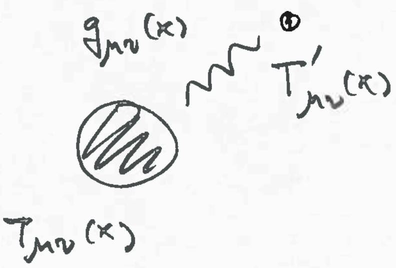
LECTURE 1

Linearized Einsteins Equations

We start by considering a heavy source, eg our own sun, described by a stress energy tensor $T_{\mu\nu}(x)$. The source will set up a gravitational field around it, equivalently, curve space time $\eta_{\mu\nu} \to g_{\mu\nu}(x)$. We can probe the metric $g_{\mu\nu}(x)$ using a probe such as a light massive observer, eg a planet, or a massless observer, eg a light ray, described by a stress energy tensor $T'_{\mu\nu}(x)$. In this way we can determine the gravitational potential and the light bending induced by a heavy source.



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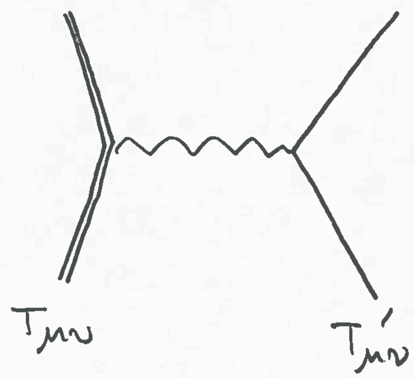


In Feynman diagrams we describe this as a stress – stress interactions with a gravitons exchange.

For weak gravitational fields we can write:

$$ds^2 = g_{\mu\nu}(x)\,dx^\mu dx^\nu = (\eta_{\mu\nu} + M_{Pl}^{-1} h_{\mu\nu})\,dx^\mu dx^\nu,$$

where $M_{Pl}^{-1} h_{\mu\nu} \ll 1$.

We have introduced a Planck mass M_{Pl} such that the perturbation acquires a dimension of mass.

The linear theory should work far away from the source but we know that this picture will break down at the Schwarzschild radius $R_S = 2GM$ and at smaller scales we need the full non-linear theory.

Equations of motion

The action for the gravitational field and the matter is given by $S = S_{EH} + S_M$ where

$$S_{EH} = M_{Pl}^2 \int d^4x \sqrt{-g}\, R$$

S_M = matter action

and as we will see from the Newtonian potential $M_{Pl}^{-2} = 16\pi G$.

Einstein's equations follow from the variation of $g_{\mu\nu}$:

$$\delta S_{EH} \equiv M_{Pl}^2 \int d^4x \sqrt{-g}\, G_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta S_M \equiv -\frac{1}{2} \int d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

where

$$G_{\mu\nu} \equiv M_{Pl}^{-2} \frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \ldots = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}},$$

see for example Wald, App E.

Thus we get the familiar Einstein's equations

$$G_{\mu\nu} = \frac{1}{2M_{Pl}^2} T_{\mu\nu} \quad , \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Sometimes it is convenient to work with the trace reversed form. First $G = \frac{1}{2M_{Pl}^2} T$, then $G = R - 2R = -R \Rightarrow$

$$R_{\mu\nu} = \frac{1}{2M_{Pl}^2}\left(T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} 2M_{Pl}^2 R\right) = \frac{1}{2M_{Pl}^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right).$$

The equation of motion for the matter must imply $\nabla_\mu T^{\mu\nu}$ for consistency, since $\nabla_\mu G^{\mu\nu} = 0$ by the Bianchi identity.

Let us now work out the first order equations. We have that

$$G^{(1)}_{\mu\nu} = M_{Pl}^{-1} \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \, ,$$

where the Einstein operator is defined by

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2}\Big(\partial^2 h_{\mu\nu} - 2\partial_{(\mu}\partial^\alpha h_{\nu)\alpha} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h)\Big) \, ,$$

where all indices are raised and lowered using the Minkowski metric $\eta_{\mu\nu} = (-1,+1,+1,+1)$. Then we get the equation

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = \frac{1}{2M_{Pl}} T_{\mu\nu} \, ,$$

where consistency now requires that $\partial^\mu T_{\mu\nu} \equiv 0$.

Indeed we have that $\partial^\mu \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \equiv 0$ from

$$
\begin{aligned}
-2\partial^\mu \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} &= \partial^\mu \partial^2 h_{\mu\nu} - 2\partial^\mu \partial_{(\mu}\partial^\alpha h_{\nu)\alpha} \\
&\quad + \cancel{\partial^\mu \partial_\mu \partial_\nu h} + \partial_\nu \partial^\alpha \partial^\beta h_{\alpha\beta} - \cancel{\partial_\nu \partial^2 h} \\
&= \cancel{\partial^\mu \partial^2 h_{\mu\nu}} - \cancel{\partial^2 \partial^\alpha h_{\nu\alpha}} - \cancel{\partial^\mu \partial_\nu \partial^\alpha h_{\mu\alpha}} \\
&\quad + \cancel{\partial_\nu \partial^\alpha \partial^\beta h_{\alpha\beta}} \\
&= 0
\end{aligned}
$$

To solve for the gravitational field induced by $T_{\mu\nu}$ we would now like to be able to invert the operator $\mathcal{E}_{\mu\nu}{}^{\alpha\beta}$ so that we could write

$$
h_{\mu\nu} \stackrel{?}{=} \frac{1}{2M_{Pl}} \mathcal{E}^{-1}_{\mu\nu}{}^{\alpha\beta} T_{\alpha\beta} .
$$

this is not possible due to gauge invariance. Indeed the matrix $\mathcal{E}_{\mu\nu}{}^{\alpha\beta}$ has zero-eigenvalues corresponding to the zero-eigenvectors $\xi_{\alpha\beta} = 2\partial_{(\alpha}\xi_{\beta)}$, that is

$$
\mathcal{E}_{\mu\nu}{}^{\alpha\beta} \partial_{(\alpha}\xi_{\beta)} = 0 , \quad \text{identically,}
$$

and thus $\mathcal{E}$ cannot be inverted.

