^{\alpha\beta}(\lambda)\varepsilon^{*}_{\mu\nu}(\lambda) = \delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta^{\alpha\beta} + \text{momentum terms}$$

The momentum terms all vanish when we calculate the stress-energy interactions, using energy conservation $k^{\mu}\tilde{T}_{\mu\nu}(k) = 0$.
This gives exactly the propagator obtained in the Fierz-Pauli theory, thus FP indeed describes the propagation of a massive spin-2 field.

Notice that $k^{\alpha}\varepsilon_{\alpha\beta}(\vec{k},\lambda) = 0$ and $\eta^{\alpha\beta}\varepsilon_{\alpha\beta}(\vec{k},\lambda) = 0$, which translates into the conditions $\partial^{\mu}h_{\mu\nu} = 0$ and $h = 0$ (equivalently $\partial^{\mu}(h_{\mu\nu} - \eta_{\mu\nu}h) = 0$ since $h = 0$), which were exactly the conditions we imposed on the $h_{\mu\nu}$ in the FP case.

Massless representations

In the massless case $k^2 = 0$ and we choose a standard momentum $k^{\mu} = (w; 0, 0, w)$.

In this case

$$W^0 = \frac{1}{2}\varepsilon^{0ij3}S_{ij}w = wS_{12} \equiv wS^3$$

$$W^3 = \frac{1}{2}\varepsilon^{3ij0}S_{ij}(-w) = -wS_{12} \equiv -wS^3 = -W^3$$

$$W^1 = \frac{1}{2}\varepsilon^{1ij0}S_{ij}(-w) + \frac{1}{2}\varepsilon^{1\mu\nu3}S_{\mu\nu}w$$
$$= -wS_{23} - wS_{20} \equiv w(K^2 - S^1)$$

$$W^2 = \frac{1}{2}\varepsilon^{2ij0}S_{ij}(-w) + \frac{1}{2}\varepsilon^{2\mu\nu3}S_{\mu\nu}w$$
$$= wS_{31} + wS_{01} \equiv w(K^1 + S^2),$$

where we have defined

$$\vec{S} = (S^{23}, S^{31}, S^{12}), \quad \vec{k} = (S^{10}, S^{20}, S^{30}),$$

see Weinberg vol 1.

Now if we use the algebra of $\vec{S}, \vec{k}$

$$[S^i, S^j] = i\varepsilon^{ijk} S_k$$
$$[k^i, k^j] = -i\varepsilon^{ijk} S_k$$
$$[S^i, k^j] = i\varepsilon^{ijk} S_k$$

we get that

$$[W^1, W^2] = 0$$
$$[W^1, S^3] = iW^2$$
$$[W^2, S^3] = -iW^1$$

This is the algebra of rotations, using S^3, and translations, using W^1, W^2, in $\mathbb{R}^2$. What are the representations of this algebra? Have we found more momenta? The eigenvalues w^1, w^2 of W^1, W^2 are continous and it seems like we are not describing any particles we know of in nature. The way out of this dilemma is to declare that $W_{1,2}$ are not observables and we require that $W_{1,2}$ acting on a physical state corresponds to the same physical state. Thus we only look at representations of S^3. Notice that S^3 is the helicity $S^3 = \frac{1}{|\vec{k}|}\vec{k}\cdot\vec{S}$ of the particle and S^3 can take the values $S^3 = 0, \pm 1/2, \pm 1, \ldots$ (including the double cover).

