Introduction to Unconventional Superconductivity

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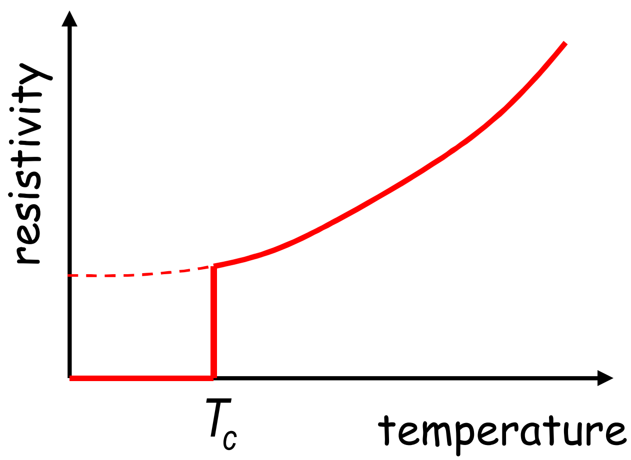
Topics:

1. Conventional superconductivity
2. Overview of unconventional superconductors
3. Group theory and Landau theory
4. Weak Coupling Theory determination of Landau theory
5. Homogeneous group states - weak coupling and beyond
6. Inhomogeneous states – topological defects
7. Recent symmetry-based developments in superconductivity

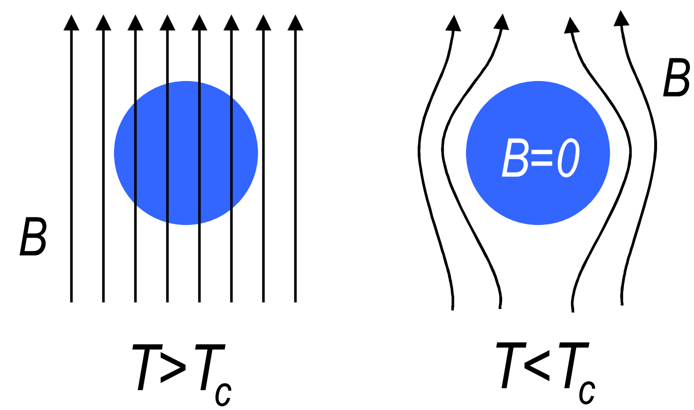
Special thanks to Manfred Sigrist for providing many powerpoint slides

Superconductivity

Electrical resistance (1911)



Field expulsion (1933)
Meissner-Ochsenfeld effect



Meissner effect not a consequence of perfect conductivity

Inside a perfect conductor: E=0

$$\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} = \nabla \times \boldsymbol{E} = 0$$

Symmetry Breaking and Ginzburg-Landau theory

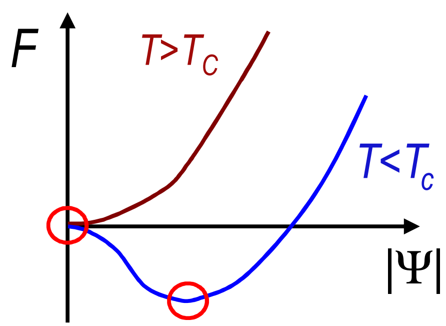
Phase transition with spontaneously broken symmetry : macroscopic wavefunction

Order parameter: Ψ $\begin{cases} = 0 & T > T_c \\ \neq 0 & T < T_c \end{cases}$

$\Psi = \Psi(\vec{r}, T) = |\Psi|\, e^{i\alpha}$

Free energy functional: $$F[\Psi] = \int d^3r \left[a(T)|\Psi|^2 + b|\Psi|^4 \right]$$

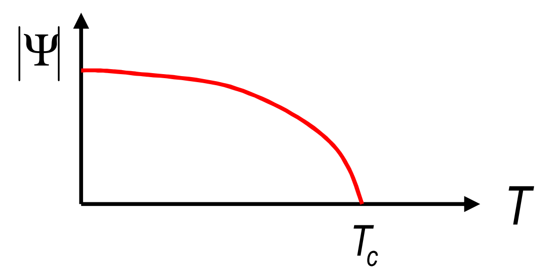
uniform phase: $a(T) = a'(T - T_c)$ $a', b > 0$



$$|\Psi|^2 = \frac{a'(T_c - T)}{2b}$$

fixed phase

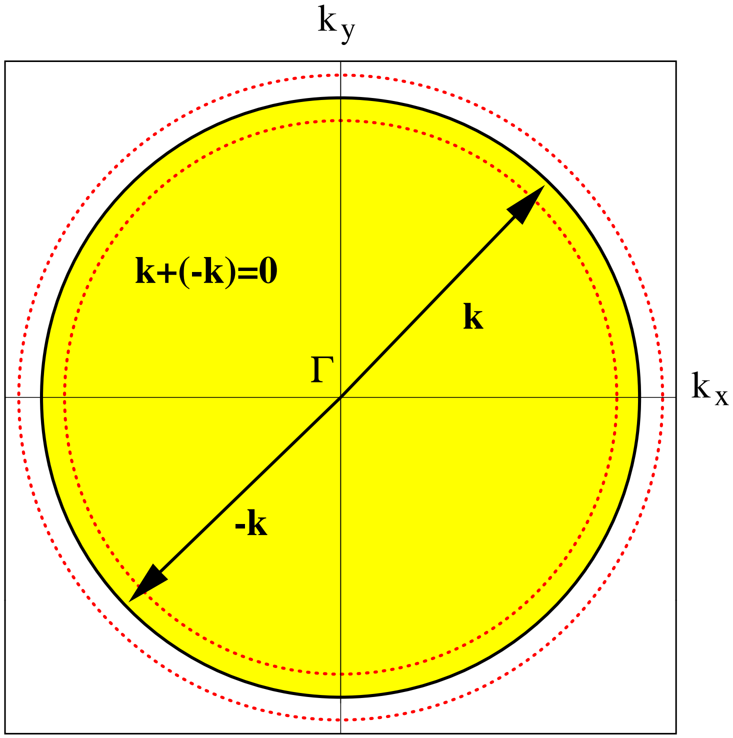
broken *U(1)* gauge symmetry



Magnetic fields: $K\left|\vec{D}\Psi\right|^2 + \frac{1}{8\pi}\left(\vec{\nabla} \times \vec{A}\right)^2$

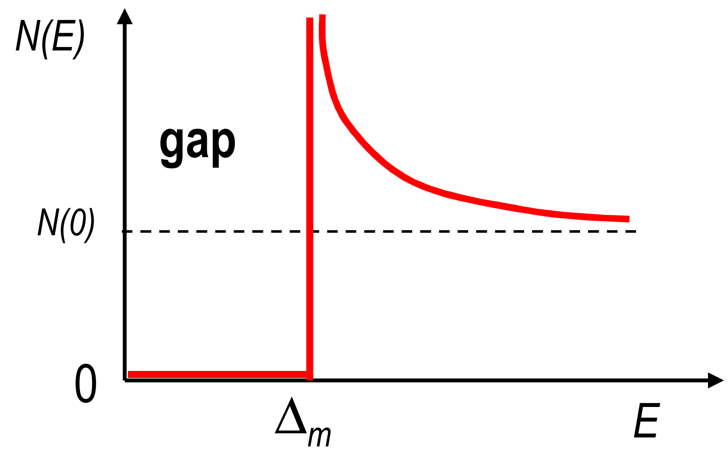
Minimal coupling : $\vec{D} = \vec{\nabla} + i\frac{2e}{\hbar c}\vec{A}$

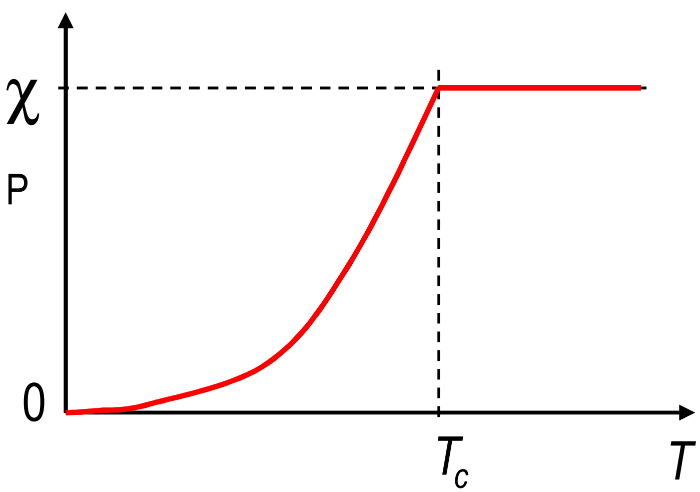
Spin Singlet Cooper Pair and Energy Gap



$$\Delta_m = V \sum_k < c_{-k\uparrow} c_{k\downarrow} >$$

Ψ and Δ are proportional to each other



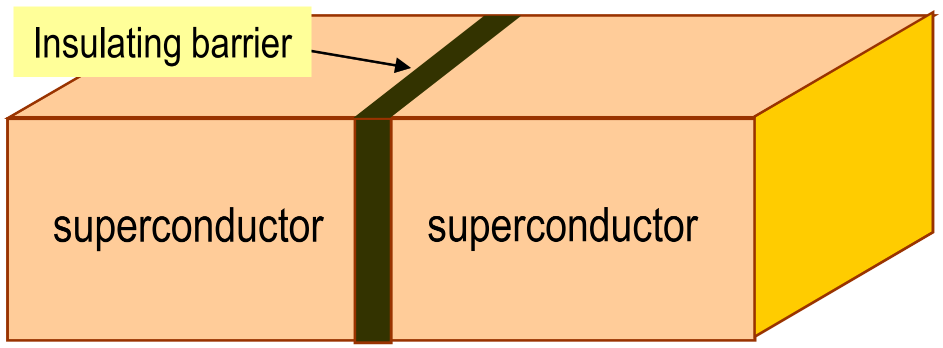


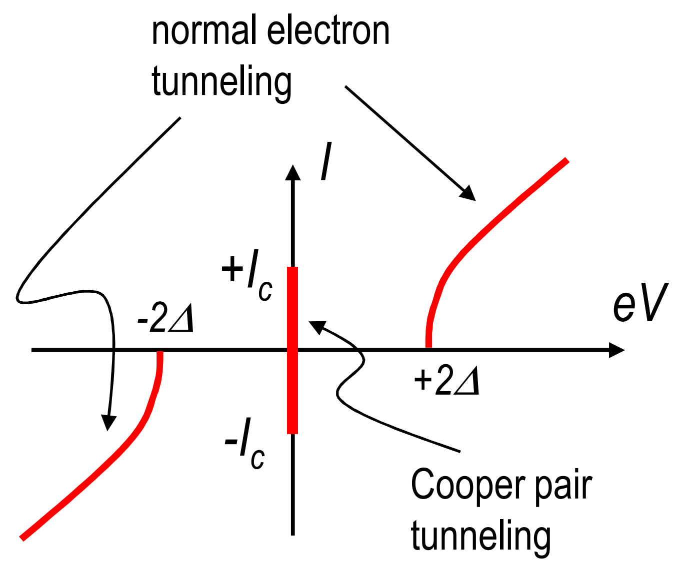
$$C(T) \approx N(0) k_B \left(\frac{\Delta_m}{k_B T}\right)^2 \sqrt{2\pi k_B T \Delta_m} e^{-\Delta_m / k_B T}$$