<u>Gauge invariance</u>

In the full non-linear theory there is a gauge invariance under general coordinate transformations (GCT)

$$
x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x) , \quad \tilde{x}^\mu = \tilde{x}^\mu(x) \text{ general.}
$$

In the linear theory, we have invariance under infinitesimal GCT, that is when

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x) \quad , \quad \xi^\mu \text{ infinitesimal}$$

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(\tilde{x})$$

Now we have that $d\tilde{s}^2 = ds^2$ where

$$d\tilde{s}^2 \equiv (\eta_{\mu\nu} + M_{Pl}^{-1}\tilde{h}_{\mu\nu}(\tilde{x}))\,d\tilde{x}^\mu d\tilde{x}^\nu$$

$$\begin{aligned} ds^2 &\equiv (\eta_{\mu\nu} + M_{Pl}^{-1} h_{\mu\nu}(x))\,dx^\mu dx^\nu \\ &= (\eta_{\mu\nu} + M_{Pl}^{-1} h_{\mu\nu}) \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}\, d\tilde{x}^\alpha d\tilde{x}^\beta \\ &\simeq (\eta_{\mu\nu} + M_{Pl}^{-1} h_{\mu\nu})(\delta^\mu_\alpha - \partial_\alpha \xi^\mu)(\delta^\nu_\beta - \partial_\beta \xi^\nu)\, d\tilde{x}^\alpha d\tilde{x}^\beta \\ &\simeq (\eta_{\alpha\beta} + M_{Pl}^{-1} h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha)\, d\tilde{x}^\alpha d\tilde{x}^\beta , \end{aligned}$$

where we have neglected higher orders ξ^2, ξh etc.

Now we have that also $\tilde{h}_{\mu\nu}(\tilde{x}) \simeq \tilde{h}_{\mu\nu}(x)$ in this approximation so that the gauge invariance takes the form

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) - 2\partial_{(\mu}\xi_{\nu)}(x) ,$$

where we have incorporated a factor of M_{Pl}^{-1} into ξ_μ.

Now since $\varepsilon_{\mu\nu}{}^{\alpha\beta}\partial_{(\alpha}\xi_{\beta)} = 0$ identically we have a gauge invariance of the form

$$\varepsilon_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} \to \varepsilon_{\mu\nu}{}^{\alpha\beta}\tilde{h}_{\alpha\beta} \overset{!}{=} \varepsilon_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} ,$$

even for <u>finite</u> ξ_μ.

Now let us exploit the gauge invariance to impose some convenient gauge on $h_{\mu\nu}$. First notice that in terms of the quantity $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ then

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2}\left(\partial^2 \bar{h}_{\mu\nu} + 2\partial_{(\mu}\partial^{\alpha}\bar{h}_{\nu)\alpha} - \eta_{\mu\nu}\partial^{\alpha}\partial^{\beta}\bar{h}_{\alpha\beta}\right),$$

so a particularly convenient choice would be a Lorenz-like condition $\partial^{\alpha}\bar{h}_{\alpha\beta} = 0$ such that $\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2}\partial^2 \bar{h}_{\mu\nu}$.

This can be imposed as follows:

Under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$ we have that

$$\partial^{\mu}\bar{h}_{\mu\nu} \to \partial^{\mu}\bar{h}_{\mu\nu} + \partial^{\mu}\delta\bar{h}_{\mu\nu},$$

where $\delta\bar{h}_{\mu\nu} = \delta h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta h = -2\partial_{(\mu}\xi_{\nu)} + \eta_{\mu\nu}\partial\cdot\xi$

such that $\partial^{\mu}\delta\bar{h}_{\mu\nu} = -\partial^2\xi_{\nu} - \partial_{\nu}\cancel{\partial\cdot\xi} + \partial_{\nu}\cancel{\partial\cdot\xi} = -\partial^2\xi_{\nu}$

So we can accomplish $\partial^{\mu}\tilde{h}_{\mu\nu} = 0$ if we choose

$$-\partial^2\xi_{\nu} = \partial^{\mu}\delta\bar{h}_{\mu\nu} \overset{!}{=} -\partial^{\mu}\bar{h}_{\mu\nu} \Rightarrow \xi_{\nu} = \frac{1}{\partial^2}(\partial^{\mu}\bar{h}_{\mu\nu})$$

In this gauge we have that (dropping the $\sim$ on $\tilde{h}_{\mu\nu}$)

$$\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2}\partial^2\bar{h}_{\mu\nu}, \quad \partial^{\mu}\bar{h}_{\mu\nu} = 0.$$

We now comment on some other gauge choices.

Other gauge choices

Consider putting instead $\partial^\mu(h_{\mu\nu} - a\eta_{\mu\nu}h)$ to zero. Under a gauge transformation we have that

$$\delta\left[\partial^\mu(h_{\mu\nu} - a\eta_{\mu\nu}h)\right] = -\partial^2\xi_\nu - \partial_\nu(\partial\cdot\xi) + 2a\partial_\nu(\partial\cdot\xi)$$
$$= -\partial^2\xi_\nu + (2a-1)\partial_\nu(\partial\cdot\xi)$$
$$\equiv Q_\nu{}^\mu\xi_\nu ,$$

where $Q_\nu{}^\mu = -\partial^2\delta_\nu^\mu + (2a-1)\partial_\nu\partial^\mu$. We can impose the gauge condition as long as $Q(a)_\nu{}^\mu$ is invertible. $Q(a)$ is invertible as long as $a \neq 1$. For $a=1$ $Q(1)$ has a zero eigenvector $\partial_\mu\phi$. Indeed we have

$$Q(1)_\mu{}^\nu\partial_\nu\phi = -\partial^2\partial_\nu\phi + \partial_\nu\partial^2\phi = 0 .$$

This can be understood from gauge invariance again, since to first order, the Ricci scalar is given by

$$R^{(1)} = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h = \partial^\mu\partial^\nu(h_{\mu\nu} - \eta_{\mu\nu}h),$$

where we used that to first order the Riemann tensor is given by

$$R^{(1)}_{\alpha\mu\beta\nu} = \frac{1}{2}\left(\partial_\mu\partial_\beta h_{\alpha\nu} + \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\alpha\partial_\beta h_{\mu\nu} - \partial_\mu\partial_\nu h_{\alpha\beta}\right),$$

see for example Misner, Thorne & Wheeler.

