Novel exact results on lattice models with the Kitaev interactions

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Outline

Topological phases and lattice models
- Toric code
- Kitaev honeycomb lattice model

Novel exact solution of the Kitaev model

Exact solution on torus and topological phase transition
- topological degeneracy on torus

Other applications
- vortex and edge modes
- non-Abelian statistics
- novel solution of Yao-Kivelson model

References

Phys. Rev. B (2010), arXiv:1009.4951
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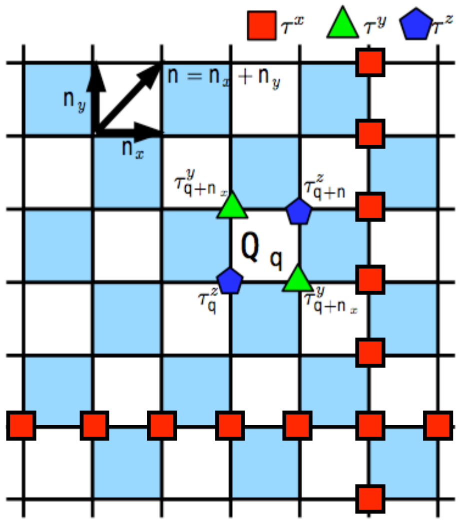
$\mathcal{D}(Z_2)$ topological phase "Toric code"



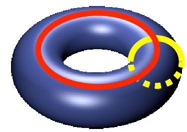
H_{TC} =-$J\,\Sigma_p\, Q_p$ $\quad [Q_p, Q_q] = 0 \quad [H_{TC}, Q_p] = 0$

$Q_p\, |\{Q_p\}\rangle = Q_p\, |\{Q_p\}\rangle$ where $Q_p = \pm 1$

Ground state: $\quad Q_p\, |\{Q_p\}\rangle = |\{Q_p\}\rangle$ on plane

on torus $\quad Q_p\, |\{Q_p\}, l_x, l_y\rangle = |\{Q_p\}, l_x, l_y\rangle$

$l_x, l_y = \pm 1 \quad \Rightarrow$ **Degeneracy(T²) = 4**



Particle set

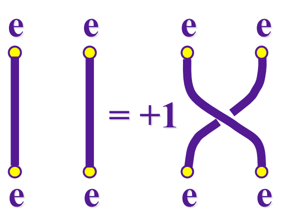
- trivial topological charge **1**
- electric charge **e**
- magnetic charge **m**
- fermion **ε**

Fusion rules

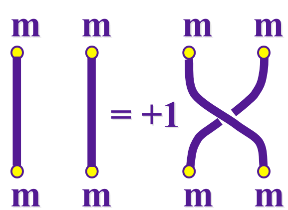
e x e = m x m = ε x ε = 1
e x m = ε
e x ε = m
m x ε = e

Braiding

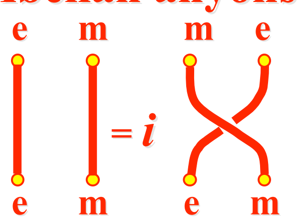
bosons



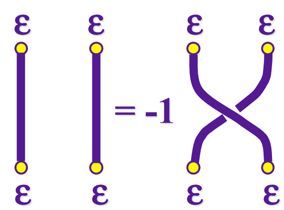
bosons



Abelian anyons



fermions



Ising topological phase

Particle set

- trivial topological charge **1**
- non-Abelian (Ising) anyon **σ**
- fermion **ε**

Degeneracy of torus **Deg(T^2) = 3**

Fusion rules

$\varepsilon \times \varepsilon = 1$

$\varepsilon \times \sigma = \sigma$

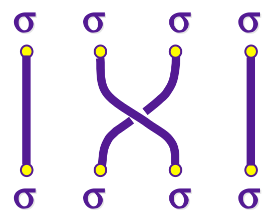
$\sigma \times \sigma = 1 + \varepsilon$

Braidings

non-Abelian anyons

$$|\psi_1^{\sigma\sigma}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\psi_\varepsilon^{\sigma\sigma}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \exp\left\{i\frac{\pi}{8}\right\} & 0 \\ 0 & \exp\left\{-i\frac{3\pi}{8}\right\} \end{pmatrix}$$

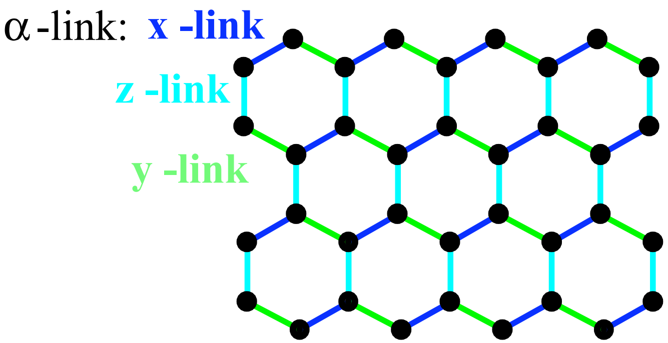


Realizations of non-Abelian anyons σ

- **Ising lattice models - Kitaev honeycomb model, Yao-Kivelson model**
- e/4 charge quasiparticle in fractional quantum Hall state at the filling ν=5/2
- (half-)vortices in p-wave superconductors

Kitaev honeycomb lattice model

$$H_0 = - \Sigma_\alpha J_\alpha \Sigma_{i,j}\, \sigma^\alpha_i \sigma^\alpha_j = - \Sigma_\alpha J_\alpha \Sigma_{i,j} K^\alpha_{ij}$$



A.Y.Kitaev, Ann. Phys. 321, 2 (2006).

Phase diagram:

- **phase A**

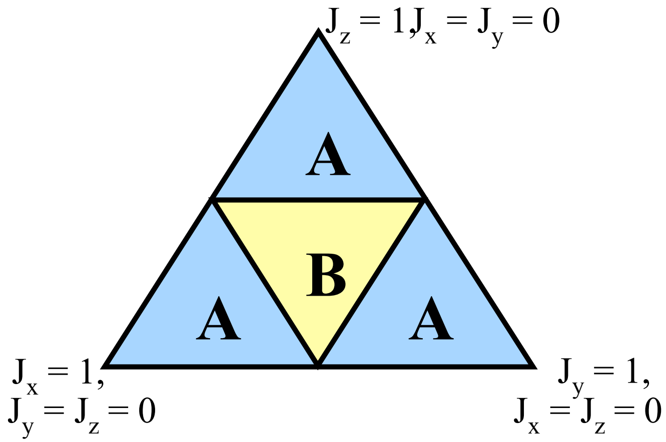
- can be mapped perturbatively onto **the toric code**:

$$|\uparrow\rangle_{eff} = |\uparrow\uparrow\rangle \quad |\downarrow\rangle_{eff} = |\downarrow\downarrow\rangle \qquad H_{\text{eff}} = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p, \qquad Q_p = \sigma^y_{\text{left}(p)} \sigma^y_{\text{right}(p)} \sigma^z_{\text{up}(p)} \sigma^z_{\text{down}(p)}$$

with the ground state stabilized by the plaquette operators:

$$\boldsymbol{Q}_{\mathbf{p}}\, |\{Q_p\}\rangle = |\{Q_p\}\rangle$$

- **phase B** - gapless.