Specific heat

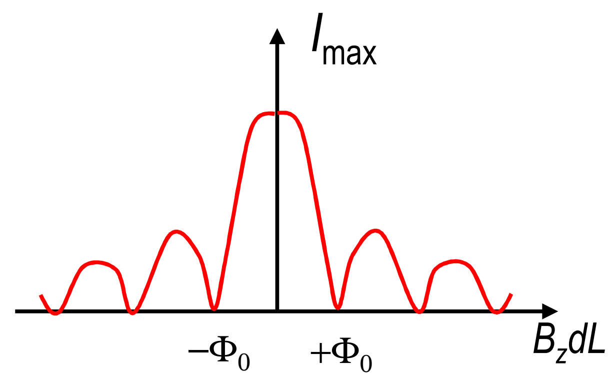
Spin Susceptibility

Josephson Effect and Tunneling

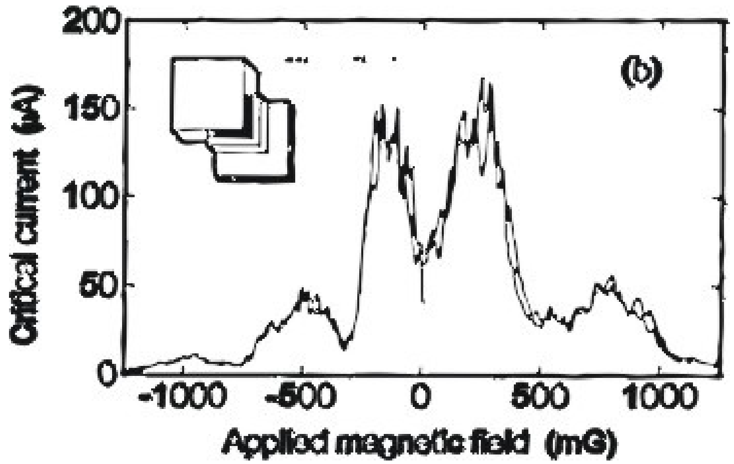


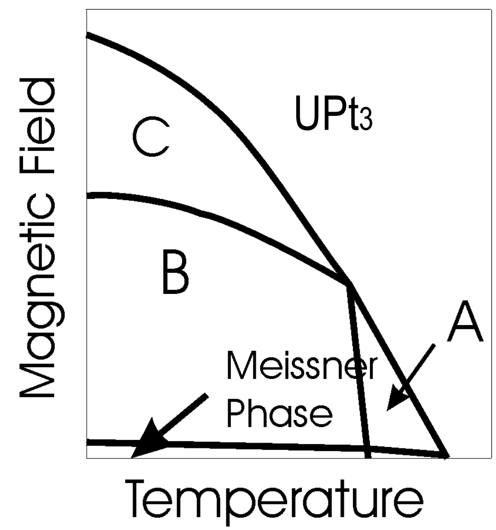


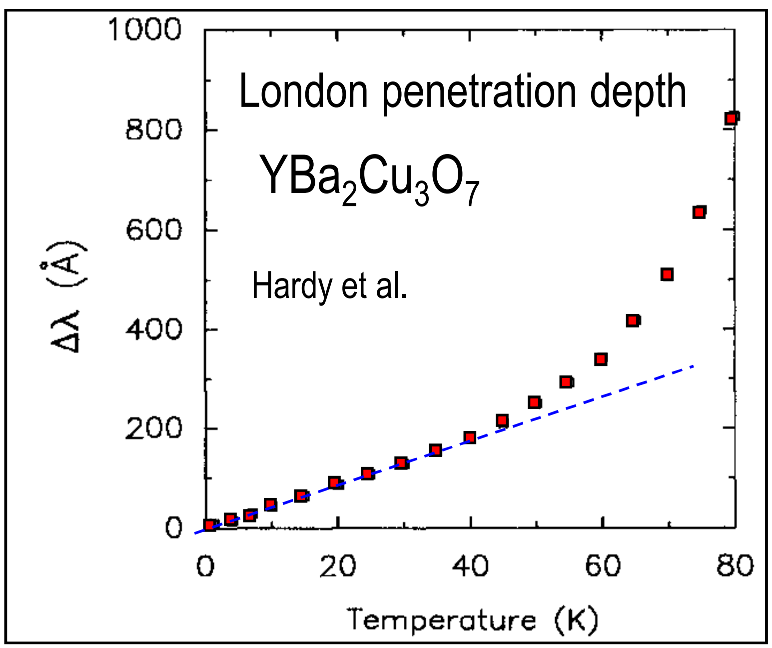
Ic is suppressed by a magnetic field in the junction:

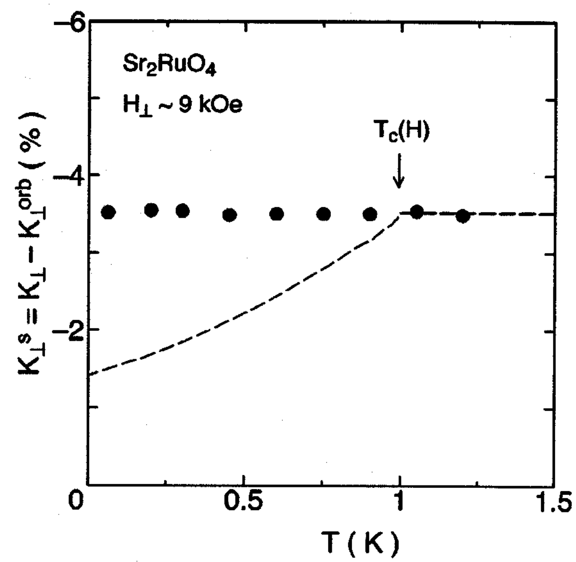


Unconventional behavior in new superconductors







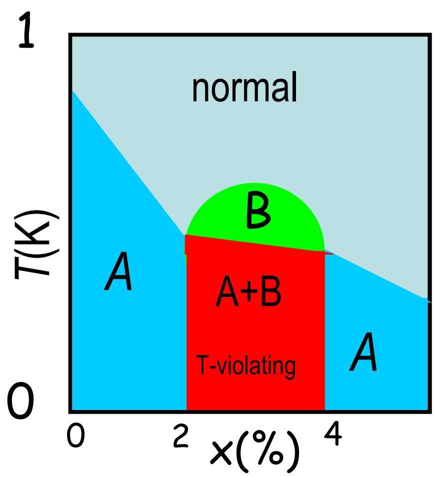


The novel superconductors

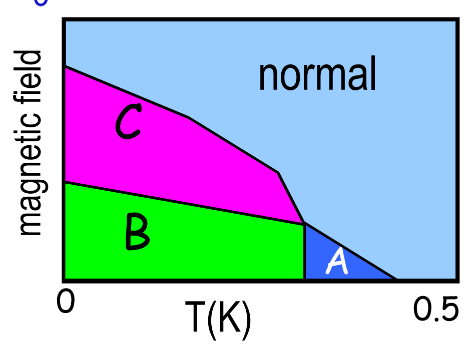
Heavy Fermion superconductors:

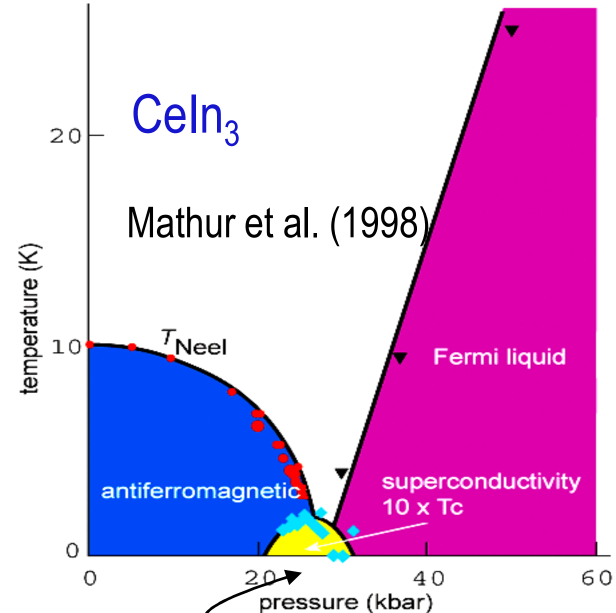
$CeCu_2Si_2$ Steglich et al. (1979)

$U_{1-x}Th_xBe_{13}$ Ott et al. (1983)



UPt_3 Stewart et al. (1984)

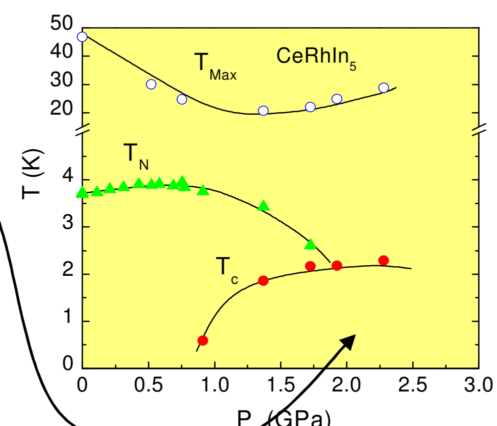




Quantum
Critical point

AF ⟷ PM

$CeRhIn_5$ Thompson et al. (2001)



The novel superconductors

High-temperature superconductors

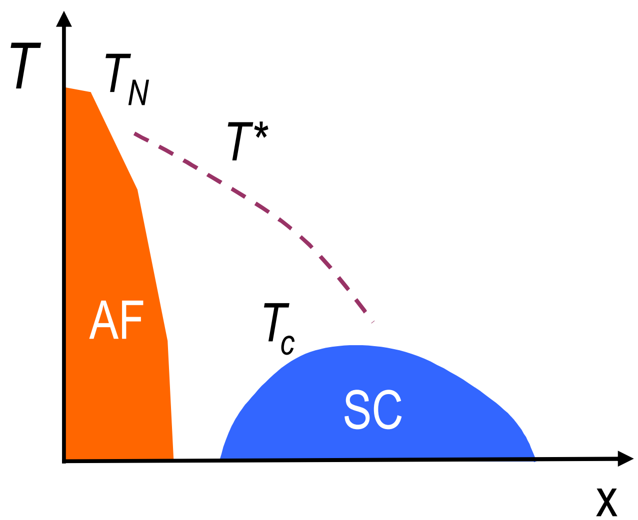
Layered perovskite cooper-oxides

Müller & Bednorz (1986)

$La_{2-x}Sr_xCuO_4$ T_c=45K

$YBa_2Cu_3O_{6+\delta}$ T_c=92K

$HgBa_2Ca_2Cu_3O_9$ T_c=133.5K

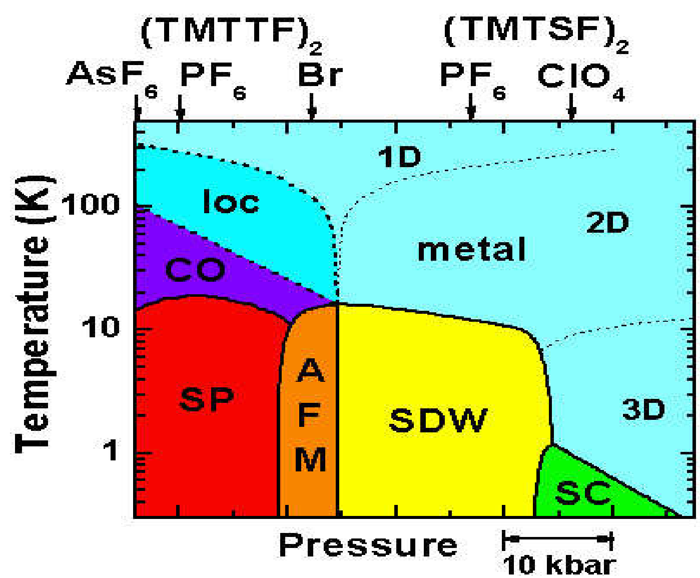


Organic superconductors

Jerome, Bechtgard et al (1980)

$(TMTSF)_2M$ ($M=PF_6$, SbF_6, ReO_4,…) T_c ~ 1K

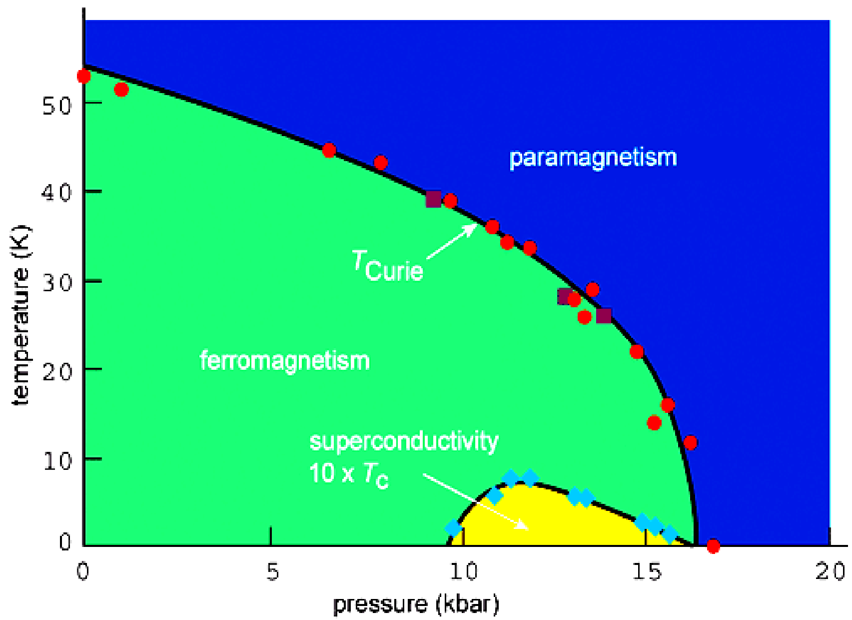
$(BEDT\text{-}TTF)_2M$ ….. T_c ~ 10K



The novel superconductors

Ferromagnetic superconductors:

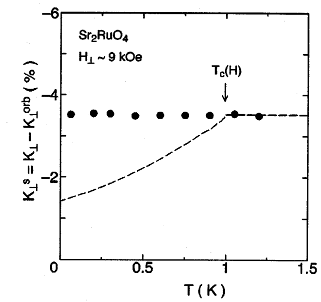
UGe_2 Saxena et a. (2000)