The $a=1$ will come back and haunt us in the massive theory.

Solution in Lorenz Gauge

Let us now return to the equation for $h_{\mu\nu}$, which reads in Lorenz gauge

$$-\frac{1}{2}\partial^2 \bar{h}_{\mu\nu} = \frac{1}{2M_{pl}} T_{\mu\nu},$$

or in its tracereversed form

$$\partial^2 h_{\mu\nu} = -\frac{1}{M_{pl}} \bar{T}_{\mu\nu} = -\frac{1}{M_{pl}} \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right).$$

This equation we easily solve using Green's functions

$$\begin{aligned} h_{\mu\nu}(x) &= -\frac{1}{M_{pl}} \frac{1}{\partial^2} \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T \right)(x) \\ &= -\frac{1}{M_{pl}} \int d^4y \left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} \right)\left(\frac{1}{\partial^2} \delta^{(4)}(x-y) \right) T_{\alpha\beta}(y) \\ &\equiv -\frac{1}{M_{pl}} \int d^4y \, G_{\mu\nu}{}^{\alpha\beta}(x-y) \, T_{\alpha\beta}(y), \end{aligned}$$

were we have defined the graviton propagator

$$G_{\mu\nu}{}^{\alpha\beta}(x-y) \equiv \left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} \right) G(x-y)$$

and $G(x-y)$ denotes the standard scalar propagator

$$\begin{aligned} G(x-y) &\equiv \frac{1}{\partial^2} \delta^{(4)}(x-y) = \frac{1}{\partial^2} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2} e^{ip(x-y)}. \end{aligned}$$

We now calculate the metric set up by a point source.

Metric around point source

Consider now a stationary point source at the origin $\vec{x}=0$,
$T_{\mu\nu}(\vec{x},t) = \delta^0_\mu \delta^0_\nu M \delta^{(3)}(\vec{x})$. Then we get that

$$h_{\mu\nu}(\vec{x}) = -\frac{1}{M_{pl}} \int d^3y\, G_{\mu\nu}{}^{\alpha\beta}(\vec{x}-\vec{y}) T_{\alpha\beta}(\vec{y})$$

$$= -\frac{1}{M_{pl}} \left(\delta^\alpha_{(\mu}\delta^\beta_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\right) \delta^0_\alpha \delta^0_\beta M\, G(\vec{x})$$

$$= -\frac{M}{M_{pl}} \left(\delta^0_\mu \delta^0_\nu + \frac{1}{2}\eta_{\mu\nu}\right) G(\vec{x}) \,,$$

where we have defined the three-dimensional Green's function

$$G(\vec{x}) = \frac{1}{\vec{\nabla}^2}\delta^{(3)}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{-\vec{p}^2} e^{i\vec{p}\cdot\vec{x}} = -\frac{1}{4\pi r} \,,$$

where $r = |\vec{x}|$. Now $\delta^0_\mu \delta^0_\nu + \frac{1}{2}\eta_{\mu\nu} = \frac{1}{2}\mathrm{diag}(1,1,1,1)$ thus

$$M_{pl}^{-1} h_{\mu\nu}(\vec{x}) = + \frac{M}{8\pi M_{pl}^2 r} \cdot \mathbb{1}_{\mu\nu} \,, \quad \mathbb{1} = \mathrm{diag}(1,1,1,1)$$

$$\equiv -2\Phi(\vec{x}) \cdot \mathbb{1}_{\mu\nu} \,,$$

where we have defined

$$\Phi(\vec{x}) = -\frac{M}{16\pi M_{pl}^2 r} = -\frac{GM}{r} \,, \quad \text{with } M_{pl}^2 = \frac{1}{16\pi G}$$

Thus the metric takes the form:

$$ds^2 = (\eta_{\mu\nu} + M_{pl}^{-1} h_{\mu\nu})\, dx^\mu dx^\nu = -(1+2\Phi)dt^2 + (1-2\Phi)d\vec{x}^2$$

Break down of linearity

The metric $ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\vec{x}^2$, $\Phi = -\frac{GM}{r}$, is the linearized form of the Schwarzschild metric in isotropic coordinates.

We see that the linear theory breaks down when

$$1 = M_{pl}^{-1} h_{00} = -2\Phi = \frac{2GM}{r}$$

that is at $r = 2GM \equiv R_S$, the Schwarzschild radius.

Does this mean that we have to non-linearly complete the theory? Can't we just stick with linear gravity and define that as gravity? Who cares if $h_{\mu\nu} \gtrsim 1$ or not? Indeed, so far it seems that nothing depended on the size of $h_{\mu\nu}$, eg the gauge transformations could actually be made finite, even in the linear theory!

The answer is that a spin-2 theory of gravity has to be non-linearly completed otherwise it is inconsistent, that is we cannot stick with a linear theory!

It is inconsistent for at least three reasons:

i) Predicts the wrong perihelion shift

ii) Contradicts the strong equivalence principle which states that gravity should feel all energy, even its own.

iii) $\partial_\mu T^{\mu\nu} = 0$ is inconsistent with $\nabla_\mu T^{\mu\nu} = 0$!

Let me expand on the last point iii). Consider a point particle in a gravitational field with stress energy tensor

$$T^{\mu\nu}(x) = m\int d\tau\, \dot{q}^\mu \dot{q}^\nu \delta^{(4)}(x-q(\tau)).$$

Let us now check energy conservation

$$\begin{aligned}
\partial_\mu T^{\mu\nu}(x) &= m\int d\tau\, \dot{q}^\mu \dot{q}^\nu \partial_\mu \delta^{(4)}(x-q(\tau)) \\
&= -m\int d\tau\, \dot{q}^\mu \dot{q}^\nu \frac{\partial}{\partial q^\mu} \delta^{(4)}(x-q(\tau)) \\
&= -m\int d\tau\, \dot{q}^\nu \frac{d}{d\tau} \delta^{(4)}(x-q(\tau)) \\
&= +m\int d\tau\, \ddot{q}^\nu \delta^{(4)}(x-q(\tau))
\end{aligned}$$

This is zero only if $\ddot{q}^\nu = 0$, that is the point particle feel no gravity, which is certainly inconsistent.