In the presence of magnetic field:

$$H = H_0 + H_1 = H_0 + \Sigma_i \Sigma_{\alpha=x,y,z} B_\alpha \sigma_{\alpha,i}$$

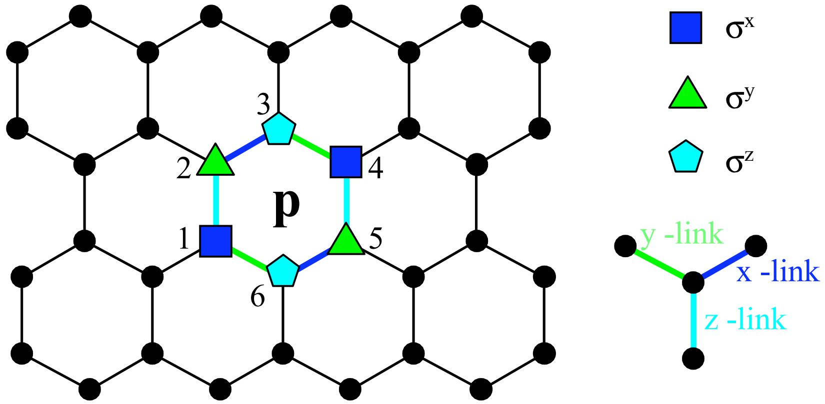
- parity and time-reversal symmetry are broken
- **phase B** acquires a gap and becomes **non-abelian topological phase of the Ising type**

The leading P and T breaking term in perturbation theory occurs at the third order:

$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} \qquad \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x \sigma_6^y \sigma_5^z + \sigma_2^z \sigma_3^y \sigma_4^x + \sigma_1^y \sigma_2^x \sigma_3^z + \sigma_4^y \sigma_5^x \sigma_6^z + \sigma_3^x \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_1^z \sigma_6^x$$

Vortex operators in the honeycomb model

$$W_p = K^z_{1,2} K^x_{2,3} K^y_{3,4} K^z_{4,5} K^x_{5,6} K^y_{6,1} =$$
$$= \sigma^x_1 \sigma^y_2\, \sigma^z_3 \sigma^x_4\, \sigma^y_5 \sigma^z_6$$



The vortex operators commute with H_0

$$[H_0, W_p] = 0$$

$$H_0 = -\sum_\alpha J_\alpha \sum_{i,j} K^\alpha_{ij}$$

$$K^\beta_{k+1,k+2} K^\alpha_{k,k+1} = -K^\alpha_{k,k+1} K^\beta_{k+1,k+2} \qquad (K^\alpha_{k,k+1})^2 = 1$$

and they also commute with the perturbative magnetic field term in the Hamiltonian:

$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} \qquad \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x \sigma_6^y \sigma_5^z + \sigma_2^z \sigma_3^y \sigma_4^x + \sigma_1^y \sigma_2^x \sigma_3^z + \sigma_4^y \sigma_5^x \sigma_6^z + \sigma_3^x \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_1^z \sigma_6^x$$

so for each energy eigenstate, we either have no vortex (+1) or a vortex (-1) at a plaquette p

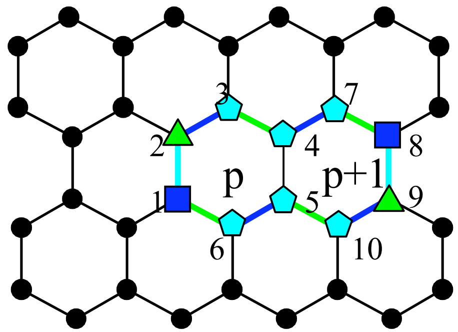
$$\boldsymbol{W}_{\mathbf{p}} |E_n\rangle = \pm 1\, |E_n\rangle$$

The Hilbert space splits into vortex sectors, i.e. subspaces with particular vortex configurations

$$L = \bigoplus_{w_1,\ldots\ldots,w_m} L_{w_1,\ldots\ldots,w_m}$$

Products of the vortex operators are loops.

$$K_{i,j}^{\alpha(1)} K_{j,k}^{\alpha(2)} \ldots K_{p,q}^{\alpha(M-1)} K_{q,i}^{\alpha(M)}$$



And all loop symmetries on torus $\Pi_{\mathbf{p}}\, W_p = 1$ can be written as

$$C_{(k,l)} = G_k F_l(W_1, W_2, \ldots, W_{N-1}).$$

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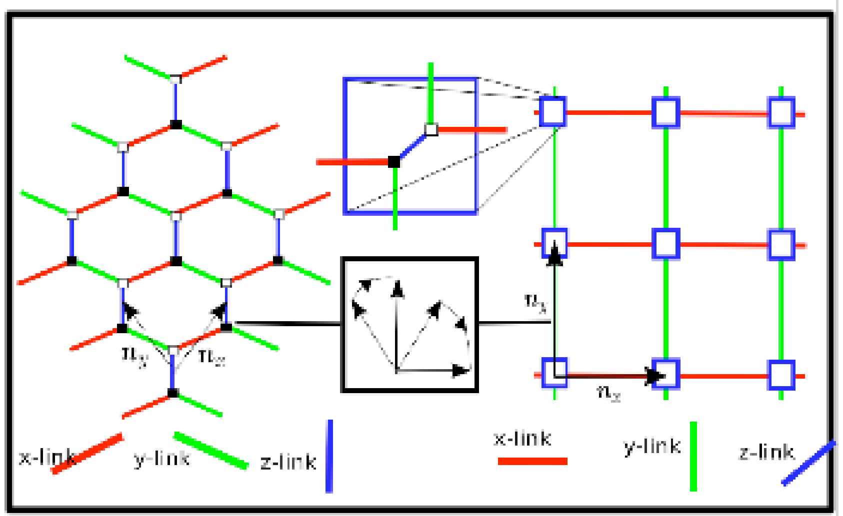
Effective spins and hardcore bosons

New perspective:

spin-hardcore boson representation

$$|\uparrow_{\blacksquare}\uparrow_{\square}\rangle = |\Uparrow, 0\rangle, \quad |\downarrow_{\blacksquare}\downarrow_{\square}\rangle = |\Downarrow, 0\rangle$$
$$|\uparrow_{\blacksquare}\downarrow_{\square}\rangle = |\Uparrow, 1\rangle, \quad |\downarrow_{\blacksquare}\uparrow_{\square}\rangle = |\Downarrow, 1\rangle$$

Schmidt, Dusuel, and Vidal (2008)



Pauli operators:

$$\begin{aligned}
\sigma^x_{q,\blacksquare} &= \tau^x_q (b^\dagger_q + b_q) \quad , \quad \sigma^x_{q,\square} = b^\dagger_q + b_q, \\
\sigma^y_{q,\blacksquare} &= \tau^y_q (b^\dagger_q + b_q) \quad , \quad \sigma^y_{q,\square} = i\,\tau^z_q (b^\dagger_q - b_q), \\
\sigma^z_{q,\blacksquare} &= \tau^z_q \quad , \quad \sigma^z_{q,\square} = \tau^z_q (I - 2 b^\dagger_q b_q),
\end{aligned}$$

Vortex and plaquette operators:

$$\boxed{W_q = (I - 2N_q)(I - 2N_{q+n_y}) Q_q}$$

$$N_q = b^\dagger_q b_q \qquad Q_q = \tau^z_q \, \tau^y_{q+n_x} \tau^y_{q+n_y} \tau^z_{q+n}$$

In the A_z-phase, $J_z >> J_x, J_y$, the bosons are energetically suppressed, thus at low energy

$$|\{W_q\}, 0> = |\{Q_q\}>$$

the low-energy perturbative Hamiltonian equals to toric code $H_{TC} = -J_{\text{eff}} \sum_q Q_q \otimes I$

This allows to write down an orthonormal basis of the full system in terms of the **toric code operators**:

$$|\{Q_q\}, \{N_q\}>$$

where $\{Q_q\}$ lists all honeycomb plaquette operators and $\{N_q\}$ lists the position vectors of any occupied bosonic modes. **On a torus,** the homologically nontrivial symmetries must be added

$$|\{Q_q\}, l_x, l_y, \{N_q\}>$$

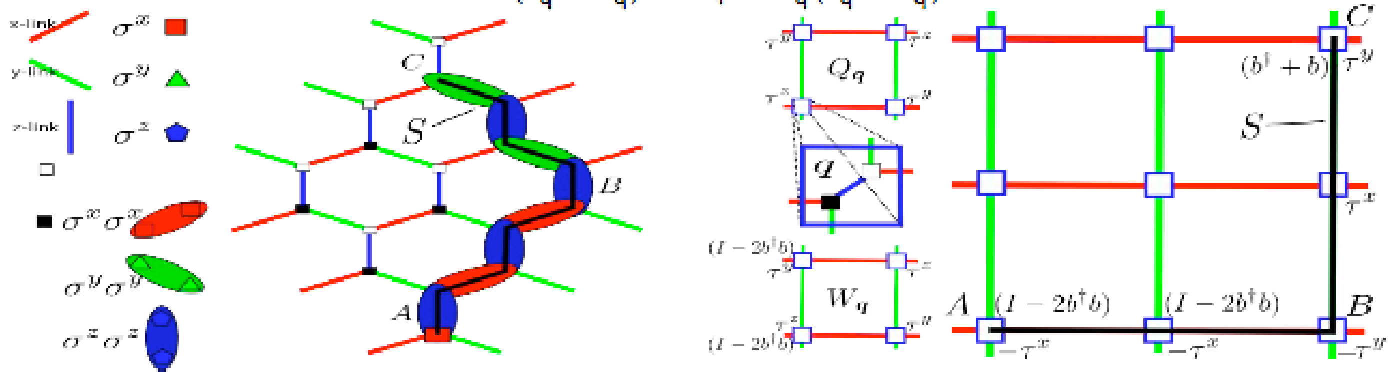
G. Kells, J.K. Slingerland, J. Vala, Phys. Rev. B **80**, 125415 (2009)