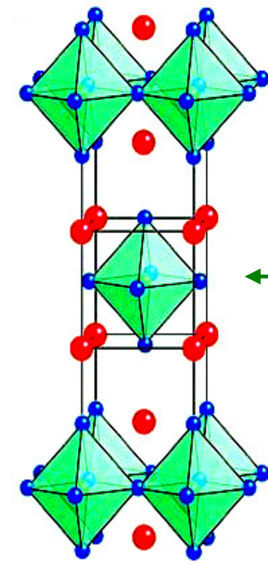


$ZrZn_2$ Pfleiderer et al. (2001)

Superconductivity within
the ferromagnetic phase

Sr_2RuO_4





← RuO_2 plane

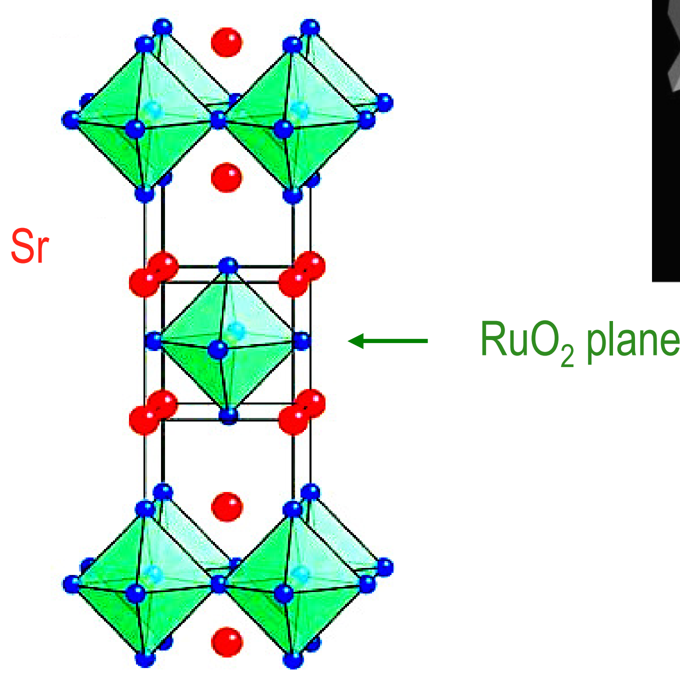
some similarities with
high-T_c superconductors,

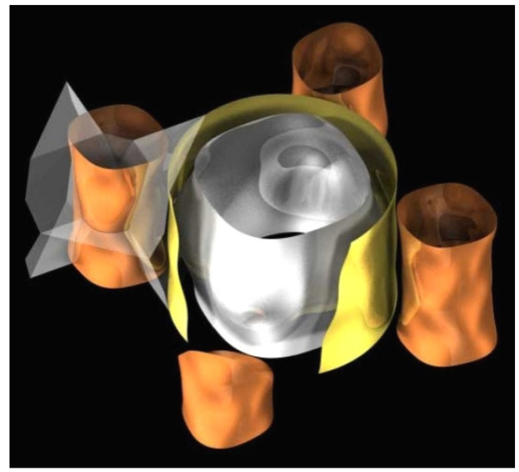
but $T_c = 1.5$ K

spin-triplet superconductor

Y. Maeno, J.G. Bednorz et al. (1994)

Sr_2RuO_4





Bergemann et al. (2000)

de Haas-van Alphen

Three metallic electron bands

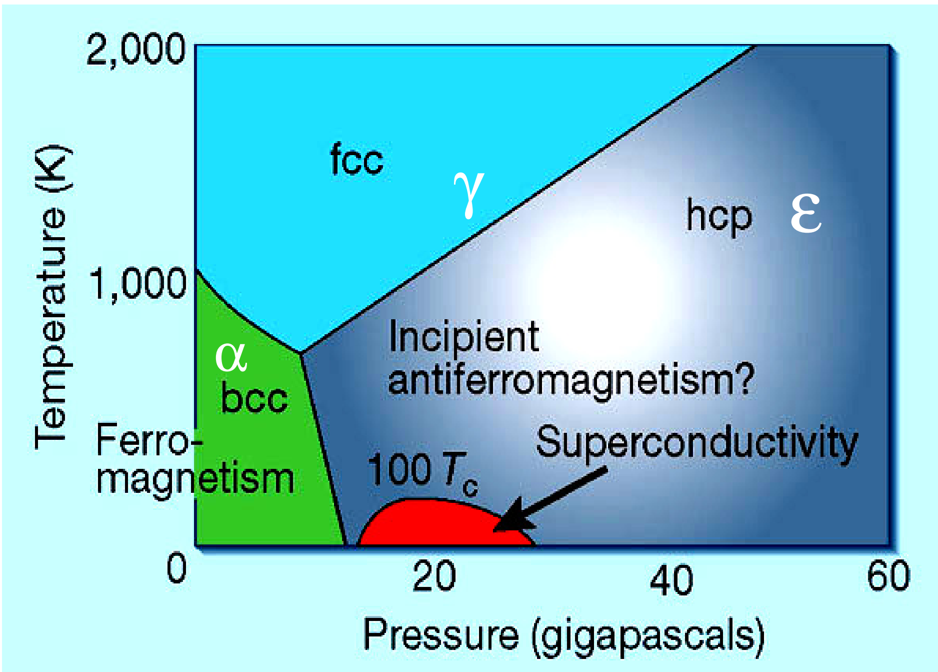
Two-dimensional Fermi liquid

Superconductivity

$T_c = 1.5$ K

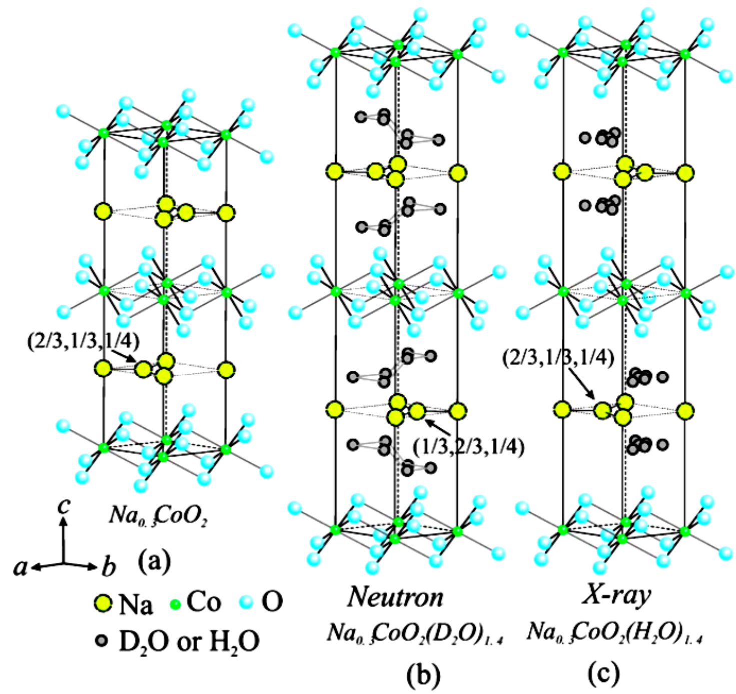
The novel superconductors - under extreme conditions

Iron under pressure



Shimizu et al. Nature 412, 316 (2001)

Hydrated Na_xCoO_4



Layered structure: triangular

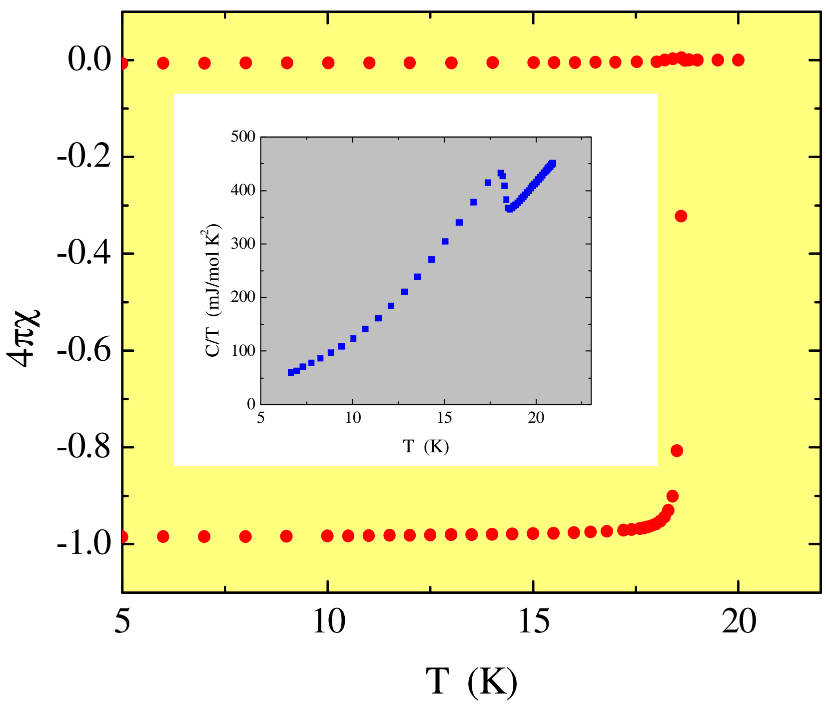
Superconductivity in a frustrated electron system

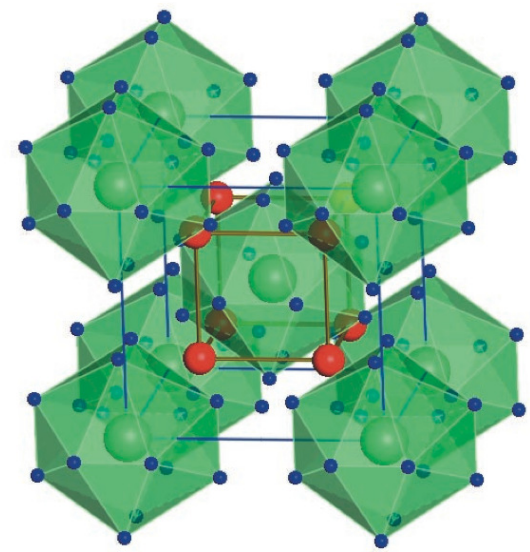
T_c ~ 5 K

Takada et al., Nature 422, 53 (2003)

The novel superconductors

Skutterudite





$PuCoGa_5$ T_c = 18 K

Thompson et al. (Los Alamos)

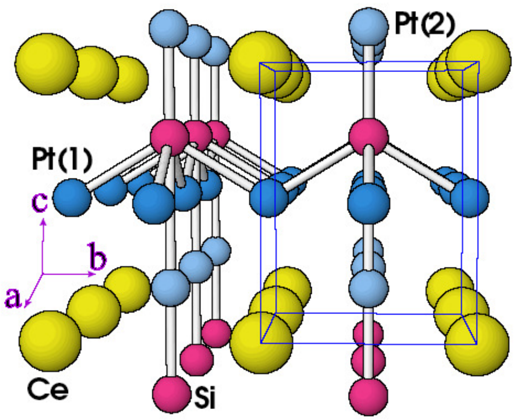
$PrOs_4Sb_{12}$ T_c = 1.8 K

Bauer et al. PRB 65, R100506 (2002)

Multiple phases

The novel superconductors - no inversion symmetry

No paramagnetic limiting



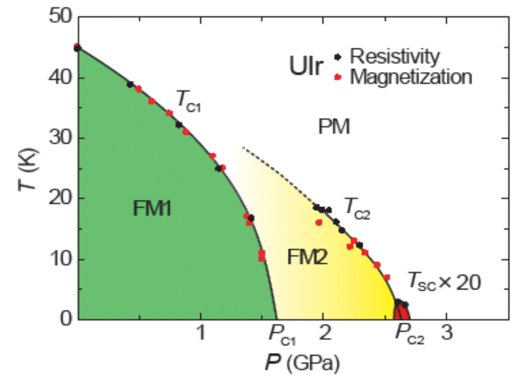
$CePt_3Si$

T_c=0.8 K

H_{c2} exceeds drastically
the paramagnetic limit

Bauer et al. PRL 92, 027003 (2004)

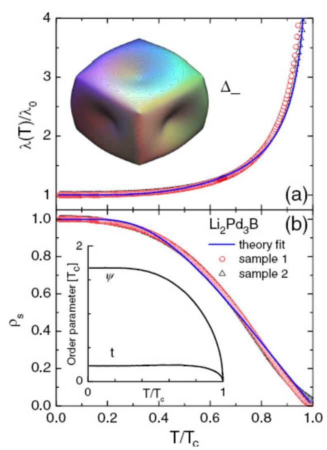
Ferromagnetic quantum phase transition

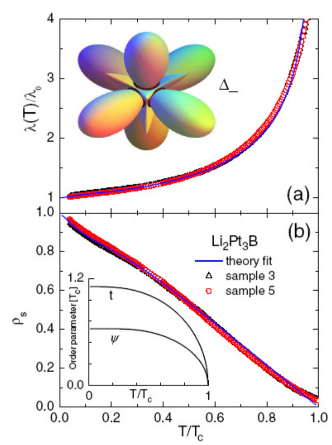


T_c=0.15 K

UIr

Akazawa et al. J.Phys. Condens. Matter 16, L29 (2004)