Thus we have to non-linear complete the equations of motion. Starting from $\mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = \frac{1}{2M_{Pl}} T_{\mu\nu}$ this seems like a hard task but using gauge invariance we can get a unique theory up to two derivatives, which is exactly Einstein's theory $G_{\mu\nu} = \frac{1}{2M_{Pl}} T_{\mu\nu}$, consistent with $D_\mu T^{\mu\nu} = 0$.

Effective interaction

Let us now return to the source-probe calculation we set up in the beginning of this lecture. The interaction of the probe with the gravitational field can be read-off from

$$S_{probe}[g_{\mu\nu}] = S_{probe}[\eta_{\mu\nu}] + \int d^4x \frac{\delta S}{\delta g^{\alpha\beta}(x)}[\eta_{\mu\nu}]\, \delta g^{\alpha\beta}(x) + \ldots,$$

where $\frac{\delta S}{\delta g^{\alpha\beta}(x)}[\eta_{\mu\nu}] = -\frac{1}{2} T'_{\alpha\beta}(x)$ and $\delta g^{\alpha\beta}(x) = -M_{pl}^{-1} h^{\alpha\beta}(x)$.

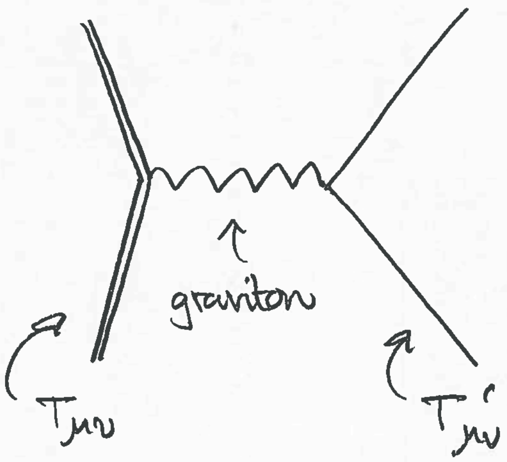
Thus we get the interaction

$$S_{int} = \frac{1}{2M_{pl}} \int d^4x\, T'_{\mu\nu}(x)\, h^{\mu\nu}(x)$$

Now, we have already obtained an expression for $h^{\mu\nu}(x)$ in terms of the source $T_{\mu\nu}(x)$, thus the interaction between the source and the probe takes the form:

$$S_{int} = -\frac{1}{2M_{pl}^2} \int d^4x \int d^4y\, T'_{\mu\nu}(x)\, G^{\mu\nu\alpha\beta}(x-y)\, T_{\alpha\beta}(y)$$

In the language of Feynman diagrams this corresponds to two currents exchanging a graviton.



In momentum space we have that

$$G_{\mu\nu\alpha\beta}(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{G}_{\mu\nu\alpha\beta}(p) e^{ip(x-y)}$$

so that the effective interaction becomes

$$S_{int} \Rightarrow -\frac{1}{2M_{pl}^2} \int \frac{d^4p}{(2\pi)^4} \left(\int d^4x \, e^{ipx} T'_{\mu\nu}(x) \right) \tilde{G}^{\mu\nu\alpha\beta}(p)$$
$$\times \left(\int d^4y \, e^{-ipy} T_{\alpha\beta}(y) \right)$$
$$\equiv -\frac{1}{2M_{pl}^2} \int \frac{d^4p}{(2\pi)^4} \tilde{T}'_{\mu\nu}(-p) \tilde{G}^{\mu\nu\alpha\beta}(p) \tilde{T}_{\alpha\beta}(p)$$

static limit

Consider now the static limit where $T_{\mu\nu}(x)$, $T'_{\mu\nu}(x)$ are both independent of time. Then we have that:

$$\tilde{T}_{\mu\nu}(p) = \int dx^0 e^{ip^0x^0} \int d^3\vec{x} \, e^{i\vec{p}\cdot\vec{x}} \, T_{\mu\nu}(\vec{x})$$
$$= 2\pi \delta(p^0) \tilde{T}_{\mu\nu}(\vec{p})$$

and likewise

$$\tilde{T}'_{\mu\nu}(-p) = 2\pi \delta(-p^0) \tilde{T}_{\mu\nu}(-\vec{p}) .$$

Then we get for the effective interaction

$$S_{int} = -\frac{1}{2M_{pl}^2} \cdot \int \frac{dp^0}{2\pi} 2\pi \delta(p^0) 2\pi \delta(-p^0) \int \frac{d^3p}{(2\pi)^3} \tilde{T}'_{\mu\nu}(-\vec{p})$$
$$\times \tilde{G}^{\mu\nu\alpha\beta}(\vec{p}, p^0) \tilde{T}_{\alpha\beta}(\vec{p})$$

We can now perform the p^0 integral and use that $2\pi(p^0=0) = \int dt\, e^{it\cdot(p^0=0)} = \int dt$, then we get

$$S_{int} = -\frac{1}{2M_{pl}^2}\int dt \int \frac{d^3\vec{p}}{(2\pi)^3}\, \tilde{T}'_{\mu\nu}(-\vec{p})\, \tilde{G}^{\mu\nu\alpha\beta}(\vec{p})\, \tilde{T}_{\alpha\beta}(\vec{p})$$

Consider now two point particles, the source at $\vec{x}=0$ and the probe at $\vec{x}=\vec{q}$, that is

$$T_{\mu\nu}(\vec{x}) = M\delta^0_\mu\delta^0_\nu\delta^{(3)}(\vec{x}), \quad T'_{\mu\nu}(\vec{x}) = m\delta^0_\mu\delta^0_\nu\delta^{(3)}(\vec{x}-\vec{q})$$

Then we get for the Fourier transforms

$$T_{00}(\vec{x}) = M\delta^{(3)}(\vec{x}) = \int\frac{d^3\vec{p}}{(2\pi)^3} M e^{i\vec{p}\cdot\vec{x}}$$

$$\Rightarrow \tilde{T}_{00}(\vec{p}) = M$$

$$T'_{00}(\vec{x}) = m\,\delta^{(3)}(\vec{x}-\vec{q}) = \int\frac{d^3\vec{p}}{(2\pi)^3} m e^{-i\vec{q}\cdot\vec{p}} e^{i\vec{p}\cdot\vec{x}}$$

$$\Rightarrow \tilde{T}'_{00}(-\vec{p}) = m\, e^{+i\vec{q}\cdot\vec{p}}$$

This gives us that

$$S_{int} = -\frac{mM}{2M_{pl}^2}\int dt \int\frac{d^3\vec{p}}{(2\pi)^3}\, \tilde{G}_{0000}(\vec{p})\, e^{i\vec{q}\cdot\vec{p}}$$

$$\equiv -\frac{mM}{2M_{pl}^2}\int dt\, G_{0000}(\vec{q})$$

We now compare this to the action for a point particle with position $\vec{q}$.