Jordan-Wigner transformation

Bosonic and effective spin Hamiltonian can be written in terms of fermions and vortices by applying a Jordan-Wigner transformation

$$
\begin{aligned}
\mathsf{H} \;=\; & -J_x \sum_{\mathsf{q}} (b_{\mathsf{q}}^\dagger + b_{\mathsf{q}}) \tau^x_{\mathsf{q}+\mathsf{n}_x} (b^\dagger_{\mathsf{q}+\mathsf{n}_x} + b_{\mathsf{q}+\mathsf{n}_x}) \\
& - J_y \sum_{\mathsf{q}} i\tau^z_{\mathsf{q}} (b_{\mathsf{q}}^\dagger - b_{\mathsf{q}}) \tau^y_{\mathsf{q}+\mathsf{n}_y} (b^\dagger_{\mathsf{q}+\mathsf{n}_y} + b_{\mathsf{q}+\mathsf{n}_y}) \\
& - J_z \sum_{\mathsf{q}} (I - 2 b^\dagger_{\mathsf{q}} b_{\mathsf{q}}).
\end{aligned}
$$

$$(c_{\mathsf{q}}^\dagger + c_{\mathsf{q}}) = S_{\mathsf{q}} = S'_{\mathsf{q}}(b_{\mathsf{q}}^\dagger + b_{\mathsf{q}})$$



$$
\begin{aligned}
\mathsf{H} \;=\; & J_x \sum_{\mathsf{q}} \mathsf{X}_{\mathsf{q}} (c_{\mathsf{q}}^\dagger - c_{\mathsf{q}})(c^\dagger_{\mathsf{q}+\mathsf{n}_x} + c_{\mathsf{q}+\mathsf{n}_x}) \\
& + J_y \sum_{\mathsf{q}} \mathsf{Y}_{\mathsf{q}} (c_{\mathsf{q}}^\dagger - c_{\mathsf{q}})(c^\dagger_{\mathsf{q}+\mathsf{n}_y} + c_{\mathsf{q}+\mathsf{n}_y}) \\
& + J_z \sum_{\mathsf{q}} (2c^\dagger_{\mathsf{q}} c_{\mathsf{q}} - I),
\end{aligned}
$$

where on a plane

$$\mathsf{Y}_{\mathsf{q}} = I \qquad \mathsf{X}_{q_x, q_y} \equiv \prod_{q'_y=0}^{q_y-1} \mathsf{W}_{q_x, q'_y}$$

G. Kells, J.K. Slingerland, J. Vala, Phys. Rev. B **80**, 125415 (2009)

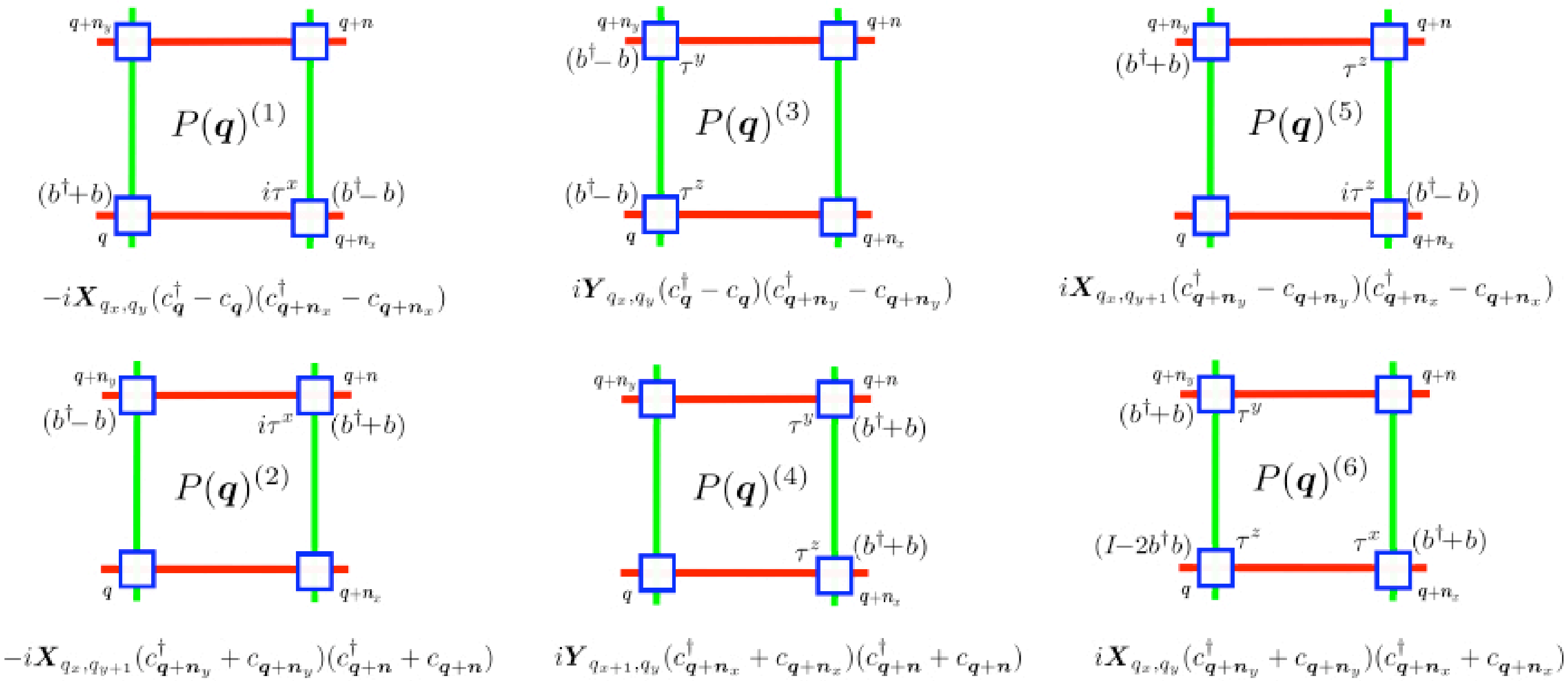
Magnetic field

- breaks parity invariance and time-reversal symmetry
- opens a gap in phase B and turns it into non-abelian topological phase of Ising type

$$H_1 = -\kappa \sum_{\boldsymbol{q}} \sum_{l=1}^{6} P(\boldsymbol{q})^{(l)}$$

$$\sum_{l=1}^{6} P(\boldsymbol{q})^{(l)} = \sigma_1^x \sigma_6^y \sigma_5^z + \sigma_2^z \sigma_3^y \sigma_4^x + \sigma_1^y \sigma_2^x \sigma_3^z + \sigma_4^y \sigma_5^x \sigma_6^z + \sigma_3^x \sigma_4^z \sigma_5^y + \sigma_2^y \sigma_1^z \sigma_6^x$$

- H_1 commutes with the plaquette operators, so stabilizer formalism can still be used



G. Kells, J.K. Slingerland, J. Vala,
Phys. Rev. B **80**, 125415 (2009)

Vortex-free sector and the ground state

Transformation to the momentum representation

$$c_{\boldsymbol{q}} = M^{-1/2} \sum c_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{q}}$$

$$H = \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}} + \frac{1}{2}(\Delta c^\dagger_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} + \Delta^* c_{-\boldsymbol{k}} c_{\boldsymbol{k}}) \right] - M J_z$$

$$\begin{aligned}
\xi_{\boldsymbol{k}} &= \varepsilon_{\boldsymbol{k}} - \mu \\
\Delta_{\boldsymbol{k}} &= \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}} \\
\mu &= -2J_z \\
\varepsilon_{\boldsymbol{k}} &= 2J_x \cos(k_x) + 2J_y \cos(k_y) \\
\alpha_{\boldsymbol{k}} &= 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y)) \\
\beta_{\boldsymbol{k}} &= 2J_x \sin(k_x) + 2J_y \sin(k_y).
\end{aligned}$$

The effect of the magnetic field is contained fully in the α_k term.