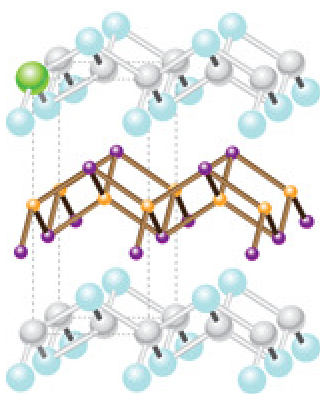


Li_2Pt_3B Li_2Pd_3B

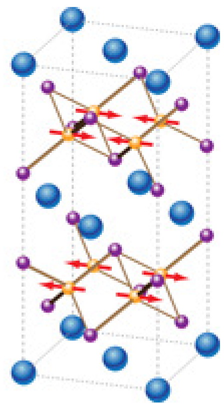
Yuan et al.
PRL **97**, 017006 (2006)

Interface superconductors (LaAlO3 on SrTiO3) Caviglia et al ; Ionic liquid/ solid interface SC (ZrNCl) Ye et al

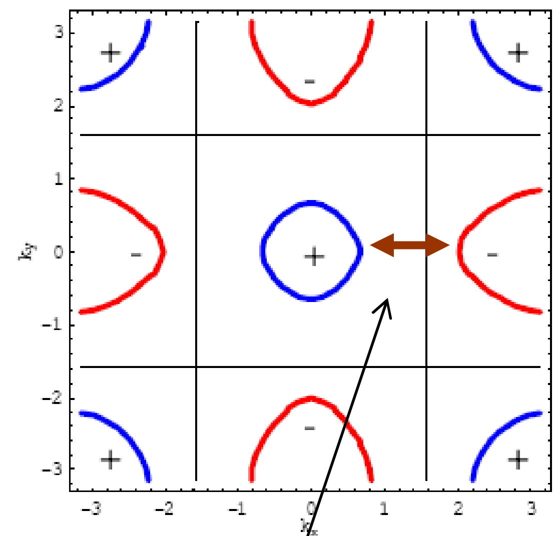
The novel superconductors – FeAs

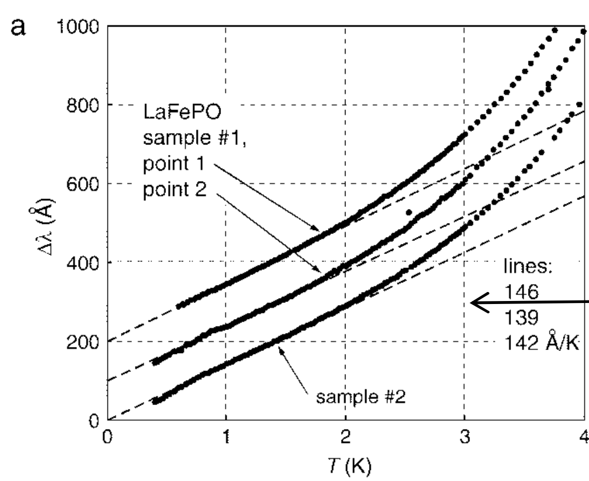


LaOFeAs



BaFe2As2





Most materials likely to have +/- gap
Some materials appear to have nodes

Landau Theory and Group Theory

Basis of order parameters

Landau: order parameters belong to irreducible representations of the normal state symmetry group

$$\psi(\vec{k}) = \sum_m \eta_m \psi_m(\vec{k})$$

$\{\psi_1(\vec{k})\psi_2(\vec{k})\ldots\}$ basis set of irred. rep.

Set up a free energy functional as a scalar function of η_m { transform according to the representation

$$F[\eta_m] = \int d^3r[a\sum_m |\eta_m|^2 + \sum_{m_1,\ldots m_4} b_{m_1,\ldots m_4} \eta^*_{m_1}\eta^*_{m_2}\eta_{m_3}\eta_{m_4}$$

Each representation has a different Tc!

invariant under all symmetry operations of rotations, time reversal and *U(1)*-gauge

$a = a'(T - T_c)$, $\boldsymbol{b}_{\boldsymbol{m}1,\boldsymbol{m}2,\boldsymbol{m}3,\boldsymbol{m}4}$ real constant

Overview of Group Theory:

A group is a set of elements {a,b,c,..} with a multiplication rule that assigns to each ordered pair a,b of G, another element ab.

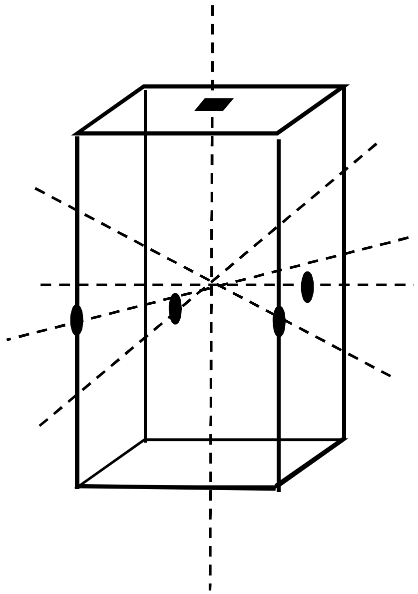
G1: a,b,c in G then a(bc)=(ab)c (composition)
G2: ae=ea=a (identity)
G3: $a\, a^{-1} = a^{-1}a = e$ (inverse)

Specific example:

Superconductor with tetragonal crystal structure

Example of a tetragonal crystal with spin orbit coupling

Point group: D_{4h}



D_{4h} contains inversion

→ even and odd representations

4 one-dim., 1 two-dim. representation

Character table for D_4

Γ	E	C_2	$2C_4$	$2C_2'$	$2C_2''$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
B_1	1	1	-1	1	-1
B_2	1	1	-1	-1	1
E	2	-2	0	0	0

Ginzburg-Landau free energy functionals:

1-dimensional representations:

$$F[\Psi] = \int d^3r \left[a(T)|\Psi|^2 + b|\Psi|^4 \right]$$

like conventional SC

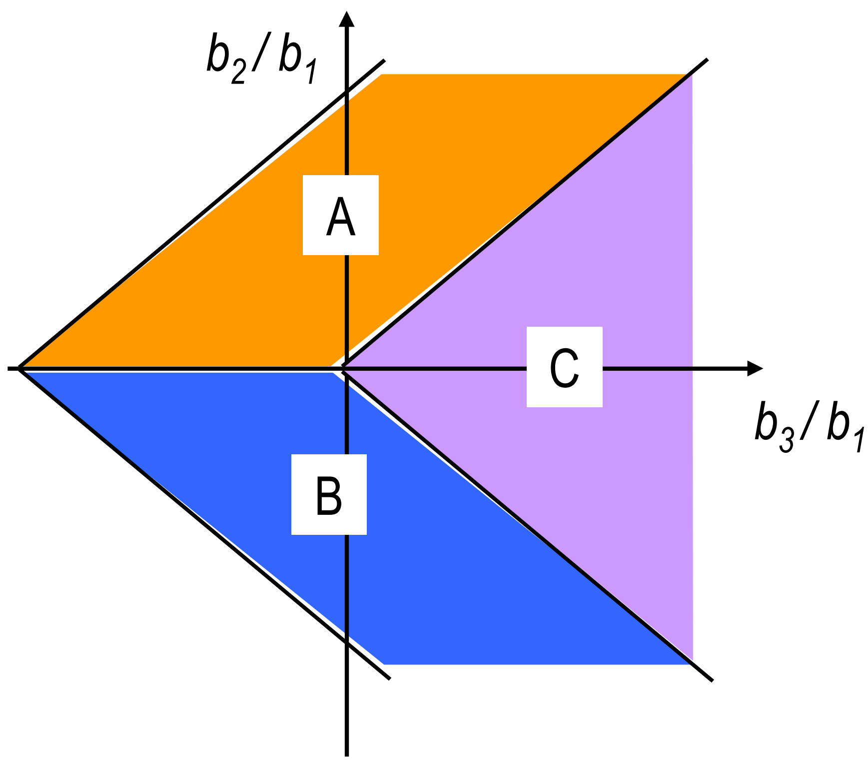
2-dimensional representations:

$$F[\vec{\eta}] = \int d^3r \left[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\right\} + b_3|\eta_x|^2|\eta_y|^2 \right]$$

Possible homogeneous superconducting phases

Higher-dimensional order parameters are interesting: $\vec{\eta} = (\eta_x, \eta_y)$

$$F[\vec{\eta}] = \int d^3r \left[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\right\} + b_3|\eta_x|^2|\eta_y|^2 \right]$$



phase		broken symmetry
A	(1 , i)	$U(1), \mathcal{K}$
B	(1 , 1)	$U(1), D_{4h} \rightarrow D_{2h}$
C	(1 , 0)	$U(1), D_{4h} \rightarrow D_{2h}$

$\mathcal{K}$ → magnetism

$D_{4h} \rightarrow D_{2h}$ → crystal deformation

Degeneracy: 2

domain formation possible

Generalized BCS theory: Microscopic calculation of symmetry properties and gap functions

Generalized formulation of the BCS mean field theory

BCS Hamiltonian:

$$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s} + \frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} c^{\dagger}_{\vec{k}s_1} c^{\dagger}_{-\vec{k}s_2} c_{-\vec{k}'s_3} c_{\vec{k}'s_4}$$

Mean field Hamiltonian:

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s} - \frac{1}{2} \sum_{\vec{k},s_1,s_2} \left[\Delta_{\vec{k},s_1 s_2} c^{\dagger}_{\vec{k}s_1} c^{\dagger}_{-\vec{k}s_2} + \Delta^{*}_{\vec{k},s_1 s_2} c_{\vec{k}s_1} c_{-\vec{k}s_2} \right]$$

$$-\frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} \langle c^{\dagger}_{\vec{k}s_1} c^{\dagger}_{-\vec{k}s_2} \rangle \langle c_{-\vec{k}'s_3} c_{\vec{k}'s_4} \rangle$$

Self-consistence equations:

$$\Delta_{\vec{k},ss'} = - \sum_{\vec{k}',s_3 s_4} V_{\vec{k},\vec{k}';ss's_3 s_4} \langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4} \rangle$$

$$\Delta^{*}_{\vec{k},ss'} = - \sum_{\vec{k}'s_1 s_2} V_{\vec{k}',\vec{k};s_1 s_2 s' s} \langle c^{\dagger}_{\vec{k}'s_1} c^{\dagger}_{-\vec{k}'s_2} \rangle$$

gap: 2x2-matrix

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k}\uparrow\uparrow} & \Delta_{\vec{k}\uparrow\downarrow} \\ \Delta_{\vec{k}\downarrow\uparrow} & \Delta_{\vec{k}\downarrow\downarrow} \end{pmatrix}$$

Structure of the gap function

$$\Delta_{\vec{k},ss'} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4}\langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4}\rangle$$

$$\Delta^*_{\vec{k},ss'} = -\sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's}\langle c^\dagger_{\vec{k}'s_1} c^\dagger_{-\vec{k}'s_2}\rangle$$

Gap function: 2x2 matrix in spin space

Even parity spin singlet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k},\uparrow\uparrow} & \Delta_{\vec{k},\uparrow\downarrow} \\ \Delta_{\vec{k},\downarrow\uparrow} & \Delta_{\vec{k},\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} 0 & \psi(\vec{k}) \\ -\psi(\vec{k}) & 0 \end{pmatrix} = i\hat{\sigma}_y \psi(\vec{k})$$

represented by scalar function $\psi(\vec{k}) = \psi(-\vec{k})$ even

Odd parity spin triplet

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} -d_x(\vec{k}) + i d_y(\vec{k}) & d_z(\vec{k}) \\ d_z(\vec{k}) & d_x(\vec{k}) + i d_y(\vec{k}) \end{pmatrix} = i\left(\vec{d}(\vec{k})\cdot\hat{\vec{\sigma}}\right)\hat{\sigma}_y$$

represented by vector function $\vec{d}(\vec{k}) = -\vec{d}(-\vec{k})$ odd

Classification of gap functions

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^{\dagger}_{\vec{k}s} c_{\vec{k}s} - \frac{1}{2} \sum_{\vec{k},s_1,s_2} \left[\Delta_{\vec{k},s_1 s_2} c^{\dagger}_{\vec{k}s_1} c^{\dagger}_{-\vec{k}s_2} + \Delta^{*}_{\vec{k},s_1 s_2} c_{\vec{k}s_1} c_{-\vec{k}s_2} \right]$$

$$-\frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} \langle c^{\dagger}_{\vec{k}s_1} c^{\dagger}_{-\vec{k}s_2} \rangle \langle c_{-\vec{k}'s_3} c_{\vec{k}'s_4} \rangle$$