The action for a point particle with coordinate $\vec{q}$ takes the form

$$S[\vec{q}] = \int dt\, L(\vec{q}) = \int dt \left(K(\vec{q}) - V(\vec{q}) \right),$$

where L is the particle Lagrangian, K its kinetic energy and V its potential energy. Thus we can identify:

$$V(\vec{q}) = \frac{mM}{2M_{pl}^2} G_{00}{}^{00}(\vec{q})$$

Now we have that

$$\begin{aligned} G_{00}{}^{00}(\vec{q}) &= \left(\delta^0_{(0}\delta^0_{0)} - \tfrac{1}{2}\eta_{00}\eta^{00}\right) G(\vec{q}) \\ &= \tfrac{1}{2} G(\vec{q}) = \tfrac{1}{2} \cdot \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{-\vec{p}^2} e^{i\vec{p}\cdot\vec{q}} \\ &= -\frac{1}{8\pi|\vec{q}|} \end{aligned}$$

therefore we get for the potential energy

$$V(\vec{q}) = -\frac{mM}{16\pi M_{pl}^2 |\vec{q}|} = -\frac{mMG}{|\vec{q}|}, \quad M_{pl}^2 = \frac{1}{16\pi G}$$

We have indeed found that

$$V(r) = -G\frac{Mm}{r},$$

as we set out to prove. This leads to an attractive $1/r^2$ force as it should. We now go over to massive gravity and compare our results.

Gauge invariance and constraints

When we turn on the mass of the graviton we explicitly break the gauge invariance of the massless theory. This leads to new degrees of freedom that we before could gauge away. To get the right number of propagating degrees of freedom, 5 in the case of a massive graviton, the dynamics of the 10 fields $h_{\mu\nu}$ should impose 5 constraints. Before doing massive gravity, let us see how it works for massive vectors.

Maxwell Field

The 2 propagating ± 1 helicities of the massless photon is described by the action

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}^2 \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Naively A_μ carries 4 propagating degrees of freedom but, as we now show, 2 are removed from gauge invariance and a constraint.

The theory is invariant under $A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu \Lambda$ thus we can remove 1 degree of freedom in Λ.

Also, A_0 is not a propagating degree of freedom but rather a Lagrange multiplier as we now see:

The momentum canonically conjugate to A_μ is given by

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial\partial_0 A_\mu} = -F^{\alpha\beta}\delta^0_\alpha\delta^\mu_\beta = -F^{0\mu}$$

Thus we see that $\Pi^0 = 0$, that is A^0 has no momentum, and is a Lagrange multiplier that instead imposes a constraint. The equation of motion is determined from the variation of A_μ

$$\delta S = -\frac{1}{2}\int d^4x\, F^{\mu\nu}\delta F_{\mu\nu} = -\int d^4x\, F^{\mu\nu}\partial_\mu \delta A_\nu$$
$$= +\int d^4x\, \partial_\mu F^{\mu\nu}\delta A_\nu$$

$$\therefore \quad \partial_\mu F^{\mu\nu} = 0$$

Specifically the A_0 variation implies that

$$0 = \partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i(\partial^i A^0 - \partial^0 A^i)$$
$$= \partial_i(\partial_i A^0 + \partial_0 A^i) \equiv -\partial_i E^i,$$

where $\vec{E} = -\vec{\nabla}A^0 - \dot{\vec{A}}$.

So we see that A_0 implies the constraint

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \text{Gauss' law}$$

Thus we conclude that the photon carries 2 propagating degrees of freedom. In empty space a particularly useful gauge is the so called radiation gauge where $A_0 = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$.

This gauge also implies $\partial\cdot A = 0$ which is called Lorenz gauge.

We now turn to the massive vector field, the Proca field.

Proca Field

Now add a massterm to the Lagrangian. This should now describe a massive spin-1 particle with 3 dofs.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2 A_\mu^2 .$$

This explicitly breaks gauge invariance and no dofs can be gauged away. But Π_0 is still zero and imposes a constraint so that we indeed end up with 3 dofs.

The variation wrt A_0 now implies that

$$-\vec{\nabla}\cdot\vec{E} - m^2 A^0 = 0 ,$$

so that also longitudinal modes can be excited.

This constraint is also equivalent to $\partial_\mu A^\mu = 0$ which can be seen from the full equation of motion:

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = 0$$

Now take ∂_ν of the above then $\partial_\mu\partial_\nu F^{\mu\nu} = 0$ but $m^2\partial_\nu A^\nu = 0$ implies that $\partial\cdot A = 0$ for non-zero m^2.

Massless & Massive gravity

Here $h_{\mu\nu}$ carries 10 dofs. Now $h_{0\mu}$ are Lagrange multipliers and reduce the number of dofs by 4. In the massless case we also have 4 gauge invariances which leads to 2 dofs left as one should have. In the massive case we have no gauge invariance and it seems like we have 6 dofs. The reduction to 5 dofs is exhibited by the Fierz-Pauli construction.