G. Kells, J.K. Slingerland, J. Vala, Phys. Rev. B **80**, 125415 (2009)

The Hamiltonian can be diagonalized by Bogoliubov transformation:

$$\gamma_{\boldsymbol{k}} = u_{\boldsymbol{k}} c_{\boldsymbol{k}} - v_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}} \qquad |u_{\boldsymbol{k}}|^2 + |v_{\boldsymbol{k}}|^2 = 1$$

resulting in the BCS Hamiltonian

$$\boxed{\mathsf{H} = \sum_{n=1}^{M} E_n(\gamma^\dagger_n \gamma_n - 1/2)}$$

$$\begin{aligned}
E_{\boldsymbol{k}} &= \sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_{\boldsymbol{k}}|^2} \\
u_{\boldsymbol{k}} &= \sqrt{1/2(1 + \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})} \\
v_{\boldsymbol{k}} &= i\sqrt{1/2(1 - \xi_{\boldsymbol{k}}/E_{\boldsymbol{k}})}
\end{aligned}$$

the ground state is BCS state with the **vacuum** given here **explicitly** in terms of **toric code stabilizers**

$$\boxed{|gs\rangle_{HC} = \prod \left(u_{\mathsf{k}} + v_{\mathsf{k}} c^\dagger_{\mathsf{k}} c^\dagger_{-\mathsf{k}}\right) \boxed{|\{Q_{\mathsf{q}}\}, \{\emptyset\}\rangle}} \rightarrow \boxed{|\{1,1,\ldots,1\},\{0\}>}$$

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Other vortex sectors on torus

To address an arbitrary vortex configuration we rewrite the general Hamiltonian

$$\mathsf{H} = \frac{1}{2}\sum_{\mathsf{qq}'} \begin{bmatrix} c_{\mathsf{q}}^{\dagger} & c_{\mathsf{q}} \end{bmatrix} \begin{bmatrix} \xi_{\mathsf{qq}'} & \Delta_{\mathsf{qq}'} \\ \Delta_{\mathsf{qq}'}^{\dagger} & -\xi_{\mathsf{qq}'}^{T} \end{bmatrix} \begin{bmatrix} c_{\mathsf{q}'} \\ c_{\mathsf{q}'}^{\dagger} \end{bmatrix}$$

To specify a particular vortex sector, the operators X_{q} and Y_{q} are replaced by their eigenvalues in that sector; for example for H_0 we obtain

$$\begin{aligned} \xi_{\boldsymbol{qq}'} &= 2J_z\delta_{\boldsymbol{q},\boldsymbol{q}'} + J_x\boldsymbol{X}_{\boldsymbol{q}}(\delta_{\boldsymbol{q},\boldsymbol{q}'-\boldsymbol{n}_x} + \delta_{\boldsymbol{q}-\boldsymbol{n}_x,\boldsymbol{q}'}) \\ &+ J_y\boldsymbol{Y}_{\boldsymbol{q}}(\delta_{\boldsymbol{q},\boldsymbol{q}'-\boldsymbol{n}_y} + \delta_{\boldsymbol{q}-\boldsymbol{n}_y,\boldsymbol{q}'}) \\ \Delta_{\boldsymbol{qq}'} &= J_x\boldsymbol{X}_{\boldsymbol{q}}(\delta_{\boldsymbol{q},\boldsymbol{q}'-\boldsymbol{n}_x} - \delta_{\boldsymbol{q}-\boldsymbol{n}_x,\boldsymbol{q}'}) \\ &+ J_y\boldsymbol{Y}_{\boldsymbol{q}}(\delta_{\boldsymbol{q},\boldsymbol{q}'-\boldsymbol{n}_y} - \delta_{\boldsymbol{q}-\boldsymbol{n}_y,\boldsymbol{q}'}). \end{aligned}$$

On **torus,** these terms include **periodicity**, i.e. the terms connecting the sites $(0, q_y)$ and $(N_x - 1, q_y)$, and $(q_x, 0)$ and $(q_x, N_y - 1)$, and thus the **homologically nontrivial symmetries**

$$\begin{aligned} X_{q_x,q_y} &= \textstyle\prod_{q_y'=0}^{q_y-1} W_{q_x,q_y'} && (q_y \neq 0 \text{ and } q_x \neq N_x - 1) \\ X_{q_x,q_y} &= 1 && (q_y = 0 \text{ and } q_x \neq N_x - 1) \\ X_{q_x,q_y} &= -l_0^{(x)} \textstyle\prod_{q_y'=0}^{q_y-1} W_{q_x,q_y} && (q_y \neq 0 \text{ and } q_x = N_x - 1) \\ X_{q_y,q_x} &= -l_0^{(x)} && (q_y = 0 \text{ and } q_x = N_x - 1) \end{aligned}$$

$$\begin{aligned} Y_{q_x,q_y} &= 1 && (q_y \neq N_y - 1) \\ Y_{q_x,q_y} &= -l_{q_x}^{(y)} && (q_y = N_y - 1) \end{aligned}$$

$$l_{q_x}^{(y)} = l_0^{(y)} \textstyle\prod_{q_y=0}^{N_{y-1}} \prod_{q_x'=0}^{q_x-1} W_{q_x',q_y}$$

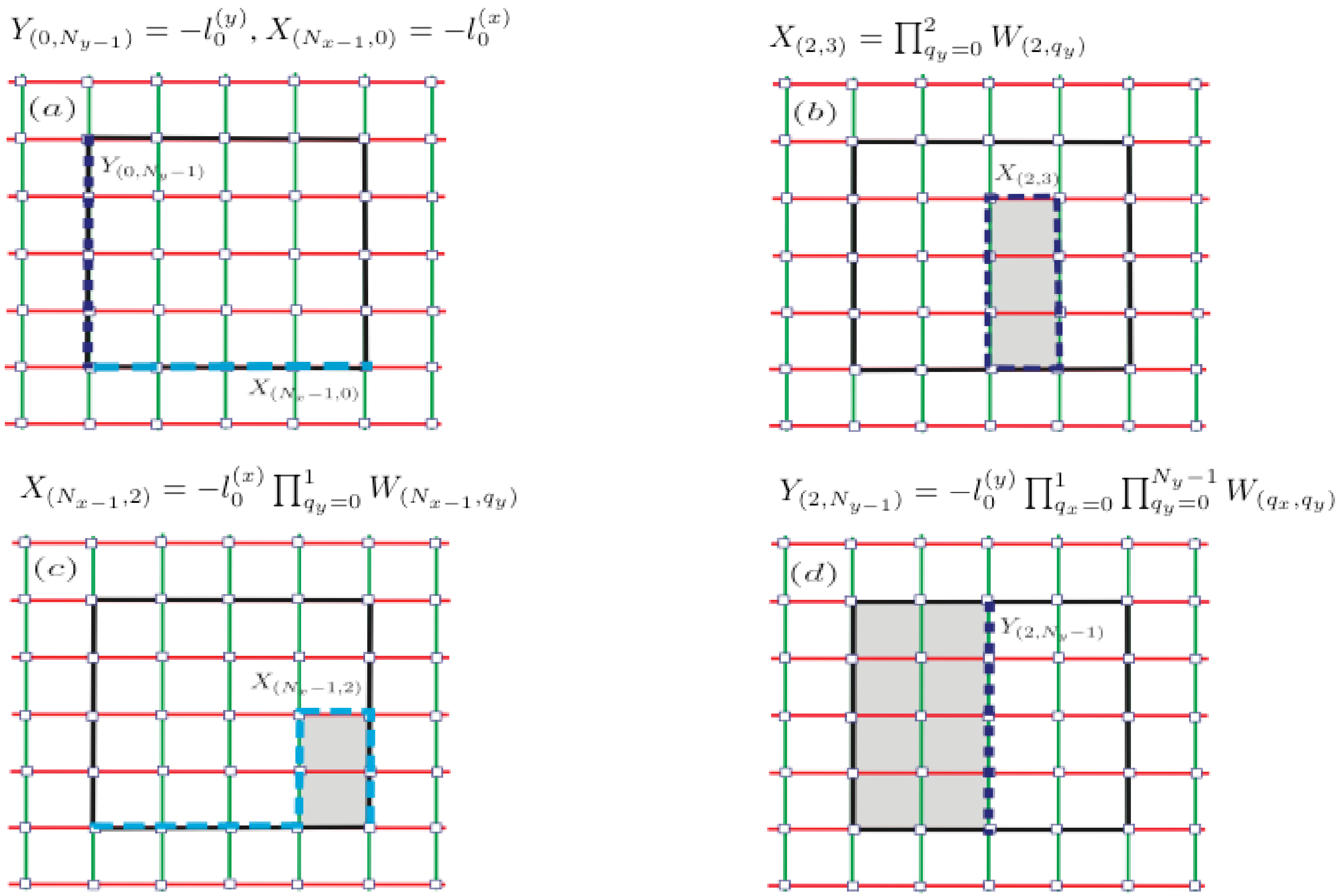
In order to include the magnetic field H_1 we have to add also