For a symmetry g of H: $$H_{mf} = g^{+} H_{mf}\, g$$

We know how the operators c transform under g and thus can deduce how the gap function transforms

Symmetry operations

Symmetries of normal phase: $G = \underbrace{G_o}_{\text{orbital rotation}} \times \underbrace{G_s}_{\text{spin rotation}} \times \underbrace{K}_{\text{time reversal}} \times \underbrace{U(1)}_{\text{gauge}}$

symmetry operation	
orbital rotation	$g c^{+}_{\vec{k}s} = c^{+}_{\hat{R}_o \vec{k}s}$ $\hat{R}_o$ orbital rotation
spin rotation	$g c^{+}_{\vec{k}s} = \sum_{s'} D_{ss'} c^{+}_{\vec{k}s'}$ $\hat{D} = e^{i\vec{\theta}\cdot\hat{\vec{\sigma}}/2}$
time reversal (antiunitary)	$\hat{K} c_{\vec{k}s} = \sum_{s'} (-i\hat{\sigma}_y)_{ss'} c_{-\vec{k}s'}$
U(1) gauge	$\hat{\Phi} c^{+}_{\vec{k}s} = e^{i\phi/2} c^{+}_{\vec{k}s}$

presence of strong spin-orbit coupling → spin and lattice rotation go together

Symmetry operations

Symmetries of normal phase: $G = G_o \times G_s \times K \times U(1)$

G_o: orbital rotation; G_s: spin rotation; K: time reversal; $U(1)$: gauge

Parity: $\psi(\vec{k}) = \psi(-\vec{k}) \quad \vec{d}(-\vec{k}) = -\vec{d}(\vec{k})$

symmetry operation	spin singlet	spin triplet
orbital rotation	$g_o\,\psi(\vec{k}) = \psi(\hat{R}_o\vec{k})$	$g_o\,\vec{d}(\vec{k}) = \vec{d}(\hat{R}_o\vec{k})$
spin rotation	$g_s\,\psi(\vec{k}) = \psi(\vec{k})$	$g_s\,\vec{d}(\vec{k}) = \hat{R}_s\vec{d}(\vec{k})$
time reversal	$\hat{K}\psi(\vec{k}) = \psi^*(\vec{k})$	$\hat{K}\,\vec{d}(\vec{k}) = \vec{d}^*(\vec{k})$
U(1) gauge	$\Phi\,\psi(\vec{k}) = e^{i\phi}\psi(\vec{k})$	$\Phi\,\vec{d}(\vec{k}) = e^{i\phi}\vec{d}(\vec{k})$

presence of strong spin-orbit coupling → spin and lattice rotation go together

Spin triplet pairing: $g\vec{d}(\vec{k}) = \hat{R}_s\vec{d}(\hat{R}_o\vec{k})$

identical 3D rotations $\begin{cases} \hat{R}_o \\ \hat{R}_s \end{cases}$

Example of a tetragonal crystal with spin orbit coupling

Point group: D_{4h}

4 one-dim., 1 two-dim. representation
even (g) / odd (u) parity

Γ	$\psi(\vec{k})$	Γ	$\vec{d}(\vec{k})$
A_{1g}	1	A_{1u}	$\hat{x}k_x + \hat{y}k_y$
A_{2g}	$k_x k_y (k_x^2 - k_y^2)$	A_{2u}	$\hat{y}k_x - \hat{x}k_y$
B_{1g}	$k_x^2 - k_y^2$	B_{1u}	$\hat{x}k_x - \hat{y}k_y$
B_{2g}	$k_x k_y$	B_{2u}	$\hat{y}k_x + \hat{x}k_y$
E_g	$\{k_x k_z, k_y k_z\}$	E_u	$\{\hat{z}k_x, \hat{z}k_y\}$ $\{\hat{x}k_z, \hat{y}k_z\}$

Conventional: A_{1g}

Unconventional: everything else

only one representation is relevant for the superconducting phase transition

Generalized BCS theory: Microscopic calculation of the Landau Energy

Generalized formulation of the BCS mean field theory

BCS Hamiltonian:

$$\mathcal{H} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} + \frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} c_{-\vec{k}'s_3} c_{\vec{k}'s_4}$$

Mean field Hamiltonian:

$$\mathcal{H}_{mf} = \sum_{\vec{k},s} \xi_{\vec{k}} c^\dagger_{\vec{k}s} c_{\vec{k}s} - \frac{1}{2} \sum_{\vec{k},s_1,s_2} \left[\Delta_{\vec{k},s_1 s_2} c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} + \Delta^*_{\vec{k},s_1 s_2} c_{\vec{k}s_1} c_{-\vec{k}s_2} \right]$$

$$-\frac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{s_1,s_2,s_3,s_4} V_{\vec{k},\vec{k}';s_1 s_2 s_3 s_4} \langle c^\dagger_{\vec{k}s_1} c^\dagger_{-\vec{k}s_2} \rangle \langle c_{-\vec{k}'s_3} c_{\vec{k}'s_4} \rangle$$

Self-consistence equations:

$$\Delta_{\vec{k},ss'} = - \sum_{\vec{k}',s_3 s_4} V_{\vec{k},\vec{k}';ss's_3 s_4} \langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4} \rangle$$

$$\Delta^*_{\vec{k},ss'} = - \sum_{\vec{k}'s_1 s_2} V_{\vec{k}',\vec{k};s_1 s_2 s's} \langle c^\dagger_{\vec{k}'s_1} c^\dagger_{-\vec{k}'s_2} \rangle$$

gap: 2x2-matrix

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k}\uparrow\uparrow} & \Delta_{\vec{k}\uparrow\downarrow} \\ \Delta_{\vec{k}\downarrow\uparrow} & \Delta_{\vec{k}\downarrow\downarrow} \end{pmatrix}$$

Generalized BCS theory

1 Bogolyubov transformation:

Mean field Hamiltonian: $H_{mf} = \sum_{\vec{k}} C_{\vec{k}}^{+} \hat{X}_{\vec{k}} C_{\vec{k}} + K$ $\qquad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

with $C_{\vec{k}} = \begin{pmatrix} c_{\vec{k}\uparrow} \\ c_{\vec{k}\downarrow} \\ c_{-\vec{k}\uparrow}^{+} \\ c_{-\vec{k}\downarrow}^{+} \end{pmatrix}$ and $\hat{X}_{\vec{k}} = \frac{1}{2} \begin{pmatrix} \xi_{\vec{k}} \hat{\sigma}_0 & \hat{\Delta}_{\vec{k}} \\ \hat{\Delta}_{\vec{k}}^{+} & -\xi_{\vec{k}} \hat{\sigma}_0 \end{pmatrix}$ assumption: *unitary* $\hat{\Delta}_{\vec{k}}^{+} \hat{\Delta}_{\vec{k}} = \left| \Delta_{\vec{k}} \right|^2 \hat{\sigma}_0$

Show: Unitary Bogolyubov transformation $\hat{U}_{\vec{k}} = \begin{pmatrix} \hat{u}_{\vec{k}} & \hat{v}_{\vec{k}} \\ \hat{v}_{-\vec{k}}^{*} & \hat{u}_{-\vec{k}}^{*} \end{pmatrix}$, $\hat{U}_{\vec{k}}^{+} \hat{U}_{\vec{k}} = \hat{1}$ ➡ $A_{\vec{k}} = \hat{U}_{\vec{k}}^{+} C_{\vec{k}}$

$$H_{mf} = \sum_{\vec{k}} A_{\vec{k}}^{+} \hat{E}_{\vec{k}} A_{\vec{k}} + K$$

$$\hat{E}_{\vec{k}} = \frac{1}{2} \begin{pmatrix} E_{\vec{k}} \hat{\sigma}_0 & 0 \\ 0 & -E_{\vec{k}} \hat{\sigma}_0 \end{pmatrix}$$

$$\hat{u}_{\vec{k}} = \frac{(E_{\vec{k}} + \xi_{\vec{k}}) \hat{\sigma}_0}{\left\{ 2E_{\vec{k}} (E_{\vec{k}} + \xi_{\vec{k}}) \right\}^{1/2}}, \quad \hat{v}_{\vec{k}} = \frac{-\hat{\Delta}_{\vec{k}}}{\left\{ 2E_{\vec{k}} (E_{\vec{k}} + \xi_{\vec{k}}) \right\}^{1/2}}$$

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + \left| \Delta_{\vec{k}} \right|^2}$$

Self-consistent gap equation

Bogolyubov transformation → Quasiparticle spectrum

$$E_{\vec{k}} = \sqrt{\xi_{\vec{k}}^2 + |\Delta_{\vec{k}}|^2} \qquad |\Delta_{\vec{k}}|^2 = \frac{1}{2}\mathrm{tr}\left(\hat{\Delta}_{\vec{k}}^{\dagger}\hat{\Delta}_{\vec{k}}\right)$$

Note: quasiparticle gap is ***k***-dependent

$$\hat{\Delta}_{\vec{k}} = \begin{pmatrix} \Delta_{\vec{k}\uparrow\uparrow} & \Delta_{\vec{k}\uparrow\downarrow} \\ \Delta_{\vec{k}\downarrow\uparrow} & \Delta_{\vec{k}\downarrow\downarrow} \end{pmatrix}$$

Self-consistence equation:

$$\Delta_{\vec{k},ss'} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';ss's_3s_4}\langle c_{\vec{k}'s_3} c_{-\vec{k}'s_4}\rangle$$

$$\Delta^*_{\vec{k},ss'} = -\sum_{\vec{k}'s_1s_2} V_{\vec{k}',\vec{k};s_1s_2s's}\langle c^{\dagger}_{\vec{k}'s_1} c^{\dagger}_{-\vec{k}'s_2}\rangle$$

$$\Delta_{\vec{k},s_1s_2} = -\sum_{\vec{k}',s_3s_4} V_{\vec{k},\vec{k}';s_1s_2s_3s_4}\frac{\Delta_{\vec{k}',s_4s_3}}{2E_{\vec{k}}}\tanh\left(\frac{E_{\vec{k}}}{2k_BT}\right)$$

Transition temperature

Pairing interaction: $V_{\vec{k},\vec{k}';s_1s_2s_3s_4} = J^0_{\vec{k},\vec{k}'}\hat{\sigma}^0_{s_1s_4}\hat{\sigma}^0_{s_2s_3} + J_{\vec{k},\vec{k}'}\hat{\vec{\sigma}}_{s_1s_4}\cdot\hat{\vec{\sigma}}_{s_2s_3}$

density-density　　spin-spin

Self-consistence equation:

even parity spin singlet

$$\psi(\vec{k}) = -\sum_{\vec{k}'}\underbrace{(J^0_{\vec{k},\vec{k}'} - 3J_{\vec{k},\vec{k}'})}_{=v^s_{\vec{k},\vec{k}'}}\frac{\psi(\vec{k}')}{2E_{\vec{k}'}}\tanh\left(\frac{E_{\vec{k}'}}{2k_BT}\right)$$

$T \to T_c$

$$-\lambda\psi(\vec{k}) = -N(0)\langle v^s_{\vec{k},\vec{k}'}\psi(\vec{k}')\rangle_{\vec{k}',FS}$$

odd parity spin triplet

$$\vec{d}(\vec{k}) = -\sum_{\vec{k}'}\underbrace{(J^0_{\vec{k},\vec{k}'} + J_{\vec{k},\vec{k}'})}_{=v^t_{\vec{k},\vec{k}'}}\frac{\vec{d}(\vec{k}')}{2E_{\vec{k}'}}\tanh\left(\frac{E_{\vec{k}'}}{2k_BT}\right)$$

$T \to T_c$

$$-\lambda\vec{d}(\vec{k}) = -N(0)\langle v^t_{\vec{k},\vec{k}'}\vec{d}(\vec{k}')\rangle_{\vec{k}',FS}$$

eigenvalue λ ➡ $k_BT_c = 1.14\epsilon_c e^{-1/\lambda}$

From self-consistent gap equation to free energy

$$\psi(\vec{k}) = -\sum_{\vec{k}'} \underbrace{(J^0_{\vec{k},\vec{k}'} - 3J_{\vec{k},\vec{k}'})}_{= v^s_{\vec{k},\vec{k}'}} \frac{\psi(\vec{k}')}{2E_{\vec{k}'}} \tanh\left(\frac{E_{\vec{k}'}}{2k_B T}\right)$$

Go from the above gap equation to $\dfrac{\partial F}{\partial \eta_i^*}$ with

$$F[\vec{\eta}] = \int d^3r \left[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\} + b_3|\eta_x|^2|\eta_y|^2 \right]$$