The Fierz-Pauli Action for Massive Gravity

The action for the massless graviton in the Linearized theory is obtained by working out the Einstein-Hilbert action to second order in $M_{pl}^{-1} h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$.

$$S_{EH}[g] = S_{EH}[\eta] + \int d^4x\, \delta g^{\alpha\beta}(x) \frac{\delta S}{\delta g^{\alpha\beta}(x)}[\eta]$$

$$+ \frac{1}{2!} \int d^4x \int d^4y\, \delta g^{\alpha\beta}(x)\, \delta g_{\mu\nu}(y) \frac{\delta^2 S_{EH}}{\delta g^{\alpha\beta}(x)\delta g_{\mu\nu}(y)}[\eta] + \dots$$

where $\delta g_{\mu\nu} = M_{pl}^{-1} h_{\mu\nu}$, $\delta g^{\mu\nu} = - M_{pl}^{-1} h^{\mu\nu}$.

Now we have that $S_{EH}[\eta] = 0$. Further more

$\frac{\delta S}{\delta g^{\alpha\beta}} \equiv M_{pl}^2 G_{\alpha\beta}$ and $G_{\alpha\beta}[\eta] = 0$.

For the second order term we have

$$\frac{\delta}{\delta g^{\alpha\beta}(x)} \frac{\delta}{\delta g_{\mu\nu}(y)} (S_{EH}) \Rightarrow \frac{\delta}{\delta g_{\mu\nu}(y)} \left(M_{pl}^2 G_{\alpha\beta}(x)\right)$$

$$= \frac{\delta}{\delta g_{\mu\nu}(y)} \left(M_{pl}^2 \cdot \mathcal{E}_{\alpha\beta}{}^{\gamma\delta}(M_{pl}^{-1} h_{\gamma\delta}(x)) + \dots\right)$$

$$= M_{pl}^2\, \mathcal{E}_{\alpha\beta}{}^{\mu\nu}(\delta^{(4)}(x-y)) + \dots$$

The dots go away when evaluated on $\eta_{\mu\nu}$.

Thus we find that

$$S_{EH}^{(2)} = \frac{1}{2!}\int d^4x \int d^4y \left(-M_{pl}^{-1} h^{\alpha\beta}(x)\right)\left(M_{pl}^{-1} h_{\mu\nu}(y)\right) \times \left(M_{pl}^2 \mathcal{E}_{\alpha\beta}{}^{\mu\nu}\left(\delta^{(4)}(x-y)\right)\right)$$

Now partial integrate twice and use the δ-function to get:

$$S_{EH}^{(2)} = -\frac{1}{2}\int d^4x\, h^{\alpha\beta} \mathcal{E}_{\alpha\beta}{}^{\mu\nu} h_{\mu\nu}$$

This is the Einstein-Hilbert action to second order around $\eta_{\mu\nu}$.
There is another way of arriving at this action using that the action should be gauge invariant under the transformation $h_{\mu\nu} \to \hat{h}_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$. One writes down the most general second order differential operator

$$\mathcal{E}'_{\alpha\beta}{}^{\mu\nu} = a\,\partial^2 \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + b\,\eta_{\alpha\beta}\partial^\mu\partial^\nu + \ldots$$

and fix the coefficients $a, b, \ldots$ (up to overall normalization) such that $\mathcal{E}'_{\alpha\beta}{}^{\mu\nu}$ kills $2\partial_{(\mu}\xi_{\nu)}$. Again one arrives at $\mathcal{E}'_{\alpha\beta}{}^{\mu\nu} \propto \mathcal{E}_{\alpha\beta}{}^{\mu\nu}$.

Mass terms

To this action we can add a general mass term

$$S_m^{(2)} = -\frac{1}{4}\int d^4x \left(m_1^2 h_{\mu\nu}h^{\mu\nu} + m_2^2 h^2\right)$$

$$\equiv -\frac{m_g^2}{4}\int d^4x \left(h_{\mu\nu}h^{\mu\nu} - a h^2\right),$$

where $m_g^2 = m_1^2$ and $a = -m_2^2/m_1^2$.

As we will see below, only the choice $a=1$ leads to a sensible theory, this is the Fierz-Pauli massterm.

For $a \neq 1$, the theory has six degrees freedom and worse: the 6th is a ghost with negative kinetic energy. Only for $a=1$ there is no extra 6th dof and the theory is healthy.

We also add an interaction with matter

$$S_{int} = \frac{1}{2M_{pl}} \int d^4x \, T_{\mu\nu} h^{\mu\nu},$$

so that the full action is given by

$$\begin{aligned} S &= S^{(2)}_{EH} + S^{(2)}_{m} + S_{int} \\ &= \int d^4x \Big[-\frac{1}{2} h^{\mu\nu} \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} - \frac{m_g^2}{4} \left(h_{\mu\nu} h^{\mu\nu} - a h^2 \right) \\ &\qquad + \frac{1}{2M_{pl}} T_{\mu\nu} h^{\mu\nu} \Big], \end{aligned}$$

where the Einstein operator is given by

$$\begin{aligned} \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{2} \Big(& \partial^2 h_{\mu\nu} - 2 \partial_{(\mu} \partial^{\alpha} h_{\nu)\alpha} + \partial_\mu \partial_\nu h \\ & + \eta_{\mu\nu} \left(\partial^\alpha \partial^\beta h_{\alpha\beta} - \partial^2 h \right) \Big). \end{aligned}$$

Equations of motion

We now vary $h_{\mu\nu}$, to determine the equations of motion

$$-2 \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = -\frac{1}{M_{pl}} T_{\mu\nu} + m_g^2 \left(h_{\mu\nu} - a \eta_{\mu\nu} h \right).$$

We now determine the constraints. First, take the divergence of the equation using that $\partial^\mu \mathcal{E}_{\mu\nu}{}^{\alpha\beta} h_{\alpha\beta} = 0$ and that the stress-energy is conserved $\partial_\mu T^{\mu\nu}$. This implies the 4 constraints

$$0 = m_g^2 \partial^\mu (h_{\mu\nu} - a\eta_{\mu\nu} h),$$

for non-vanishing m_g^2. This reduces the 10 dofs of $h_{\mu\nu}$ to 6 dofs. Second, we take the trace of the equation. We must now evaluate

$$\begin{aligned} -2\mathcal{E}_\mu{}^{\mu\alpha\beta} h_{\alpha\beta} &= \partial^2 h - 2\partial^\mu\partial^\alpha h_{\mu\alpha} + \partial^2 h \\ &\quad + 4(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h) \\ &= -2\partial^2 h + 2\partial^\alpha\partial^\beta h_{\alpha\beta} \end{aligned}$$