The photon is thus described by the two helicity ± 1 states

$$\varepsilon_\mu(\vec{k}, \pm 1)$$

with eigenvalues ± 1 of the helicity operator

$$\frac{\vec{k}}{|\vec{k}|} \cdot (\vec{S})^\mu{}_\nu \, \varepsilon^\nu(\vec{k}, \pm 1) = \pm \varepsilon^\mu(\vec{k}, \pm 1)$$

<u>Gauge transformations</u>

Now we have that

$$(W^1)^\mu{}_\nu = -i\omega \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(W^2)^\mu{}_\nu = -i\omega \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and the action on the helicity states are given by

$$(\alpha W^1 + \beta W^2)\,\varepsilon(\vec{k}, \pm 1) = \mp \frac{1}{\sqrt{2}}(\alpha \mp i\beta)\, k^\mu \equiv \Lambda_\pm k^\mu$$

Thus we have found gauge transformations:

$$\varepsilon^\mu(\vec{k}, \pm 1) \rightarrow \varepsilon^\mu(\vec{k}, \pm 1) + \Lambda_\pm k^\mu ,$$

which is the momentum space version of

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda$$

Massless propagators

For the photons we only sum over the helicity ± 1 states. In the frame $k^\mu = (\omega; 0,0,\omega)$ the ± 1 states are complete in the 1,2 directions, that is

$$\sum_{\lambda=1}^{2} \varepsilon_\mu(\lambda)\varepsilon_\nu^*(\lambda) = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix} = \eta_{\mu\nu} - \frac{k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu}{k\cdot\bar{k}},$$

where $\bar{k}^\mu = (\omega; 0,0,-\omega)$. When we evaluate this inside a current-current interaction only the $\eta_{\mu\nu}$ term survives and we see that the massless propagator has the same tensor structure as the massive.

For the graviton we only sum over the ± 2 helicities. They are complete in the 1,2 space of symmetric traceless tensors

$$\sum_{\lambda=1}^{2} \varepsilon^{ab}(\lambda)\varepsilon_{cd}^*(\lambda) = \delta^a_{(c}\delta^b_{d)} - \frac{1}{2}\delta_{cd}\delta^{ab},$$

where $a,b,c,d = 1,2$. Now in a general frame we get that

$$\sum_{\lambda=1}^{2} \varepsilon^{\alpha\beta}(\lambda)\varepsilon_{\mu\nu}^*(\lambda) = \delta^\alpha_{(\mu}\delta^\beta_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} + \text{momenta}$$

Thus we have found the vDVZ-discontinuity: the tensor structure of the massless propagator is different from that of the massive propagator.

Massless limit

We now consider taking the massless limits of both the Proca theory and Fierz-Pauli theory. As we will find, this is easily done in a Stückelberg formulation of the theory.

Massless limit of Proca theory

Consider now taking the $m^2 \to 0$ limit of the massive vector boson. Consider the longitudinal polarization vector. In the rest frame $k^\mu = (m;0,0,0)$ we have $\varepsilon^\mu(\vec{0},0) = (0;0,0,1)$. In the boosted frame $k^\mu = (E;0,0,|\vec{k}|)$ we have that

$$\varepsilon^\mu(\vec{k},0) = \frac{1}{m}(|\vec{k}|;0,0,E)$$

such that still $k_\mu \varepsilon^\mu = 0$ and $\varepsilon_\mu \varepsilon^\mu = 1$.

Now in the limit $m \to 0$ we have that

$$\varepsilon^\mu(\vec{k},0) \simeq \frac{1}{m}(|\vec{k}|;0,0,|\vec{k}|)$$
$$k^\mu \simeq (|\vec{k}|;0,0,|\vec{k}|)$$

thus we get the amusing behavior

$$\varepsilon^\mu(\vec{k},0) \simeq \frac{1}{m} k^\mu \quad , \quad m \to 0$$

thus the longitudinal vector becomes more and more parallel to k^μ, which is consistent with $\varepsilon \cdot k = 0$ since $k^2 \simeq 0$, $m \to 0$.

Thus the $\lambda = 0$ mode in the expansion of $A_\mu(x)$ becomes more and more like $\frac{1}{m}\partial_\mu \phi(x)$ of a scalar.