Sr_2RuO_4 example:

$$\vec{d}(k) = \hat{z}[\eta_x f_x(k) + \eta_y f_y(k)]$$

$$\boldsymbol{F}[\vec{\eta}] = \int \boldsymbol{d}^3\boldsymbol{r}\left[\boldsymbol{a}|\vec{\eta}|^2 + \boldsymbol{b}_1|\vec{\eta}|^4 + \frac{\boldsymbol{b}_2}{2}\left\{\eta_{\boldsymbol{x}}^{*2}\eta_{\boldsymbol{y}}^2 + \eta_{\boldsymbol{x}}^2\eta_{\boldsymbol{y}}^{*2}\right\} + \boldsymbol{b}_3|\eta_{\boldsymbol{x}}|^2|\eta_{\boldsymbol{y}}|^2\right]$$

$$\beta_2 / \beta_1 = \gamma$$

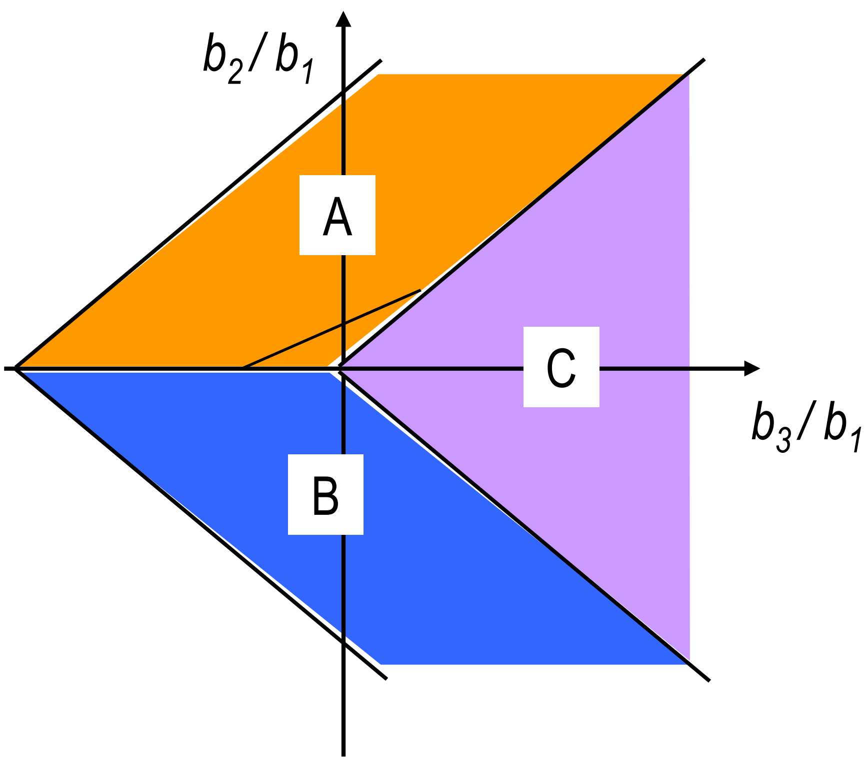
$$\beta_3 / \beta_1 = 2\gamma - 1$$

$$\gamma = \frac{< f_x^2 f_y^2 >}{< f_x^4 >}$$

Possible homogeneous superconducting phases

Higher-dimensional order parameters are interesting: $\vec{\eta} = (\eta_x, \eta_y)$

$$F[\vec{\eta}] = \int d^3r \left[a|\vec{\eta}|^2 + b_1|\vec{\eta}|^4 + \frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2 + \eta_x^2\eta_y^{*2}\right\} + b_3|\eta_x|^2|\eta_y|^2 \right]$$



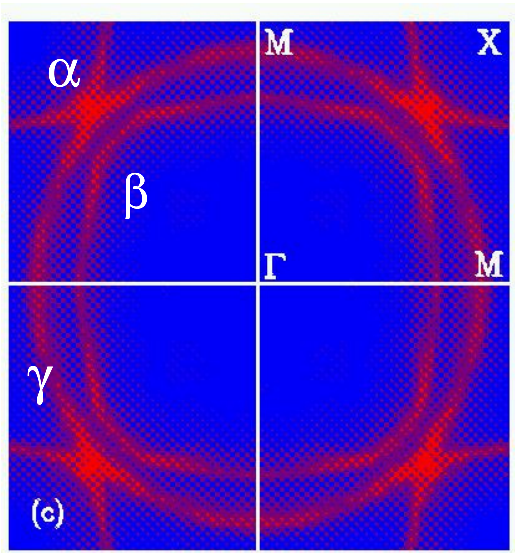
phase	$\psi(\vec{k})$	$\vec{d}(\vec{k})$	broken symmetry
A	$(k_x \pm ik_y)k_z$	$\hat{z}(k_x \pm ik_y)$	$U(1), \mathcal{K}$
B	$(k_x \pm k_y)k_z$	$\hat{z}(k_x \pm k_y)$	$U(1), D_{4h} \to D_{2h}$
C	k_xk_z, k_yk_z	$\hat{z}k_x, \hat{z}k_y$	$U(1), D_{4h} \to D_{2h}$

Degeneracy: 2

domain formation possible

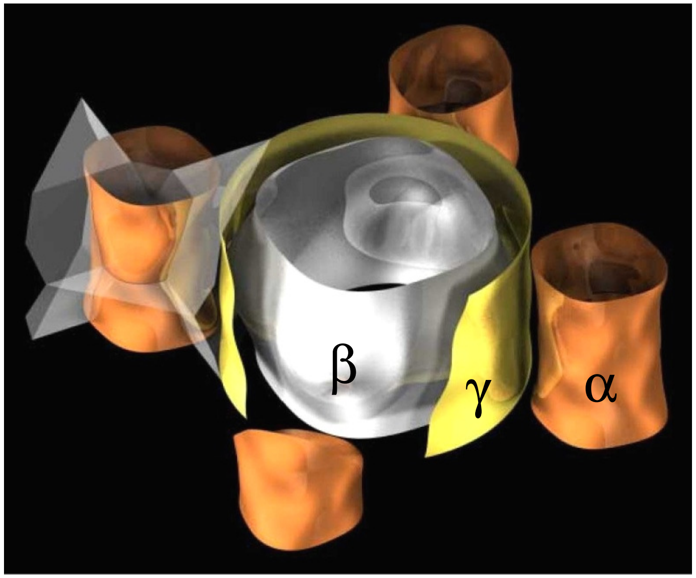
Fermi surfaces of Sr_2RuO_4

ARPES



Damascelli et al.

de Haas-van Alphen

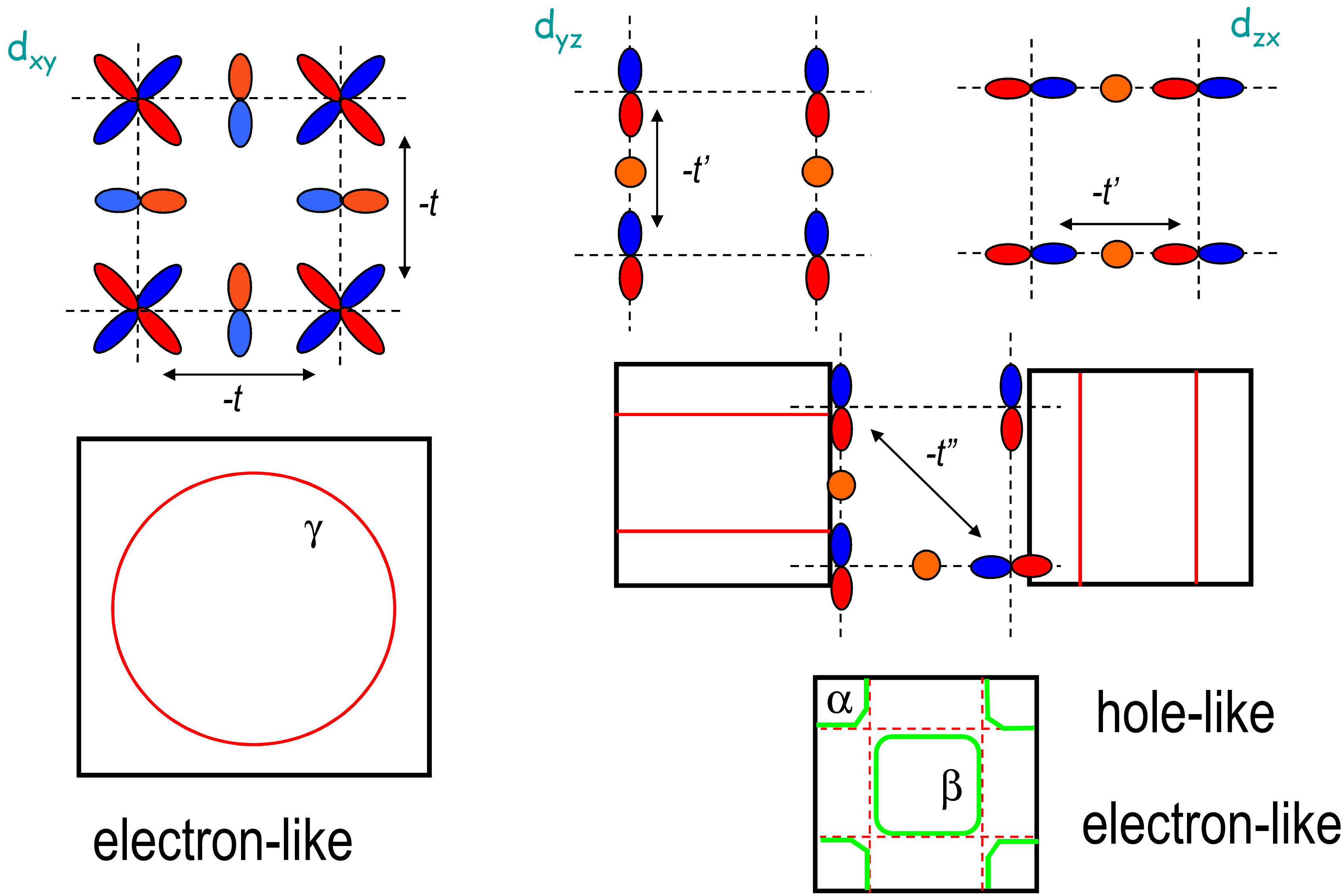


Bergemann et al.

quasi-two-dimensional Fermi liquid

Agrees very well with bandstructure calculations   Oguchi, Singh

Electronic structure of t_{2g}-orbitals

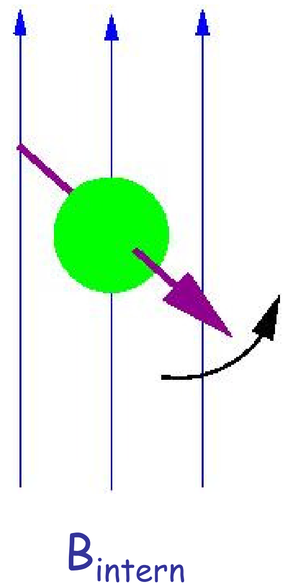


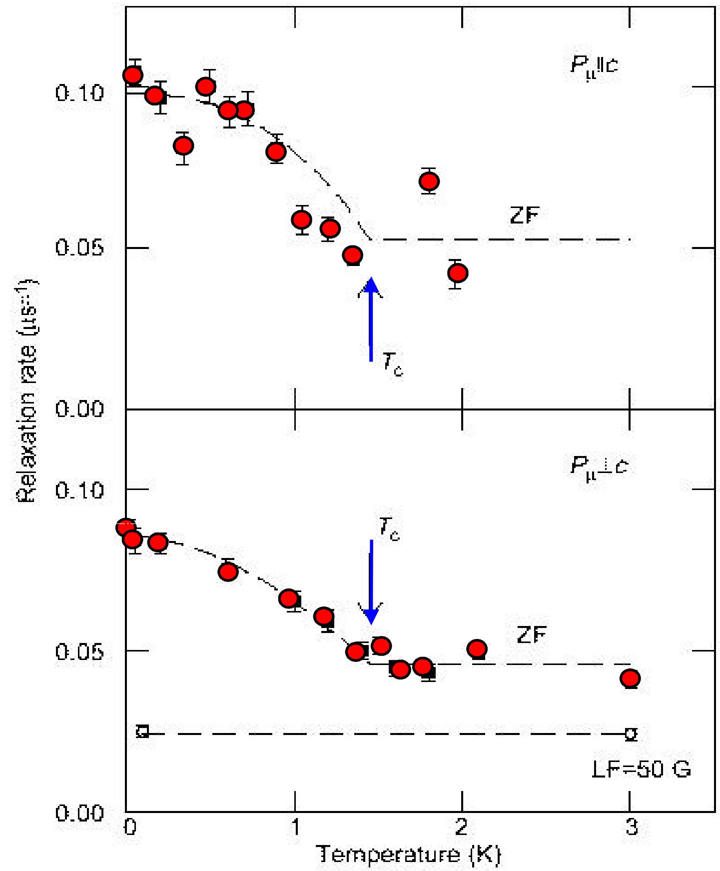
Broken Time Reversal: Sr_2RuO_4

Zero-field muon spin relaxation

Muon spin depolarized by random
Internal field

Magnetism generated
by superconductivity
(0.1 - 1 Gauss)