Now use the constraint $\partial^\mu\partial^\nu h_{\mu\nu} = a\partial^2 h$ such that

$$-2\mathcal{E}_\mu{}^{\mu\alpha\beta} h_{\alpha\beta} = -2\partial^2 h + 2a\partial^2 h = 2(a-1)\partial^2 h.$$

We must also evaluate

$$m_g^2 \eta^{\mu\nu}(h_{\mu\nu} - a\eta_{\mu\nu} h) = m_g^2 (h - 4ah) = (1-4a) m_g^2 h$$

We are now ready to take the trace, which leads to

$$2(a-1)\partial^2 h = -\frac{1}{M_{pl}} T + (1-4a) m_g^2 h$$

<u>$a=1$</u> For the case $a=1$ we get the constraint

$$3m_g^2 h = -\frac{1}{M_{pl}} T$$

thus we only have 5 propagating degrees of freedom, this is the Fierz-Pauli choice.

$\underline{a \neq 1}$ For the case $a \neq 1$ we get

$$\partial^2 h = \frac{-1}{2(a-1)} \frac{1}{M_{Pl}} T + \frac{1-4a}{2(a-1)} m_g^2 h \,,$$

$$-\partial^2 h + \frac{4a-1}{2(1-a)} m_g^2 h = \frac{1}{2(a-1)} \frac{1}{M_{Pl}} T$$

Thus in this case the trace h becomes a propagating dof

$$(-\partial^2 + m_a^2) h = \frac{-1}{2(a-1)} \frac{1}{M_{Pl}} T \,,$$

where $m_a^2 = m_g^2 \left(\frac{4a-1}{2(1-a)} \right)$ and from causality $1 \geq a \geq 1/4$.

We see that when $a \to 1$, $m_a^2 \to \infty$ and the extra mode becomes infinitely heavy and decouples from the theory, which corresponds to the Fierz-Pauli choice.

Fierz-Pauli theory $a=1$

From now on we disuss the case $a=1$ and come back later to the case $a \neq 1$. In this case the fire constraints read

$$\partial^\mu (h_{\mu\nu} - \eta_{\mu\nu} h) = 0 \quad , \quad h = -\frac{1}{3 m_g^2 M_{Pl}} T$$

For a vanishing source this just tells us that the trace h is zero. Notice that the divergence constraint above is the only constraint that we cannot impose in the massless theory. Thus the constraints do not correspond to possible gauge choice. This is in contrast to the photon case where we can impose $\partial \cdot A = 0$ in a gauge but also $\partial \cdot A = 0$ was automatic for the massive photon.

We now go on to inverting the equation of motion:

We now use the constraints in the kinetic term

$$-2\mathcal{E}_{\mu\nu}{}^{\alpha\beta}h_{\alpha\beta} = \partial^2 h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\nu\alpha} - \partial_\nu\partial^\alpha h_{\mu\alpha} + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \partial^2 h)$$

$$= \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h - \partial_\nu\partial_\mu h + \partial_\mu\partial_\nu h + \eta_{\mu\nu}(\cancel{\partial^2 h} - \cancel{\partial^2 h})$$

$$= \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h = \partial^2 h_{\mu\nu} + \frac{1}{M_{pl}}\left(\frac{1}{3m_g^2}\partial_\mu\partial_\nu T\right)$$

and in the massterm

$$m_g^2(h_{\mu\nu} - \eta_{\mu\nu}h) = m_g^2 h_{\mu\nu} + \frac{1}{M_{pl}}\left(\frac{1}{3}\eta_{\mu\nu}T\right).$$

Now collecting the above we get that

$$(\partial^2 - m_g^2)h_{\mu\nu} = -\frac{1}{M_{pl}}\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T - \frac{1}{3m_g^2}\partial_\mu\partial_\nu T\right).$$

We can compare this to the massless case where

$$\partial^2 h_{\mu\nu} = -\frac{1}{M_{pl}}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right).$$

In the limit $m_g^2 \to 0$ the left hand side of the massive equation goes continuously to the massless equation, but the right hand side does not. There is a mismatch of $-1/3$ compared to $-1/2$ and it seems that the $1/m_g^2$ term blows up in the small mass limit. Actually the $1/m_g^2$ term drops out in the effective interaction as we show below.

Inverting the equation of motion we now get

$$h_{\mu\nu}(x) = -\frac{1}{M_{pl}} \int d^4y\, G(x-y; m_g^2) \Big(T_{\mu\nu}(y) - \frac{1}{3}\eta_{\mu\nu} T(y) - \frac{1}{3m_g^2}\partial_\mu \partial_\nu T(y) \Big),$$

where we have introduced the massive scalar Green's function

$$G(x-y; m_g^2) = \frac{1}{\partial^2 - m_g^2}\delta^{(4)}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{-p^2-m_g^2} e^{ip(x-y)} .$$

Effective interaction

The interaction between the source and the probe now takes the form

$$S_{int} = \frac{1}{2M_{pl}} \int d^4x\, T'_{\mu\nu}(x)\, h^{\mu\nu}(x)$$

$$= -\frac{1}{2M_{pl}^2} \int d^4x \int d^4y\, T'_{\mu\nu}(x)\, G(x-y; m_g^2) \times \Big(T^{\mu\nu}(y) - \frac{1}{3}\eta^{\mu\nu} T(y) - \frac{1}{3m_g^2}\partial^\mu \partial^\nu T(y) \Big)$$

We now show that the derivative term drops out

$$\int d^4x \int d^4y\, T'_{\mu\nu}(x)\, G(x-y; m_g^2)\, \partial^\mu_{(y)} \partial^\nu_{(y)} T(y)$$

$$= -\int d^4x \int d^4y\, T'_{\mu\nu}(x)\, \partial^\mu_{(y)} G(x-y; m_g^2)\, \partial^\nu_{(y)} T(y)$$

$$= +\int d^4x \int d^4y\, T'_{\mu\nu}(x)\, \partial^\mu_{(x)} G(x-y; m_g^2)\, \partial^\nu_{(y)} T(y)$$

$$= -\int d^4x \int d^4y\, \partial^\mu_{(x)} T'_{\mu\nu}(x)\, G(x-y; m_g^2)\, \partial^\nu_{(y)} T(y) = 0,$$

using stress-energy conservation.