$$\begin{aligned} X_{q_x,q_y+1} &= -l_{q_x+1}^{(y)} && (q_x \neq N_x - 1) \\ X_{q_x,q_y+1} &= l_0^{(x)} l_0^{(y)} && (q_x = N_x - 1), \end{aligned}$$

$$\begin{aligned} X_{q_x,q_y} &= l_{q_x}^{(y)} \textstyle\prod_{q_y'=0}^{q_y-1} W_{q_x,q_y'} && (q_x \neq N_x - 1) \\ X_{q_x,q_y} &= l_0^{(x)} l_0^{(y)} W_{q_x,q_y} && (q_x = N_x - 1). \end{aligned}$$

Role of symmetries

On a torus, the system has N/2+1 loop symmetry generators from which all other loop symmetries can be obtained. We can specify a particular sector of the Hamiltonian by specifying the eigenvalues of the N/2-1 plaquette symmetries and 2 homologically nontrivial symmetries.



Fermionization on torus

The general Hamiltonian for an arbitrary vortex configuration

$$\mathsf{H} = \frac{1}{2}\sum_{\mathsf{qq}'} \begin{bmatrix} c_{\mathsf{q}}^\dagger & c_{\mathsf{q}} \end{bmatrix} \begin{bmatrix} \xi_{\mathsf{qq}'} & \Delta_{\mathsf{qq}'} \\ \Delta_{\mathsf{qq}'}^\dagger & -\xi_{\mathsf{qq}'}^T \end{bmatrix} \begin{bmatrix} c_{\mathsf{q}'} \\ c_{\mathsf{q}'}^\dagger \end{bmatrix}$$

presents the Bogoliubov-de Gennes eigenvalue problem

$$\begin{bmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{bmatrix} = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & -E \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^\dagger$$

with quasiparticle excitations

$$\begin{bmatrix} \gamma_1^\dagger, \ldots, \gamma_M^\dagger, & \gamma_1, \ldots, \gamma_M \end{bmatrix} = \begin{bmatrix} c_1^\dagger, \ldots, c_M^\dagger, & c_1, \ldots, c_M \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}$$

and the system thus reduces to free fermion Hamiltonian

$$\mathsf{H} = \sum_{n=1}^{M} E_n (\gamma_n^\dagger \gamma_n - 1/2)$$

Valid for
- all vortex sectors
- all homology sectors on torus

with the ground state (in momentum representation)

$$\boxed{|gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c_{\mathsf{k}}^\dagger c_{-\mathsf{k}}^\dagger \right) \boxed{|\{Q_{\mathsf{q}}\}, l_x \;, l_y \;, \{\emptyset\}\rangle}} \rightarrow \boxed{|\{1,1,\ldots,1\}, l_x, l_y, \{0\}>}$$

Topological phase transition: change of degeneracy on torus

The allowed values of momentum k_α in the various homology sectors on torus are given as

$$k_\alpha = \theta_\alpha + 2\pi n_\alpha / N_\alpha \qquad \theta_\alpha = (l_\alpha + 1)/2N_\alpha \qquad n_\alpha = 0, 1, \ldots, N_\alpha - 1$$

So thus only one of the four configurations, ($l_x = -1$, $l_y = -1$), permits the momentum to be exactly (π, π):

$$\Delta_{\pi,\pi} = \alpha_k + i\,\beta_k = 0 \quad \text{and thus} \quad E_{\pi,\pi} = (\xi_{\pi,\pi}{}^2 + |\,\Delta_{\pi,\pi}\,|^2)^{1/2} = |\xi_{\pi,\pi}\,|$$

$$\alpha_k = 4\kappa(\sin(k_x) - \sin(k_y) - \sin(k_x - k_y)) = 0 \text{ and } \beta_k = 2J_x(\sin(k_x) + 2J_y(\sin(k_y) = 0$$

At the phase transition, where $J_z = J_x + J_y$,

$\xi_{\pi,\pi} = \varepsilon_k - \mu = 2J_x \cos(k_x) + 2\,J_y \cos(k_y) - (-2\,J_z) = [-2(J_x + J_y)] + 2\,J_z$ **changes the sign**, giving

$$\xi_{\pi,\pi} / E_{\pi,\pi} = \xi_{\pi,\pi} / |\,\xi_{\pi,\pi}\,| = -1 \quad \text{and thus} \quad \boxed{u_{\pi,\pi} = [1/2\,(1 + \xi_{\pi,\pi} / E_{\pi,\pi})]^{1/2} = 0}$$

$$\boxed{c^+_{\pi,\pi}\, c^+_{-\pi,-\pi} = (c^+_{\pi,\pi})^2 = 0}$$

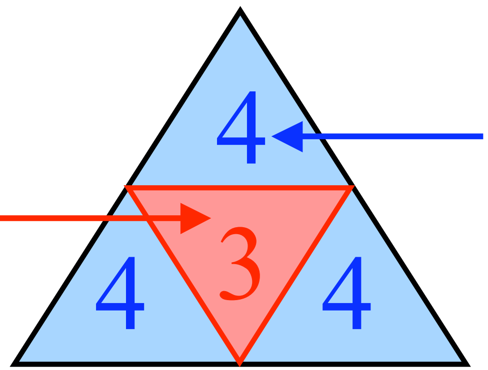
$$|g.s.\rangle = \Pi_k\,(\cancel{u_k} + v_k\, \cancel{c^+_k}\, c^+_{-k})\,|\{Q_q\}, l_x^{(0)}, l_y^{(0)}, \{0\}\rangle$$

One of the four BCS states on torus vanishes at transition to Ising phase

Non-Abelian phase

3-fold degenerate

ground state on torus



Abelian phases

4-fold degenerate

ground state on torus

G. Kells, J.K. Slingerland, J. Vala, Phys. Rev. B **80**, 125415 (2009)

Outline

Topological phases and lattice models

- Toric code
- Kitaev honeycomb lattice model

Novel exact solution of the Kitaev model

Exact solution on torus and topological phase transition

- topological degeneracy on torus

Other applications

- vortex and edge modes
- non-Abelian statistics
- novel solution of Yao-Kivelson model

References

Phys. Rev. B (2010), arXiv:1009.4951

Phys. Rev. B 81, 104429 (2010)

Phys. Rev. B 80, 125415 (2009)

Phys. Rev. Lett. 101, 240404 (2008)

