Nordita Workshop:

Quantum Matter in Low Dimensions

Unusual singular behavior of the entanglement entropy in one dimension

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- JPA 41, 2530 (2008)
- arXiv:1002.2931
- arXiv:1008.3892
- work in progress…

Outline

- Introduction: Von Neumann and Renyi Entropy as a measure of Entanglement
- Entanglement Entropy in 1-D systems
- XY in magnetic field & XYZ in zero field
- Essential Critical Point for the entropy
- Quantum Entropy in integrable models
- Conclusions

Introduction

- Classical Computing $\rightarrow$ Boolean Logic $\rightarrow$ Bits
- Quantum Information $\rightarrow$ Q-Bits $\rightarrow$ Entanglement
- How to define/measure it?
- New challenges for our understanding of Nature

Understanding Entanglement

- Consider a unique (pure) ground state
- Divide the system into two Subsystems: A & B
- If system wave-function is:

$$|\Psi^{A,B}\rangle = |\Psi^{A}\rangle \otimes |\Psi^{B}\rangle$$

$\rightarrow$ No Entanglement

(Measurements on B do not affect A state)

Understanding Entanglement (cont.)

- If the system wave-function is:

$$|\Psi^{A,B}\rangle = \sum_{j=1}^{d} \lambda_j |\Psi_j^A\rangle \otimes |\Psi_j^B\rangle$$

(with d > 1, $|\Psi_j^A\rangle$ & $|\Psi_j^B\rangle$ linearly independent):

→ Entangled (Measurements on B affect A state):

i.e. $$\langle\Psi_i^B|\Psi^{A,B}\rangle = \lambda_j |\Psi_i^A\rangle$$

Simple Example

- Two spins 1/2 in triplet state $\rightarrow S_z = 1$:

$$|\uparrow\rangle \otimes |\uparrow\rangle$$

No entanglement

- Middle component with $S_z = 0$:

$$|\uparrow\rangle \otimes |\downarrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle$$

Maximally entangled

How to measure Entanglement?

- Compute Density Matrix of subsystem:

$$\rho_A = tr_B \left(|\Psi^{A,B}\rangle\langle\Psi^{A,B}| \right)$$

- Entanglement for pure state as Quantum Entropy (Bennett, Bernstein, Popescu, Schumacher 1996):

$$S = -tr_A \left(\rho_A \ln \rho_A \right)$$

Von Neumann Entropy

More Entanglement Estimators

- Von Neumann Entropy: $S_A = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$
- Renyi Entropy: $S_\alpha = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$

  (equal to Von Neumann for $\alpha \to 1$)
- Tsallis Entropy
- Concurrence (Two-Tangle)
- …

The Von Neumann Entropy

$$S_A = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

- Quantum analog of Shannon Entropy:

$$\rho_A = \sum \lambda_j |\Psi_j^A\rangle \langle\Psi_j^A| \quad \rightarrow \quad S_A = -\sum \lambda_j \log \lambda_j$$

- Measures the amount of "information" in the given state
- Schumacher's Theorem: information in a **N-Dim** state *A* can be compressed in a **S_A-Dim** state

Entropy as a measure of entanglement

- Assume Bell State as unity of Entanglement:

$$|\text{Bell}\rangle = \frac{|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle}{\sqrt{2}}$$

- Von Neumann Entropy measures how many Bell-Pairs are contained in a given state $|\Psi^A\rangle$ (i.e. closeness of state to maximally entangled one)

Entanglement in a Spin Chain

- Consider the Ground state of a Hamiltonian:

$$H = \sum_{i=1}^{N} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - h \sum_i \sigma_i^z$$

- Block of spins in the space interval [1, n] is subsystem A
- The rest of the ground state is subsystem B.

→ Entanglement of a block of spins in the space interval [1, n] with the rest of the ground state <u>as a function of n</u>

Entropy as a Correlation Function

- We study the bi-partite entropy of the ground state of a system: $\rho_A = \mathrm{tr}_B |\Psi^{A,B}\rangle \langle \Psi^{A,B}|$

$$S(n) = -\mathrm{tr}\left(\rho_A \ln \rho_A\right)$$ Von Neumann

$$S_\alpha(n) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$ Renyi

- Multi-Point correlation function with contributions from all two-point correlators
- Highly non-trivial correlation function: new insights?