Thus the effective interaction takes the form

$$S_{int} = -\frac{1}{2M_{Pl}^2} \int d^4x \int d^4y \, T'_{\mu\nu}(x) \, G^{\mu\nu\alpha\beta}(x-y; m_g^2) \, T_{\alpha\beta}(y)$$

where the massive propagator is given by

$$G_{\mu\nu}{}^{\alpha\beta}(x-y; m_g^2) = G(x-y; m_g^2) \left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta^{\alpha\beta} \right)$$

Thus we see that the massive propagator does not reduce to the massless in the limit of small mass, indeed

$$\lim_{m_g^2 \to 0} G_{\mu\nu}{}^{\alpha\beta}(x-y; m_g^2) \neq G_{\mu\nu}{}^{\alpha\beta}(x-y) .$$

This is the famous van Dam–Veltman–Zakharov (vDVZ) discontinuity which we now explore phenomenologically.

Newtonian potential

For the static interaction between two sources m, M we have

$$V(\vec{q}) = \frac{mM}{2M_{Pl}^2} G_{0000}(\vec{q}; m_g^2) ,$$

where G_{0000} is now given by

$$\begin{aligned} G_{00}{}^{00}(\vec{q}; m_g^2) &= \left(\delta^{0}_{(0}\delta^{0}_{0)} - \frac{1}{3}\eta_{00}\eta^{00} \right) G(\vec{q}; m_g^2) \\ &= \frac{2}{3} G(\vec{q}; m_g^2) = \frac{4}{3} \cdot \left(\frac{1}{2} G(\vec{q}; m_g^2) \right) \end{aligned}$$

Thus we get that

$$V(\vec{q}) = -\frac{4}{3}\cdot\left(\frac{mM}{16\pi M_{pl}^2}\right)\frac{1}{|\vec{q}|}\,e^{-m_g|\vec{q}|}$$

To get the right Newtonian potential in the limit $m_g^2 \to 0$ we have to define a new Newton's constant

$$\frac{4}{3}\cdot\frac{1}{16\pi M_{pl}^2} \equiv G' \quad \text{compared to} \quad \frac{1}{16\pi M_{pl}^2} \equiv G \quad \text{before.}$$

Light bending

Consider now light bending around a source, i.e. we take $T'_{\mu\nu}$ to describe a lightray. In this case the trace T' is zero since Maxwell's theory is conformally invariant. Thus the anomalous terms $-1/2$ and $-1/3$ that coupled the traces of the stress-energy tensors drop out and the massive and massless theory have the same tensor structure. So that in the massless limit the massive and the massless theory gives the same light bending.

Now, the standard predictions for light bending around the sun predicts a deflection angle

$$\Delta\theta_\odot^{\text{massless}} = \frac{1}{16\pi M_{pl}^2}\frac{4M_\odot}{R_\odot} = \frac{4GM_\odot}{R_\odot}$$

and the massive theory predicts

$$\Delta\theta_\odot^{\text{massive}} = \frac{1}{16\pi M_{pl}^2}\frac{4M_\odot}{R_\odot} = \frac{3}{4}\cdot\frac{4G'M_\odot}{R_\odot}$$

The angle is the same but the prediction in terms of the Newton's constant is different.

Ghost issues for $a \neq 1$

We now return to the case $a \neq 1$ where we have an extra 6th propagating degree of freedom. Of course such a setting would not describe a single massive graviton with 5 dofs but could the theory be healthy anyway? The answer is no! This extra dof is a ghost with negative kinetic energy.

Redoing an interaction energy calculation we find a propagator, see Boulware & Deser

$$G_{\mu\nu}{}^{\alpha\beta}(x-y, m_g^2, m_a^2) = G_{\mu\nu}{}^{\alpha\beta}(x-y; m_g^2)$$
$$- \frac{1}{6} \eta_{\mu\nu} \eta^{\alpha\beta} G(x-y; m_a^2) \,,$$

where the standar FP propagator is

$$G_{\mu\nu}{}^{\alpha\beta}(x-y; m_g^2) = \left(\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta^{\alpha\beta}\right) G(x-y; m_g^2)$$

and the scalar propagator with mass m^2 is

$$G(x-y; m^2) = (\partial^2 - m^2)^{-1} \delta^{(4)}(x-y)$$

The minus sign in front of $\frac{1}{6}\eta_{\mu\nu}\eta^{\alpha\beta}$ is crucial and tells us that this extra scalar degree of freedom is a ghost!

It is intriguing to notice that if we take the limit $m_g^2 \to 0$ with constant $a \neq 1$ then massless gravity is recovered:

$$\lim_{\substack{m_g^2 \to 0 \\ a \neq 1 \\ \text{const}}} G_{\mu\nu}{}^{\alpha\beta}(x-y; m_g^2, m_a^2) = \left(\delta^\alpha_{(\mu}\delta^\beta_{\nu)} - \frac{1}{3}\eta_{\mu\nu}\eta^{\alpha\beta} - \frac{1}{6}\eta_{\mu\nu}\eta^{\alpha\beta} \right) \times G(x-y)$$

$$= \left(\delta^\alpha_{(\mu}\delta^\beta_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta} \right) G(x-y)$$

Negative sign in propagator

A free scalar is governed by the action

$$S = \frac{1}{2}\int d^4x\, \phi(\partial^2 - m^2)\phi \,,$$

and leads to the scalar propagator

$$G(x-y) = (\partial^2 - m^2)\,\delta^{(4)}(x-y) \,.$$

Thus a propagator of a field ϕ'

$$G'(x-y) = -(\partial^2 - m^2)\,\delta^{(4)}(x-y)$$

is governed by an action

$$S' = -\frac{1}{2}\int d^4x\, \phi'(\partial^2 - m^2)\phi' \,,$$

which has the wrong sign kinetic energy $\Rightarrow$ ghost!