Vortices and zero energy modes

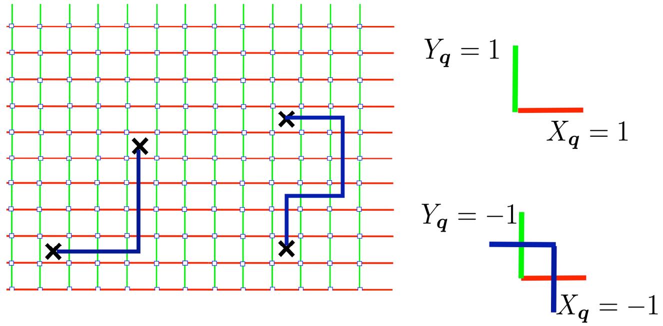
$$H = H_0 + \sum_{\boldsymbol{q}} \sum_{l} P_{\boldsymbol{q}}^{(l)}$$

$$\begin{aligned} H_0 &= J_x \sum_{\boldsymbol{q}} \boldsymbol{X}_{\boldsymbol{q}} (c_{\boldsymbol{q}}^\dagger - c_{\boldsymbol{q}})(c_{\boldsymbol{q}\rightarrow}^\dagger + c_{\boldsymbol{q}\rightarrow}) \\ &+ J_y \sum_{\boldsymbol{q}} \boldsymbol{Y}_{\boldsymbol{q}} (c_{\boldsymbol{q}}^\dagger - c_{\boldsymbol{q}})(c_{\boldsymbol{q}\uparrow}^\dagger + c_{\boldsymbol{q}\uparrow}) \\ &+ J_z \sum_{\boldsymbol{q}} (2c_{\boldsymbol{q}}^\dagger c_{\boldsymbol{q}} - I), \end{aligned}$$

$$\begin{aligned} P^{(1)} &= -iK\boldsymbol{X}_{\boldsymbol{q}}(c_{\boldsymbol{q}}^\dagger - c_{\boldsymbol{q}})(c_{\boldsymbol{q}\rightarrow}^\dagger - c_{\boldsymbol{q}\rightarrow}) \\ P^{(2)} &= -iK\boldsymbol{X}_{\boldsymbol{q}\uparrow}(c_{\boldsymbol{q}\uparrow}^\dagger + c_{\boldsymbol{q}\uparrow})(c_{\boldsymbol{q}\nearrow}^\dagger + c_{\boldsymbol{q}\nearrow}) \\ P^{(3)} &= +iK\boldsymbol{Y}_{\boldsymbol{q}}(c_{\boldsymbol{q}}^\dagger - c_{\boldsymbol{q}})(c_{\boldsymbol{q}\uparrow}^\dagger - c_{\boldsymbol{q}\uparrow}) \\ P^{(4)} &= +iK\boldsymbol{Y}_{\boldsymbol{q}\rightarrow}(c_{\boldsymbol{q}\rightarrow}^\dagger + c_{\boldsymbol{q}\rightarrow})(c_{\boldsymbol{q}\nearrow}^\dagger + c_{\boldsymbol{q}\nearrow}) \end{aligned}$$

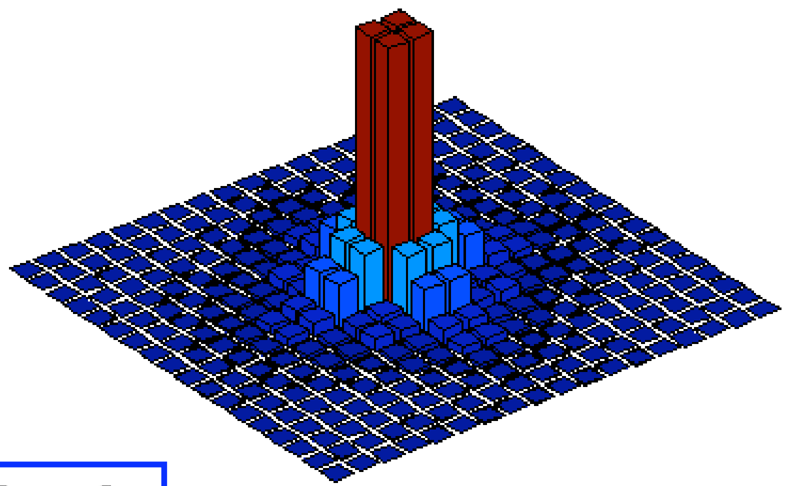
where on a plane

$$\mathsf{Y}_{\mathsf{q}} = I \qquad \mathsf{X}_{q_x, q_y} \equiv \prod_{q_y'=0}^{q_y - 1} \mathsf{W}_{q_x, q_y'}$$



$$H = \frac{1}{2} \sum_{\boldsymbol{q}\boldsymbol{q}'} \left[\begin{array}{cc} c_{\boldsymbol{q}}^\dagger & c_{\boldsymbol{q}} \end{array} \right] \left[\begin{array}{cc} \xi_{\boldsymbol{q}\boldsymbol{q}'} & \Delta_{\boldsymbol{q}\boldsymbol{q}'} \\ \Delta_{\boldsymbol{q}\boldsymbol{q}'}^\dagger & -\xi_{\boldsymbol{q}\boldsymbol{q}'}^T \end{array} \right] \left[\begin{array}{c} c_{\boldsymbol{q}'} \\ c_{\boldsymbol{q}'}^\dagger \end{array} \right]$$

$$\left[\begin{array}{cc} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{array} \right] = \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right] \left[\begin{array}{cc} E & \mathbf{0} \\ \mathbf{0} & -E \end{array} \right] \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right]$$



$$H = \sum_{n=1}^{M} E_n (a_n^\dagger a_n - \frac{1}{2})$$

$$\left[\begin{array}{cc} a_1^\dagger, ..., a_M^\dagger, & a_1, ..., a_M \end{array} \right] = \left[\begin{array}{cc} c_1^\dagger, ..., c_M^\dagger, & c_1, ..., c_M \end{array} \right] \left[\begin{array}{cc} U & V^* \\ V & U^* \end{array} \right]$$

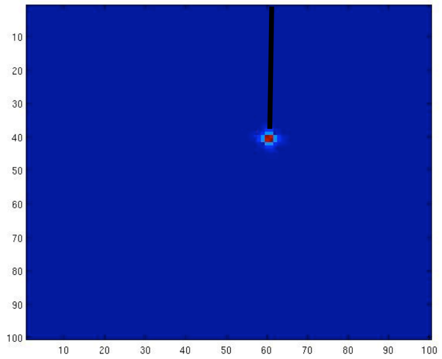
$$\boxed{\gamma_j = \sum_{n=1}^{N} \alpha_{jn} a_n^\dagger + \alpha_{j,n+N} a_n = \left[\begin{array}{cc} c_1^\dagger, ..., c_M^\dagger, & c_1, ..., c_M \end{array} \right] \left[\begin{array}{c} u_{\boldsymbol{q},j} \\ v_{\boldsymbol{q},j} \end{array} \right]}$$

Edges and their modes

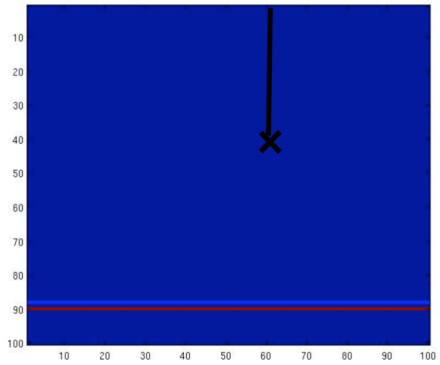
The non-Abelian phase of Kitaev models is topological insulator of the Bogoliubov-de Gennes class i.e. it is bulk insulator with conducting edges

$$a_n^\dagger = \mathcal{N} \sum_{\boldsymbol{q}} f(y-y_0) e^{\pm i k_x x} (e^{-i\theta/2} c_{\boldsymbol{q}}^\dagger + e^{+i\theta/2} c_{\boldsymbol{q}})$$