Luke, Uemura et al. (1998)

Also Polar Kerr effect (Xia, Kapitulnik, 2006)

Topological defects and spatial variations

Ginzburg-Landau free energy: spatial variations: conventional case

1-dimensional representations:

$$F\left[\eta,\vec{A}\right]=\int d^3r\left[a|\eta|^2+b|\eta|^4+K\left|\vec{D}\eta\right|^2+\frac{1}{8\pi}\left(\vec{\nabla}\times\vec{A}\right)^2\right]$$

$$a(T)=a'(T-T_c) \qquad a',b,K>0 \qquad \vec{D}=\vec{\nabla}+i\frac{2e}{\hbar c}\vec{A} \qquad \vec{B}=\vec{\nabla}\times\vec{A}$$

Ginzburg Landau equations:

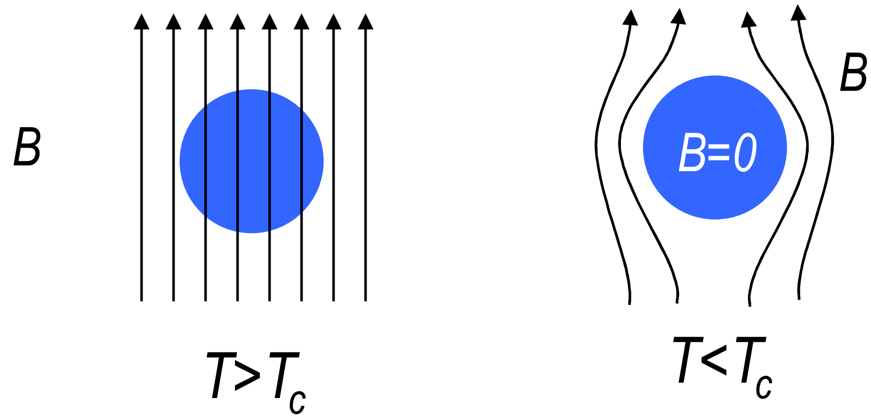
$$\left\{a+2b|\Psi|^2-K\vec{D}^2\right\}\Psi=0$$

$$\vec{\nabla}\times\vec{B}=\frac{4\pi}{c}\vec{J}_s \qquad \vec{J}_s=\frac{e}{2hi}K\left\{\Psi^*\left(\vec{D}\Psi\right)-\Psi\left(\vec{D}\Psi\right)^*\right\}$$

$$\boldsymbol{K}(\vec{\boldsymbol{n}}\cdot\vec{\boldsymbol{D}})\psi=0 \qquad \vec{\boldsymbol{n}}\times(\vec{\boldsymbol{B}}-\vec{\boldsymbol{H}})=0$$

Conventional Superconductivity

Field expulsion (1933)
Meissner-Ochsenfeld effect



Explained within GL theory by taking $|\psi|$ fixed:

$$\vec{J}_s = -\frac{4e^2}{\hbar^2 c} |\Psi_0|^2 \; K[A - \frac{\hbar c}{2e}\nabla\theta]$$

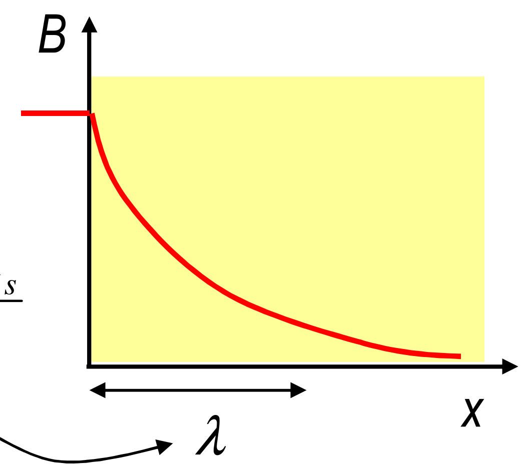
London theory (1935)

$$\left.\begin{aligned} \nabla \times \lambda^2 \vec{j} &= -\vec{B} \\ \nabla \times \vec{B} &= \frac{4\pi}{c}\vec{j} \end{aligned}\right\} \Rightarrow \boxed{\nabla^2 \vec{B} = \lambda^{-2} \vec{B}}$$

$$\lambda^{-2} = \frac{4\pi e^2 n_s}{mc^2}$$

density of superconducting electrons (n_s)

London penetration depth (λ)

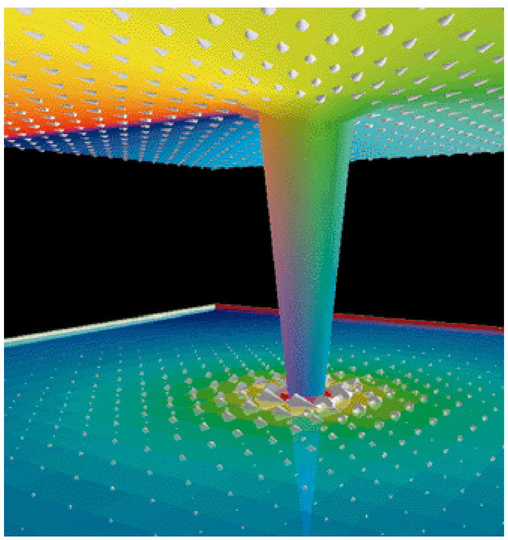


Standard Vortex

The free energy has a U(1) gauge invariance.

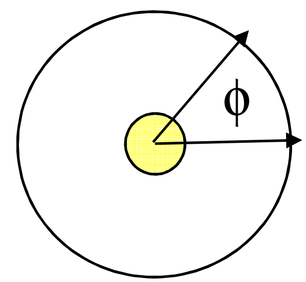
Consider a line-defect in the wave function:

$$\psi(r,\phi) = |\psi(r)|\, e^{in\phi}$$



$$\vec{j} = i\hbar m[\psi(\nabla\psi)^* - \psi^*(\nabla\psi)] - \frac{2me}{c}|\psi|^2\vec{A}$$

Far from the vortex core



$$0 = |\psi|^2\, m[\hbar n\nabla\phi - \frac{2e}{c}\vec{A}]$$

$$\oint A\cdot dl = n\Phi_0 = n\frac{hc}{2e}$$

The flux contained by a vortex is quantized

Ginzburg-Landau free energy: spatial variations

1-dimensional representations:

$$F\left[\eta,\vec{A}\right]=\int d^3r\left[a|\eta|^2+b|\eta|^4+K\left|\vec{D}\eta\right|^2+\frac{1}{8\pi}\left(\vec{\nabla}\times\vec{A}\right)^2\right]$$

$$a(T)=a'(T-T_c)\qquad a',b,K>0\qquad \vec{D}=\vec{\nabla}+i\frac{2e}{\hbar c}\vec{A}\qquad \vec{B}=\vec{\nabla}\times\vec{A}$$

2-dimensional representations:

$$F\left[\vec{\eta},\vec{A}\right]=\int d^3r\Big[a|\vec{\eta}|^2+b_1|\vec{\eta}|^4+\frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2+\eta_x^2\eta_y^{*2}\right\}+b_3|\eta_x|^2|\eta_y|^2$$

$$+K_1\left\{|D_x\eta_x|^2+|D_y\eta_y|^2\right\}+K_2\left\{|D_x\eta_y|^2+|D_y\eta_x|^2\right\}+K_3\left\{|D_z\eta_x|^2+|D_z\eta_y|^2\right\}$$

$$+\left\{K_4(D_x\eta_x)^*(D_y\eta_y)+K_5(D_x\eta_y)^*(D_y\eta_x)+cc.\right\}+\frac{1}{8\pi}\left(\vec{\nabla}\times\vec{A}\right)^2\Big]$$

Anisotropy

Diamagnetic screening: supercurrents $\vec{j} = -c\frac{\partial F}{\partial \vec{A}}$

$$j_x = 8\pi ei\left[K_1\eta_x^* D_x\eta_x + K_2\eta_y^* D_x\eta_y + K_4\eta_x^* D_y\eta_y + K_5\eta_y^* D_y\eta_x - cc.\right]$$

$$j_y = 8\pi ei\left[K_1\eta_y^* D_y\eta_y + K_2\eta_x^* D_y\eta_x + K_4\eta_y^* D_x\eta_x + K_5\eta_x^* D_x\eta_y - cc.\right]$$

$$j_x = 8\pi eiK_3\left\{\eta_x^* D_z\eta_x + \eta_y^* D_z\eta_y - cc.\right\}$$

tensorial London equation: $\nabla^2\vec{B} = \hat{\Lambda}\vec{B}$ Important for vortex lattice structure!

$$\hat{\Lambda}_A = \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix} \qquad \hat{\Lambda}_B = \begin{pmatrix} \lambda^{-2} & \tilde{\lambda}^{-2} & 0 \\ \tilde{\lambda}^{-2} & \lambda^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix} \quad \hat{\Lambda}_C = \begin{pmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda'^{-2} & 0 \\ 0 & 0 & \lambda_z^{-2} \end{pmatrix}$$

tetragonal ($\hat{\Lambda}_A$)

orthorhombic ($\hat{\Lambda}_B$, $\hat{\Lambda}_C$)

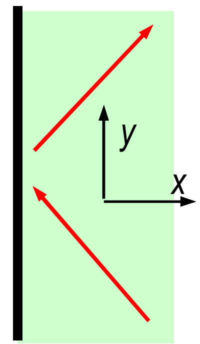
Surface properties - spontaneous supercurrents for chiral phase

surface scattering detrimental interference effects:

E_g: $\psi_x(\vec{k}) = k_z k_x$

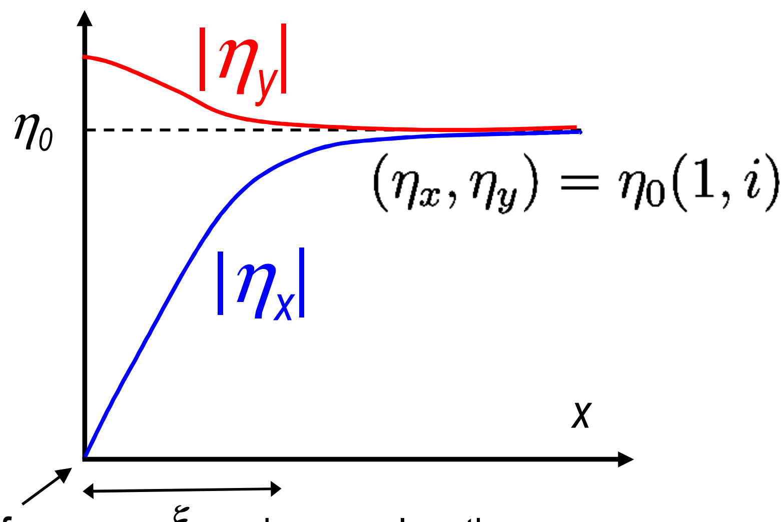
$\psi_y(\vec{k}) = k_z k_y$

$$\psi = \eta_x(\vec{r})\psi_x + \eta_y(\vec{r})\psi_y$$



specular scattering $\begin{cases} k_x \to -k_x \\ k_y \to k_y \end{cases}$

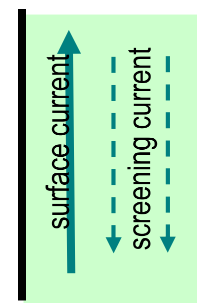
$\psi_x \to -\psi_x$	destructive
$\psi_y \to \psi_y$	constructive

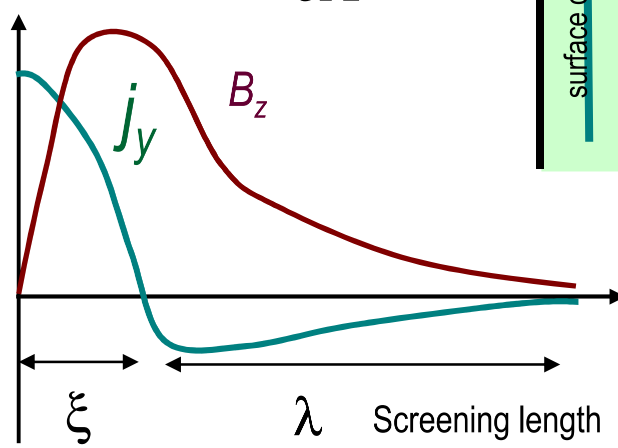


ξ coherence length

"extension of Cooper pairs"