That is, in the limit $m \to 0$ we get

$$A_\mu(x) = \sum_{\lambda=\pm 1} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_\lambda(\vec{k}) \varepsilon_\mu(\vec{k},\lambda) e^{ikx} + a_\lambda^\dagger(\vec{k}) \varepsilon_\mu^*(\vec{k},\lambda) e^{-ikx} \right)$$
$$+ \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \left(a_0(\vec{k}) \frac{k_\mu}{m} e^{ikx} + a_0^\dagger(\vec{k}) \frac{k_\mu}{m} e^{-ikx} \right)$$
$$\equiv \tilde{A}_\mu(x) + \frac{1}{m} \partial_\mu \phi(x)$$

Stückelberg formulation

consider the Proca field coupled to a source

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} m^2 A_\mu^2 - j^\mu A_\mu \right).$$

Now do a change of variables $A_\mu = \tilde{A}_\mu + \frac{1}{m} \partial_\mu \phi$ then $-\frac{1}{4} F_{\mu\nu}^2 = -\frac{1}{4} \tilde{F}_{\mu\nu}^2$ since this looks like a gauge transformation.

Furthermore, $j^\mu A_\mu = j^\mu \tilde{A}_\mu + \frac{1}{m} \partial_\mu \phi \, j^\mu$

$$= j^\mu \tilde{A}_\mu + \frac{1}{m} \partial_\mu (\phi j^\mu) - \frac{1}{m} \phi \partial_\mu j^\mu .$$

The second term vanishes inside an integral and the third vanishes from current conservation and we end up with

$$S = \int d^4x \left(-\frac{1}{4} \tilde{F}_{\mu\nu}^2 - \frac{1}{2} m^2 \left(\tilde{A}_\mu + \frac{1}{m} \partial_\mu \phi \right)^2 - j^\mu \tilde{A}_\mu \right).$$

We have introduced an extra degree of freedom ϕ but also got an extra gauge symmetry

$$\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu \Lambda \; ; \quad \phi \to - m \phi$$

so the number of degrees of freedom are the same.

But how should we treat $A_\mu, \tilde{A}_\mu, \phi$? Are all degrees of freedom that should be varied in the action?

The answer is that it does not matter what fields we vary, we might vary only A_μ or $\tilde{A}_\mu$ or $\tilde{A}_\mu$ and ϕ etc.

For example by varying $\tilde{A}_\mu$ we get

$$\partial_\mu \tilde{F}^{\mu\nu} - m^2\left(\tilde{A}^\nu + \frac{1}{m}\partial^\nu \phi\right) - j^\nu = 0 ,$$

By varying ϕ we get

$$+ m^2 \frac{1}{m}\partial_\mu\left(\tilde{A}^\mu + \frac{1}{m}\partial^\mu\phi\right) = 0 ,$$

but this is already implied by the first equation by taking its divergence.

Now lets take the $m \to 0$ limit. We have that

$$S = \int d^4x \left(-\frac{1}{4}\tilde{F}_{\mu\nu}^2 - \frac{1}{2}m^2\tilde{A}_\mu^2 - m\tilde{A}^\mu\partial_\mu\phi - \frac{1}{2}(\partial\phi)^2 - j^\mu \tilde{A}_\mu\right)$$

$$\to \int d^4x \left(-\frac{1}{4}\tilde{F}_{\mu\nu}^2 - j^\mu\tilde{A}_\mu - \frac{1}{2}(\partial\phi)^2\right) \text{ when } m \to 0 .$$

Thus, in this way we see that the 3 dofs of the Proca field goes into 2 dofs of a photon + 1 dof of a completely decoupled scalar. The dofs are preserved!

Massless limit of FP theory

In the case of FP theory we will see that Stückelberg formulation gives a smooth limit $m \to 0$ and all degrees are presered but not all degrees decouple and we end up with a scalar-tensor theory.