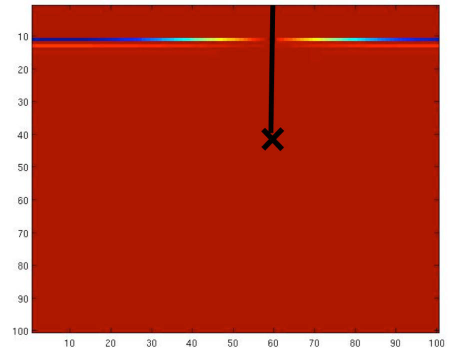
Vortex zero mode



Edge zero mode



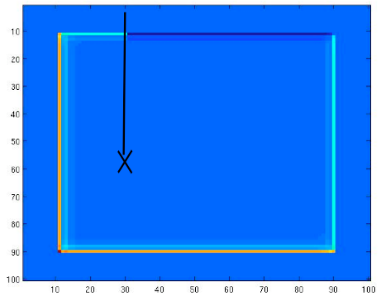
Chiral edge low energy mode



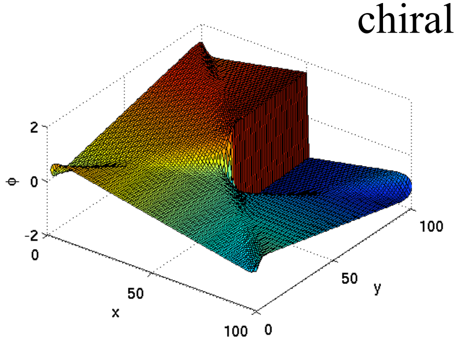
A non-Abelian domain surrounded by an Abelian domain:

edge zero mode

from bulk vortex



chiral edge low energy mode



phase dependence

around the domain

of 100x100 plaquettes

Non-Abelian fractional statistics: Berry phase

$$\mathcal{P}\exp\left\{i\oint \mathcal{A}(\lambda')d\lambda'\right\} := \lim_{M\to\infty} e^{i\mathcal{A}(\lambda_{M-1})\Delta\lambda}e^{i\mathcal{A}(\lambda_{M-2})\Delta\lambda}\cdots e^{i\mathcal{A}(\lambda_0)\Delta\lambda}$$

$$\lambda_M = \lambda_1 = \lambda(0) \qquad \Delta\lambda = \frac{|\lambda_M-\lambda_1|}{M}$$

$$[\mathcal{A}]^{ba}(\lambda')\Delta\lambda = i\langle\phi^b(\lambda)|\frac{d}{d\lambda}\phi^a(\lambda)\rangle\Delta\lambda = i\frac{\langle\phi^b(\lambda)|\phi^a(\lambda+\Delta\lambda)\rangle - \langle\phi^b(\lambda)|\phi^a(\lambda-\Delta\lambda)\rangle}{2}$$

Overlap calculations

|ϕ(i)> and |ϕ(0)> are the fermionic ground states with vortex at the positions *i* and 0 (reference) resp.

$$\begin{bmatrix}\gamma^\dagger_\leftrightarrow(i) & \gamma_\leftrightarrow(i)\end{bmatrix} := \begin{bmatrix}c^\dagger_\leftrightarrow & c_\leftrightarrow\end{bmatrix}\begin{bmatrix}U(i) & V^*(i)\\ V(i) & U^*(i)\end{bmatrix} \qquad \begin{bmatrix}\gamma^\dagger_\leftrightarrow(0) & \gamma_\leftrightarrow(0)\end{bmatrix} := \begin{bmatrix}c^\dagger_\leftrightarrow & c_\leftrightarrow\end{bmatrix}\begin{bmatrix}U(0) & V^*(0)\\ V(0) & U^*(0)\end{bmatrix}$$

$$\begin{bmatrix}\gamma^\dagger_\leftrightarrow(i) & \gamma_\leftrightarrow(i)\end{bmatrix} = \begin{bmatrix}\gamma^\dagger_\leftrightarrow(0) & \gamma_\leftrightarrow(0)\end{bmatrix}\begin{bmatrix}U(0,i) & V^*(0,i)\\ V(0,i) & U^*(0,i)\end{bmatrix}$$

$$\begin{aligned}U(i,0) &= U^\dagger(0)\,U(i) + V^\dagger(0)\,V(i)\\ V(i,0) &= V^T(0)\,U(i) + U^T(0)\,V(i)\end{aligned}$$

Thouless theorem

$$|\phi(i)\rangle = \sqrt{||\det U(i,0)||}\exp\left\{\frac{1}{2}\sum_{k,k'} Z_{kk'}(i,0)\gamma^\dagger_k(0)\gamma^\dagger_{k'}(0)\right\}|\phi(0)\rangle$$

where *Z*(i,0) is a skew-symmetric matrix given as

$$Z(i,0) = \left(V(i,0)U^{-1}(i,0)\right)^*$$

and **the overlap between two ground states** is

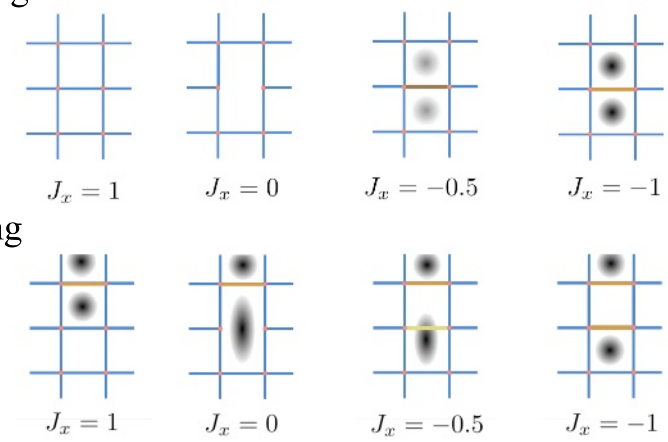
$$\langle\phi(i)|\phi(i+1)\rangle = \mathrm{Pf}\left(\begin{bmatrix}Z(i,0) & -I\\ I & -Z^*(i+1,0)\end{bmatrix}\right)$$

Braiding non-Abelian anyons

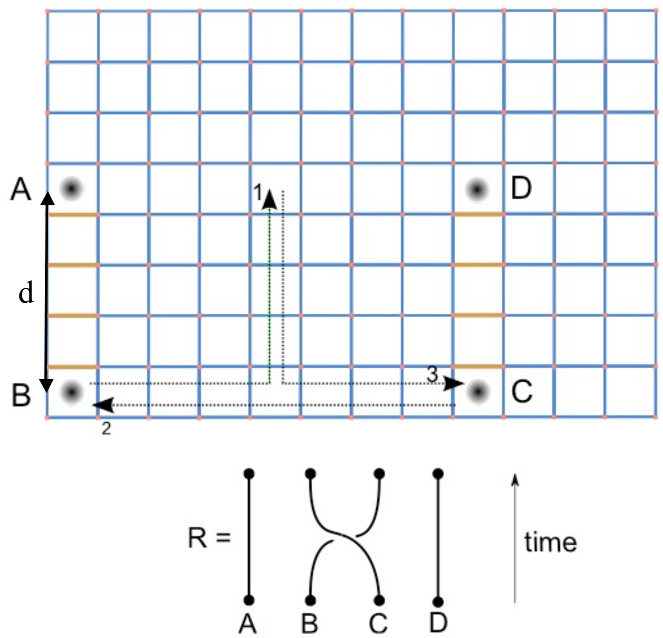
Vortices are non-Abelian anyons that can be continuously moved between lattice plaquettes by varying the spin interactions

creating vortices

moving



Braiding non-Abelian anyons



Calculation of the braiding matrix

basis: two states corresponding to two fusion channels of four vortices

$$|\psi_1^{\sigma\sigma}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\psi_\varepsilon^{\sigma\sigma}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Expectation

(pure Ising theory)

$$R = \begin{pmatrix} \exp\left\{i\frac{\pi}{8}\right\} & 0 \\ 0 & \exp\left\{-i\frac{3\pi}{8}\right\} \end{pmatrix} \cong \begin{pmatrix} 0.9239 + 0.3827i & \\ & 0.3827 - 0.9239i \end{pmatrix}$$

calculation (50000 steps)

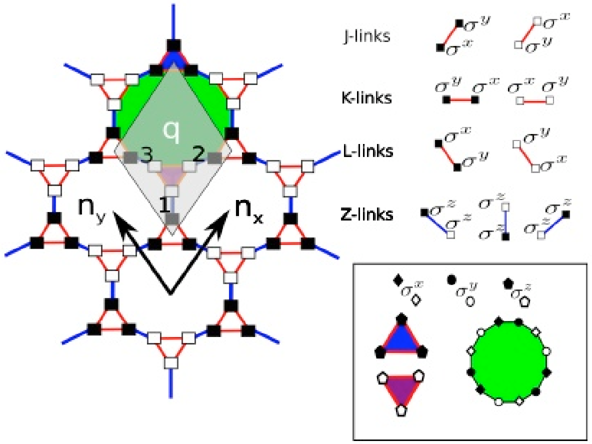
$$\mathrm{Eig}(B) = \begin{pmatrix} 0.9233 + 0.3841i & \\ & 0.3833 - 0.9236i \end{pmatrix}$$