Supercurrent: $\vec{j} = -c \dfrac{\partial F}{\partial \vec{A}}$



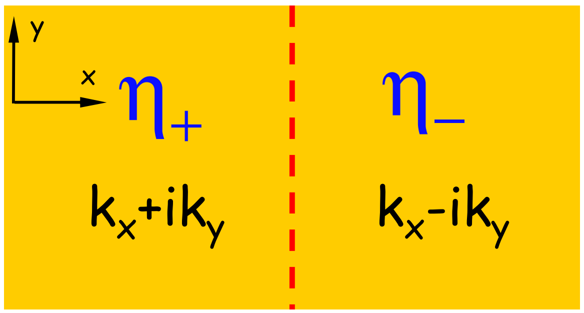


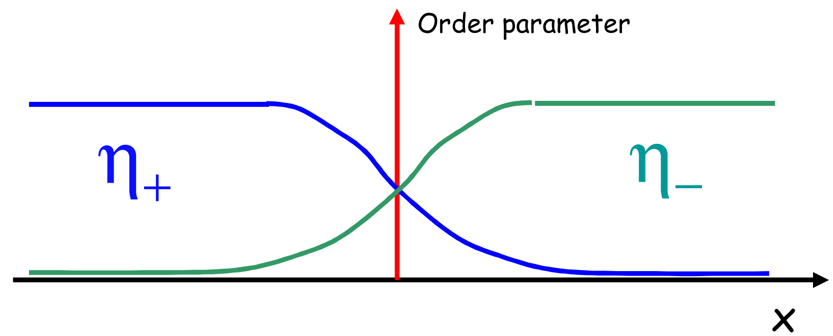
Screening length

London penetration depth

Topological defects

two degenerate phases → domains and domain walls

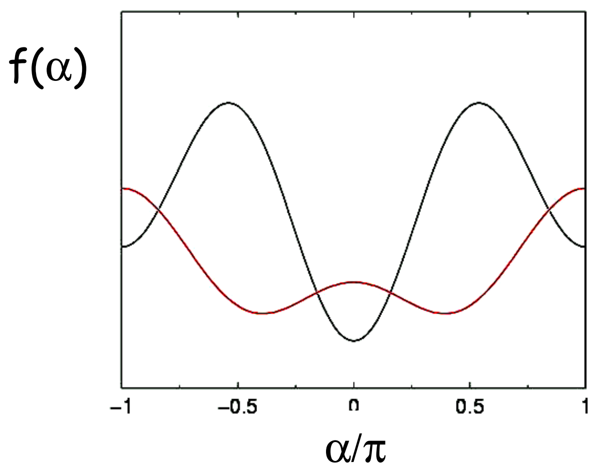




Energy of domain wall

depends on
domain wall orientation

degenerate minima
or metastable forms
of a domain wall



Analog to Josephson junction

$$\eta_\pm = |\eta_\pm| e^{i\phi_\pm}$$

relative phase

$$\alpha = \phi_+ - \phi_-$$

Simplified Theory: Δ_1 and Δ_2

$$f = \alpha\ |\Delta_1|^2 + \alpha\ |\Delta_2|^2 + \beta_1 (|\Delta_1|^2 + |\Delta_2|^2)^2$$

$$+ \beta_2 |\Delta_1|^2 |\Delta_2|^2 + \frac{1}{2m}(|\vec{D}\Delta_1|^2 + |\vec{D}\Delta_2|^2)$$

$\boldsymbol{U}(1) \times \boldsymbol{U}(1)$ symmetry

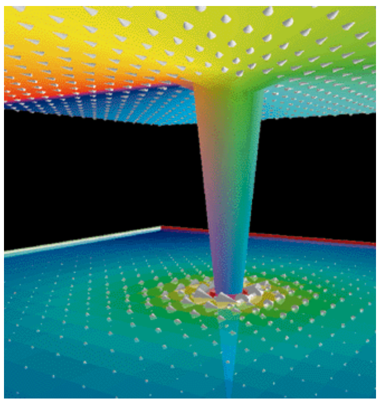
Two homogeneous solutions:

$$(\Delta_1, \Delta_2) = \Delta(1,1)/\sqrt{2}$$

$$(\Delta_1, \Delta_2) = \Delta(1,0)$$

Vortices

$$\psi_1(\boldsymbol{r},\phi) = |\psi_1(\boldsymbol{r})| e^{in\phi} \qquad \psi_2(\boldsymbol{r},\phi) = |\psi_2(\boldsymbol{r})| e^{im\phi}$$



Consider (1,0) vortex in a phase where both components have unequal magnitudes:

$$\vec{j} = i\hbar m[\psi_1(\nabla\psi_1)^* - \psi_1^*(\nabla\psi_1)] - \frac{2me}{c}(|\psi|_1^2 + |\psi|_2^2)\vec{A}$$

$$\oint A \cdot dl = \Phi_0 \frac{|\psi_1|^2}{|\psi_1|^2 + |\psi_2|^2}$$

(1,1) vortex is usual Abrikosov vortex with flux Φ_0

If one of the U(1) symmetries is broken then get confinement – $V\cos[l(\varphi_1 - \varphi_2)]$ – relative phase is not free to rotate – single fractional vortices no longer exist.

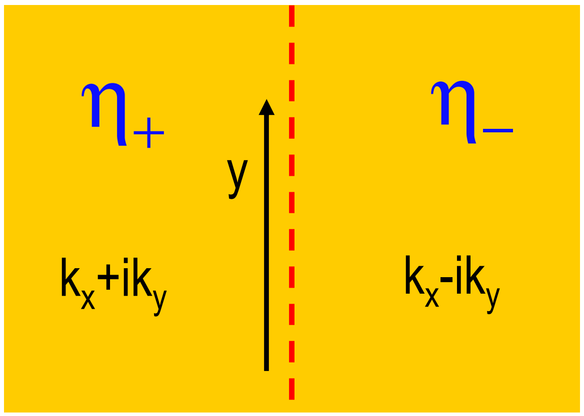
The b2 term breaks U(1)xU(1) symmetry

$$F[\vec{\eta}]=\int d^3r\left[a|\vec{\eta}|^2+b_1|\vec{\eta}|^4+\frac{b_2}{2}\left\{\eta_x^{*2}\eta_y^2+\eta_x^2\eta_y^{*2}\right\}+b_3|\eta_x|^2|\eta_y|^2\right]$$

Can fractional vortices exist with confinement?

Yes –if they appear on a domain wall. Sigrist, Ueda, and Rice, Phys. Rev. Lett. **63**, 1727 (1989).

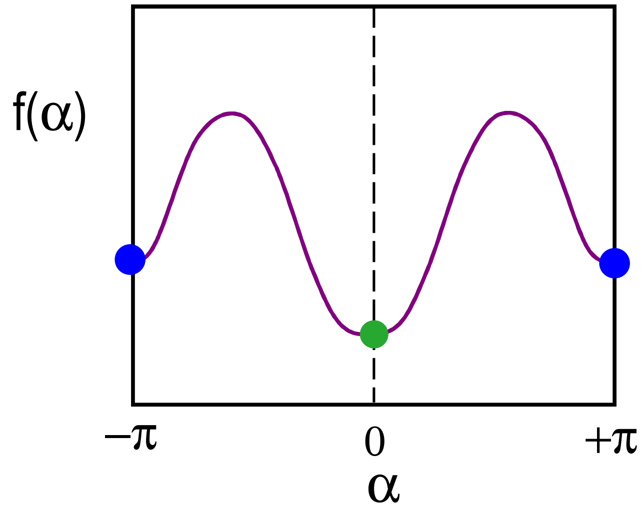
Structure of domain wall vortex

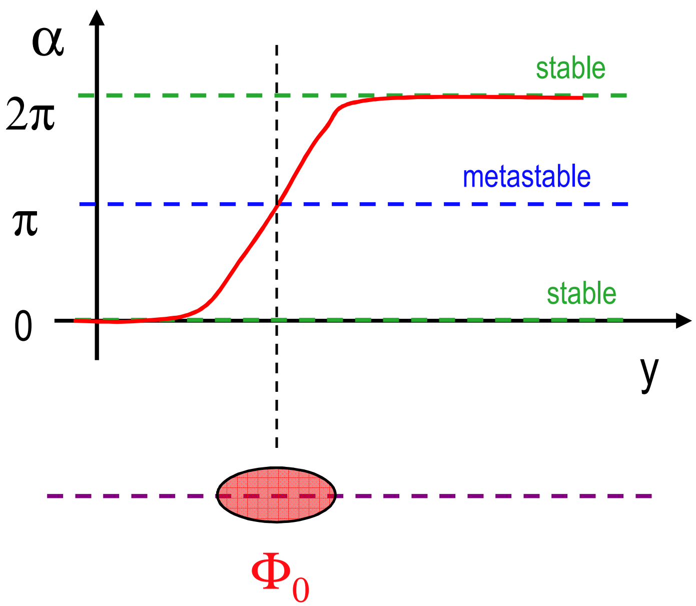


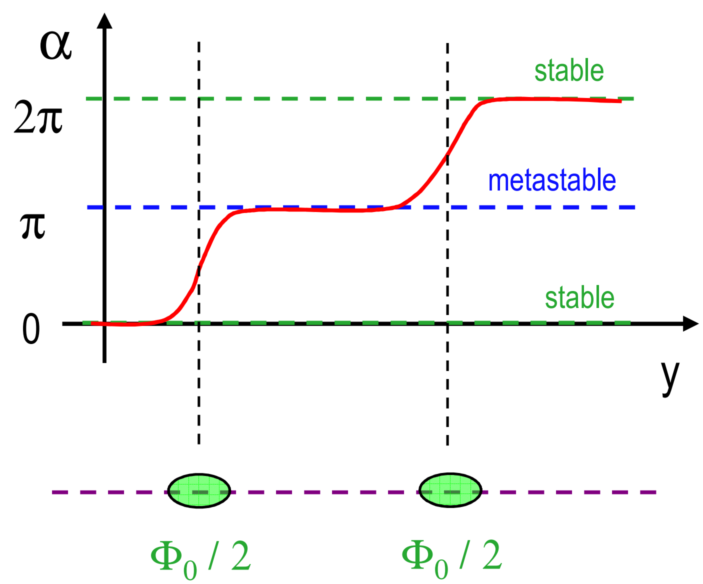
$$\eta_\pm = |\eta_\pm| e^{i\phi_\pm}$$

$$\alpha = \phi_+ - \phi_-$$

relative phase

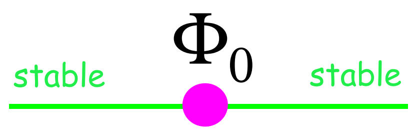


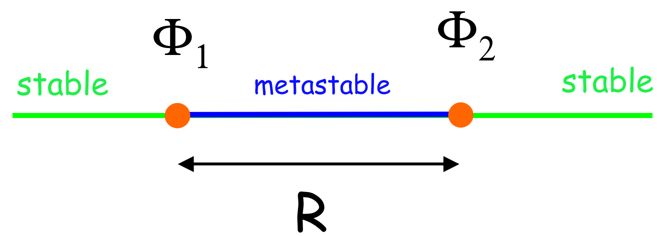




Stability of fractional vortices

decay of conventional vortex





energy gain:

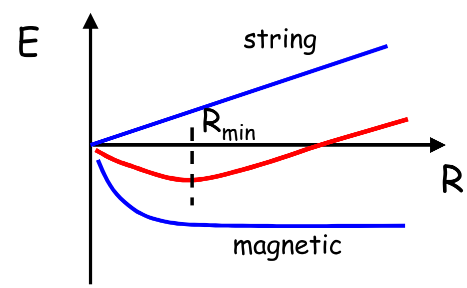
magnetic repulsion

$$\frac{\Phi_0^2}{(4\pi\lambda)^2}\ln\kappa \; > \; \frac{\Phi_1^2}{(4\pi\lambda)^2}\ln\kappa_1 + \frac{\Phi_2^2}{(4\pi\lambda)^2}\ln\kappa_2$$

energy cost:

string potential

$$(\varepsilon_{meta} - \varepsilon_{stable}) R$$



stable fractional vortex pair with R_{min}

Creates strong pinning of vortices

Spatially inhomogeneous BCS theory:

$$H = \sum_{\vec{k},s} \xi_{\vec{k}} c^{+}_{\vec{k}s} c_{\vec{k}s} + \frac{1}{2} \sum_{\vec{k},\vec{k}',s,s'} V_{\vec{k},\vec{k}'} c^{+}_{\vec{k}+q/2,s} c^{+}_{-\vec{k}+q/2s'} c_{-\vec{k}'+q/2s'} c_{\vec{k}'+q/2s}$$

BCS: Ignore q dependence in V
since q will be much less than k

Conclusions and final remarks

- Superconductivity in strongly correlated electron systems often unconventional
  - strong Coulomb repulsion favors angular momentum $l > 0$

- Unconventional order parameters give rise to new phenomena
  - quasi-particle properties, tunneling and Josephson effect
  - vortex matter, flux dynamics
  - superconducting multi-phase diagrams, competing phases
  - intrinsic magnetism and connection to competing magnetic phases

→ higher dimensional order parameters (Sr_2RuO_4, $(U,Th)Be_{13}$, UPt_3, …) are interesting

Many chapters on unconventional superconductivity are still unwritten and new materials are discovered at an accelerating pace (sample purity is mandatory!)