Consider the $m \to 0$ limit of the polarization tensors

$$\varepsilon_{\alpha\beta}(\pm 2) \to \varepsilon_\alpha(\pm 1)\varepsilon_\beta(\pm 1)$$

$$\varepsilon_{\alpha\beta}(\pm 1) \to \frac{1}{\sqrt{2}}\left(\varepsilon_\alpha(\pm 1)\frac{k_\beta}{m} + \frac{k_\alpha}{m}\varepsilon_\beta(\pm 1)\right)$$

$$\varepsilon_{\alpha\beta}(0) \to \frac{1}{\sqrt{6}}\left(\varepsilon_\alpha(+1)\varepsilon_\beta(-1) + \varepsilon_\alpha(-1)\varepsilon_\beta(+1) + 2\frac{k_\alpha k_\beta}{m^2}\right)$$

We see that the ± 2 modes survive but that ± 1 modes decouple since they are parallel with k_μ. Interestingly the 0 mode does not decouple completely. Indeed we have from photon calculation that

$$\varepsilon_\alpha(+1)\varepsilon_\beta(-1) + \varepsilon_\alpha(-1)\varepsilon_\beta(+1)$$

$$= -\varepsilon_\alpha(+1)\varepsilon^*_\beta(+1) - \varepsilon_\alpha(-1)\varepsilon^*_\beta(-1)$$

$$= -\sum_{\lambda=1}^{2}\varepsilon_\alpha(\lambda)\varepsilon^*_\beta(\lambda) = -\eta_{\mu\nu} + \text{momentum terms}$$

So there is a scalar mode proportional to the trace that survives the limit $m \to 0$ and that does not decouple.

Stückelberg treatment

From the analysis of the polarization tensors we see that in the limit $m \to 0$ then $h_{\mu\nu}$ takes the form:

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{1}{\sqrt{3}}\left(\eta_{\mu\nu} + 2\frac{\partial_\mu\partial_\nu}{m^2}\right)\phi .$$

We have taken out a factor $1/\sqrt{2}$ from A and ϕ since in our conventions the off-diagonal components of $h_{\mu\nu}$ are canonically normalized but diagonal not.

From the point of view of $h_{\mu\nu}$ the ∂A and $\partial\partial\phi$ terms look like like gauge transformations and does not effect the coupling to matter and the kinetic terms. But the $\eta_{\mu\nu}\phi$ term does not and we get new terms.

For the coupling to matter we have that

$$\frac{1}{2M_{pl}} T_{\mu\nu} h^{\mu\nu} \simeq \frac{1}{2M_{pl}} T_{\mu\nu}\left(\tilde{h}_{\mu\nu} + \frac{1}{\sqrt{3}}\eta_{\mu\nu}\phi\right)$$

$$= \frac{1}{2M_{pl}} T_{\mu\nu}\tilde{h}_{\mu\nu} + \frac{1}{2M_{pl}}\frac{1}{\sqrt{3}} T\phi ,$$

up to total derivatives and terms that vanish due to stress energy conservation.

For the kinetic term we use that

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = \mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} + \frac{1}{\sqrt{3}}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\eta_{\alpha\beta}\phi)$$

Now we have that

$$-2\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\eta_{\alpha\beta}\phi) = \eta_{\mu\nu}\partial^2\phi - 2\partial_\mu\partial_\nu\phi + 4\partial_\mu\partial_\nu\phi$$
$$+ \eta_{\mu\nu}(\partial^2\phi - 4\partial^2\phi)$$
$$= -2(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi,$$

that is $\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\eta_{\alpha\beta}\phi) = (\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi$.

Thus we get for the kinetic term

$$-\frac{1}{2}h^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta} = -\frac{1}{2}(\tilde{h}^{\mu\nu} + \frac{1}{\sqrt{3}}\eta^{\mu\nu}\phi)\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\tilde{h}_{\alpha\beta} + \frac{1}{\sqrt{3}}\eta_{\alpha\beta}\phi)$$

$$\cong -\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \frac{1}{\sqrt{3}}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\eta_{\alpha\beta}\phi) - \frac{1}{6}(\eta^{\mu\nu}\phi)\mathcal{E}_{\mu\nu}{}^{\alpha\beta}(\eta_{\alpha\beta}\phi)$$

$$= -\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \frac{1}{\sqrt{3}}\tilde{h}^{\mu\nu}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\phi$$
$$- \frac{1}{6}\phi\,\eta^{\mu\nu}(\eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\phi)$$

$$\cong -\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \frac{1}{\sqrt{3}}(\tilde{h}\partial^2\phi - \partial\partial\tilde{h}\phi)$$
$$- \frac{1}{6}\phi(4\partial^2 - \partial^2)\phi$$

$$= -\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} - \frac{1}{\sqrt{3}}(\tilde{h}\partial^2\phi - \partial\partial\tilde{h}\phi) - \frac{1}{2}\phi\partial^2\phi$$

It seems that we have kinetic mixing between $\tilde{h}$ and ϕ but let us analyse the mass term also.