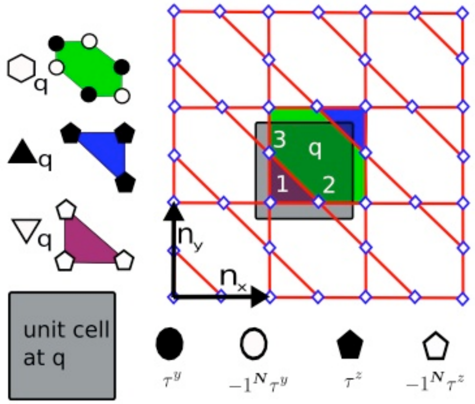
Frobenius norm (for d=9)

$$||R - \mathrm{Eig}(B)||_F = 3.7907 \times 10^{-6}$$

Other models: Yao-Kivelson

$$H = H_Z + H_J + H_K + H_L = -Z \sum_{Z \text{ links}} \sigma^z\sigma^z - J \sum_{J \text{ links}} \sigma^x\sigma^y - K \sum_{K \text{ links}} \sigma^x\sigma^y - L \sum_{L \text{ links}} \sigma^x\sigma^y.$$





G. Kells, D. Mehta, J.K. Slingerland, J. Vala, Phys. Rev. B **81**, 104429 (2010)

$$H_Z = Z\sum_{q,n} (2c^\dagger_{q,n}c_{q,n} - I)$$

$$H_J = J\sum_q [(c^\dagger_{q,1} - c_{q,1})(c^\dagger_{q,2} + c_{q,2}) + \square_{q\downarrow}(c^\dagger_{q,2} - c_{q,2})(c^\dagger_{q\rightarrow,1} + c_{q\rightarrow,1})]$$

$$H_K = K\sum_q \{-i\nabla_q(c^\dagger_{q,2} + c_{q,2})[(c^\dagger_{q,3} + c_{q,3}) - i\square_{q\downarrow}\blacktriangle_{q\downarrow}(c^\dagger_{q,2} - c_{q,2})(c^\dagger_{q\searrow,3} - c_{q\searrow,3})]\}$$

$$H_L = L\sum_q [(c^\dagger_{q,1} - c_{q,1})(c^\dagger_{q,3} + c_{q,3}) + (c^\dagger_{q,3} - c_{q,3})(c^\dagger_{q\uparrow,1} + c_{q\uparrow,1})]$$

$$\hexagon_q = \prod \tau^y_q$$

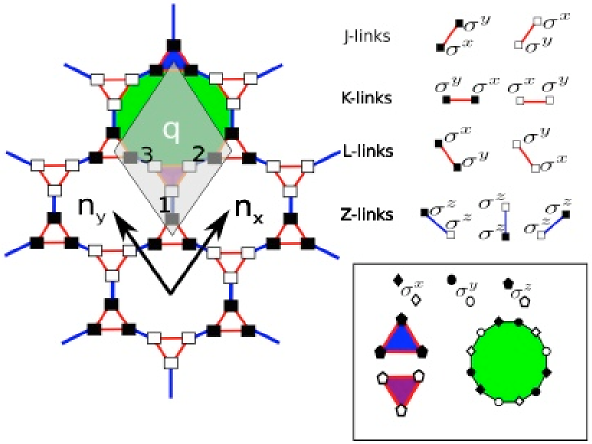
$$\triangledown_q = \prod_{\square} \tau^z_q$$

$$\blacktriangle_q = \prod_{\blacksquare} \tau^z_q$$

$$\boxed{|\,\mathrm{gs}\rangle = \prod_{\boldsymbol{k},n}(u_{\boldsymbol{k},n} + v_{\boldsymbol{k},n}a^\dagger_{\boldsymbol{k},n}a^\dagger_{-\boldsymbol{k},n})|\{\hexagon\},\{\triangledown\},\{\blacktriangle\},\{\emptyset\}\rangle}$$

Other models: Yao-Kivelson

$$H = H_Z + H_J + H_K + H_L = -Z\sum_{Z\ \text{links}} \sigma^z\sigma^z - J\sum_{J\ \text{links}}\sigma^x\sigma^y - K\sum_{K\ \text{links}}\sigma^x\sigma^y - L\sum_{L\ \text{links}}\sigma^x\sigma^y.$$

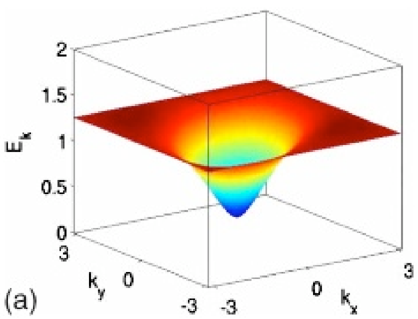


$$H = \frac{1}{2}\sum_{qq'} \left[c_q^\dagger \quad c_q\right] \begin{bmatrix} \xi_{qq'} & \Delta_{qq'} \\ \Delta^\dagger_{qq'} & -\xi^T_{qq'} \end{bmatrix} \begin{bmatrix} c_{q'} \\ c^\dagger_{q'} \end{bmatrix}$$

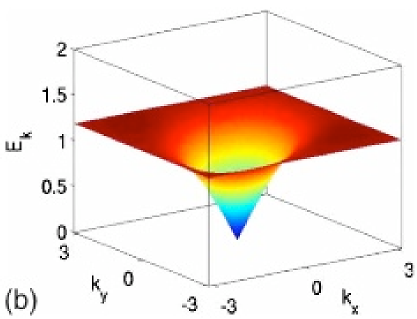
$$\boxed{|\,\mathrm{gs}\rangle = \prod_{\boldsymbol{k},n}(u_{\boldsymbol{k},n} + v_{\boldsymbol{k},n} a^\dagger_{\boldsymbol{k},n} a^\dagger_{-\boldsymbol{k},n})|\,\{\bigcirc\},\{\triangledown\},\{\blacktriangle\},\{\emptyset\}\rangle}$$

Dispersion relations:

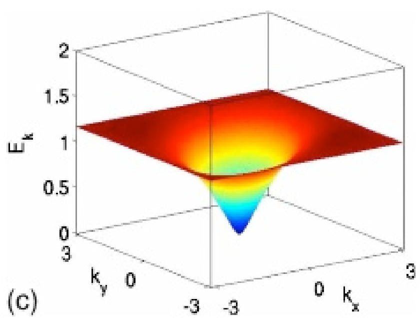
Abelian phase



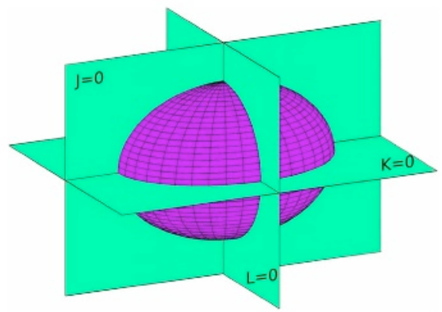
transition



Non-abelian phase



Phase diagram



Conclusions

Novel solution of the Kitaev honeycomb lattice model was presented which combines powerful wavefunction descriptions:

- BCS product
- stabilizer formalism

The novel solution represents a microscopic model of non-Abelian topological phase which

- provides closed expression for the ground state with the vacuum etate xplicitely given

$$|gs\rangle_{HC} = \prod \left(u_{\mathsf{k}} + v_{\mathsf{k}} c^{\dagger}_{\mathsf{k}} c^{\dagger}_{-\mathsf{k}}\right) \boxed{|\{Q_{\mathsf{q}}\}, \{\emptyset\}\rangle}$$

$$|gs\rangle_{HC} = \prod_{\mathsf{k}} \left(u_{\mathsf{k}} + v_{\mathsf{k}} c^{\dagger}_{\mathsf{k}} c^{\dagger}_{-\mathsf{k}}\right) \boxed{|\{Q_{\mathsf{q}}\}, l_x^{(0)}, l_y^{(0)}, \{\emptyset\}\rangle}$$

- yields important insighst into the relatins between the toric code and the Ising non-Abelian phase
- generalizes to other models, e.g. Yao-Kivelson model
- allows calculation of the vortex states and edge states
- allows direct calculation of the non-Abelian fractional statistics

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