We now look at the mass term in the $m \to 0$ limit. We use $\simeq$ as equility in this limit and $\cong$ up to total derivatives

$$-\frac{1}{4}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)$$

Let us analyze the first term

$$(h_{\mu\nu})^2 = (\tilde{h}_{\mu\nu} + \frac{2}{m}\partial_{(\mu}A_{\nu)} + \frac{1}{\sqrt{3}}(\eta_{\mu\nu} + \frac{2}{m^2}\partial_\mu\partial_\nu)\phi)^2$$

$$\simeq 2\cdot\tilde{h}_{\mu\nu}\cdot\frac{2}{\sqrt{3}}\frac{1}{m^2}\partial^\mu\partial^\nu\phi + \frac{4}{m^2}(\partial_{(\mu}A_{\nu)})^2$$

$$+ 2\cdot\frac{2}{m}\partial_{(\mu}A_{\nu)}\cdot\frac{1}{\sqrt{3}}\frac{2}{m^2}\partial_\mu\partial_\nu\phi + 2\cdot\frac{1}{3}\eta_{\mu\nu}\phi\frac{2}{m^2}\partial^\mu\partial^\nu\phi$$

$$+ \frac{1}{3}\cdot\frac{4}{m^4}(\partial_\mu\partial_\nu\phi)^2$$

$$\cong \frac{4}{\sqrt{3}}\frac{1}{m^2}\partial\partial\tilde{h}\phi + \frac{1}{m^2}(\partial_\mu A_\nu + \partial_\nu A_\mu)^2 + \frac{8}{\sqrt{3}}\frac{1}{m^3}(\partial\cdot A)\partial^2\phi$$

$$+ \frac{4}{3}\frac{1}{m^2}\phi\partial^2\phi + \frac{4}{3}\frac{1}{m^4}(\partial^2\phi)^2$$

continue with the second

$$(h)^2 = (\tilde{h} + \frac{2}{m}(\partial\cdot A) + \frac{1}{\sqrt{3}}(4 + \frac{2}{m^2}\partial^2)\phi)^2$$

$$\simeq \frac{4}{\sqrt{3}}\frac{1}{m^2}\tilde{h}\partial^2\phi + \frac{4}{m^2}(\partial\cdot A)^2 + \frac{8}{\sqrt{3}}\frac{1}{m^3}(\partial\cdot A)\partial^2\phi$$

$$+ \frac{16}{3}\frac{1}{m^2}\phi\partial^2\phi + \frac{4}{3}\frac{1}{m^4}(\partial^2\phi)^2$$

Then we get in total

$$-\frac{1}{4}m^2\left(h_{\mu\nu}h^{\mu\nu}-h^2\right)$$

$$\simeq -\frac{1}{4}m^2\Big\{\frac{4}{\sqrt{3}}\frac{1}{m^2}\partial\partial\tilde{h}\phi+\frac{1}{m^2}(\partial_\mu A_\nu+\partial_\nu A_\mu)^2+\frac{8}{\sqrt{3}}\frac{1}{m^3}\cancel{(\partial\cdot A)\partial^2\phi}$$

$$+\frac{4}{3}\frac{1}{m^2}\phi\partial^2\phi+\frac{4}{3}\cancel{\frac{1}{m^4}(\partial^2\phi)^2}-\frac{4}{\sqrt{3}}\frac{1}{m^2}\tilde{h}\partial^2\phi-\frac{4}{m^2}(\partial\cdot A)^2$$

$$-\frac{8}{\sqrt{3}}\frac{1}{m^3}\cancel{(\partial\cdot A)\partial^2\phi}-\frac{16}{3}\frac{1}{m^2}\phi\partial^2\phi-\frac{4}{3}\cancel{\frac{1}{m^4}(\partial^2\phi)^2}\Big\}$$

$$=-\frac{1}{4}\Big\{(\partial_\mu A_\nu+\partial_\nu A_\mu)^2-4(\partial\cdot A)^2$$

$$+\frac{4}{\sqrt{3}}\partial\partial\tilde{h}\phi-\frac{4}{\sqrt{3}}\tilde{h}\partial^2\phi-4\phi\partial^2\phi\Big\}$$

$$=-\frac{1}{4}\left[(\partial_\mu A_\nu+\partial_\nu A_\mu)^2-4(\partial\cdot A)^2\right]$$

$$-\frac{1}{\sqrt{3}}\partial\partial\tilde{h}\phi+\frac{1}{\sqrt{3}}\tilde{h}\partial^2\phi+\phi\partial^2\phi$$

The mixing terms $\tilde{h}\phi$ in the mass term cancels those in the kinetic terms. Furthermore we have that

$$(\partial_\mu A_\nu+\partial_\nu A_\mu)^2-4(\partial\cdot A)^2=2\partial_\mu A_\nu(\partial^\mu A^\nu+\partial^\nu A^\mu)-4(\partial\cdot A)^2$$

$$=2\partial_\mu A_\nu\partial^\mu A^\nu+2\partial_\mu A_\nu\partial^\nu A^\mu-4(\partial\cdot A)^2$$

$$\cong 2\partial_\mu A_\nu(\partial^\mu A^\nu-\partial^\nu A^\nu)+4\partial_\mu A_\nu\partial^\nu A^\mu-4(\partial\cdot A)^2$$

$$\simeq(\partial_\mu A_\nu-\partial_\nu A_\mu)(\partial^\mu A^\nu-\partial^\nu A^\mu)+4\cancel{(\partial\cdot A)^2}-4\cancel{(\partial\cdot A)^2}$$

$$\equiv F_{\mu\nu}F^{\mu\nu},\quad\text{where}\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

Thus in summary we get that

$$\lim_{m\to 0}\left(-\frac{1}{2}h^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta}-\frac{1}{4}m^2(h_{\mu\nu}^2-h^2)+\frac{1}{2M_{pl}}T_{\mu\nu}h^{\mu\nu}\right)$$

$$=-\frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta}-\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}\phi\partial^2\phi$$

$$+\frac{1}{2M_{pl}}T_{\mu\nu}\tilde{h}^{\mu\nu}+\frac{1}{2M_{pl}}\frac{T\phi}{\sqrt{3}}$$

we see that in the limit $m\to 0$ the 5 dofs of the massive graviton goes into 2 dofs of a massless graviton, 2 dofs of a massless photon that decouples and one scalar dof that do not decouple and gives rise to the vDVZ disontinuity.

Above we defined:

$$h_{\mu\nu}=\tilde{h}_{\mu\nu}+\frac{1}{m}(\partial_\mu A_\nu+\partial_\nu A_\mu)+\frac{1}{\sqrt{3}}\left(\eta_{\mu\nu}+2\frac{\partial_\mu\partial_\nu}{m^2}\right)\phi.$$

The extra scalar leads to an additional interaction on top of the massless graviton contribution

$$\delta S_{int}=-\left(\frac{1}{2M_{pl}\sqrt{3}}\right)^2\int\frac{d^4p}{(2\pi)^4}\tilde{T}'(-p)\tilde{G}(p)\tilde{T}(p)$$

$$=-\frac{1}{2M_{pl}^2}\int\frac{d^4p}{(2\pi)^4}\tilde{T}'_{\mu\nu}(-p)\left(\frac{1}{6}\eta^{\mu\nu}\eta_{\alpha\beta}\right)\tilde{G}(p)\tilde{T}^{\alpha\beta}(p),$$

which gives the vDVZ factor $-\frac{1}{2}+\frac{1}{6}=-\frac{1}{3}$!