General Behavior (Area Law)

- We study the behavior for block size $n \to \infty$

(Double scaling limit: $0 << n << N$)

$$S(n) = -\mathrm{tr}\left(\rho_A \log \rho_A\right)$$

- For gapped phases: (Vidal, Latorre, Rico, Kitaev 2003)

$$S(n) \simeq \text{Constant} + \dots$$

- For critical conformal phases: (Calabrese, Cardy 2004)

$$S(n) \simeq \frac{c}{3} \ln n + \dots$$

CFT & Entropy

$$S_\alpha(l) = \frac{1}{1-\alpha} \ln \mathrm{tr}\left(\rho_A^\alpha\right)$$

- Powers of ρ easily accessible in CFT (replica)

$$S_\alpha(l) \simeq \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln l + c'_\alpha + b_\alpha l^{-2x/\alpha} + \dots$$

Conjectured

- Close to criticality: $\xi \sim \Delta^{-1}$, $l \to \infty$

(Calabrese, Cardy, Peschel 2010)

$$S_\alpha \simeq \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln \xi + C'_\alpha + B_\alpha \xi^{-2x/\alpha} + \dots$$

Entanglement and Integrability

$$H = \sum_{i=1}^{N} \left[J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + J_z \sigma_i^z \sigma_{i+1}^z \right] - h \sum_i \sigma_i^z$$

- Take two integrable chains

    1) XY in transverse field ($J_z = 0$)

    2) XYZ in zero field ($h = 0$)

- We study the entanglement in the $n \to \infty$ limit

$\to$ entanglement of one half-line with the other

The Anisotropic XY Model

$$H = -\sum_i \left[\left(1+\gamma\right) \sigma_i^x \sigma_{i+1}^x + \left(1-\gamma\right) \sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

↓

Exact (non-local) mapping into free fermions

(Jordan-Wigner + Bogolioubov rotation)

↓

$$H = \sum_q \varepsilon_q \left(\chi_q^\dagger \chi_q - 1/2 \right) \qquad \varepsilon_q = \sqrt{(h/2 - \cos q)^2 + \gamma^2 \sin^2 q}$$

- Entanglement NOT of free fermions, but can be calculated using Toeplitz Determinants (Its et al. 2005)

Phase Diagram of the XY Model

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

$$\varepsilon_q = \sqrt{(h/2-\cos q)^2 + \gamma^2 \sin^2 q}$$

Phase Diagram:

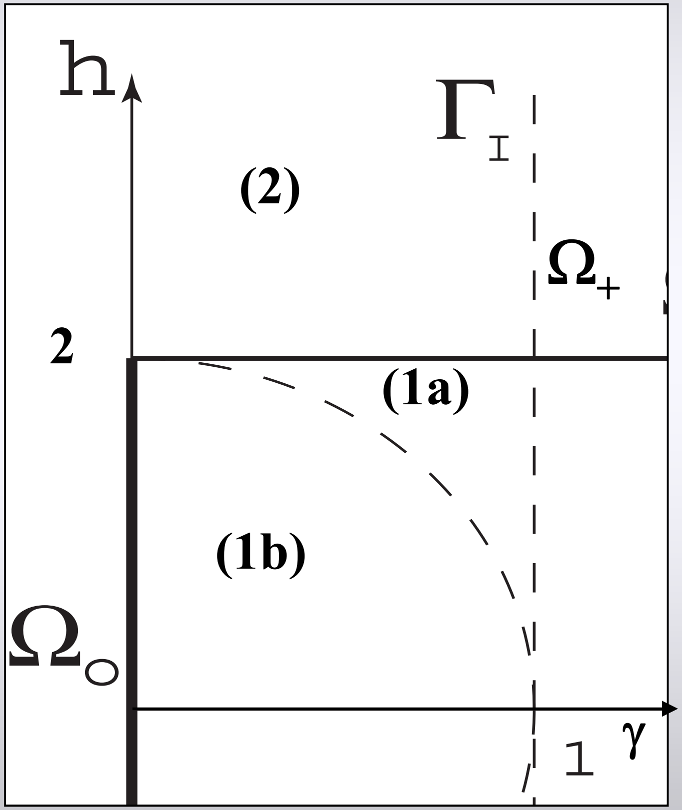
- 3 non-critical regions (2,1a,1b)
- 2 critical phases:
  - Ω_0: Isotropic XY
  - Ω_+: Critical magnetic field

For h < 2 (1a,1b):

Spectrum Doubly Degenerate:

- $|GS>_+$: Even # Excitations
- $|GS>_-$: Odd # Excitations

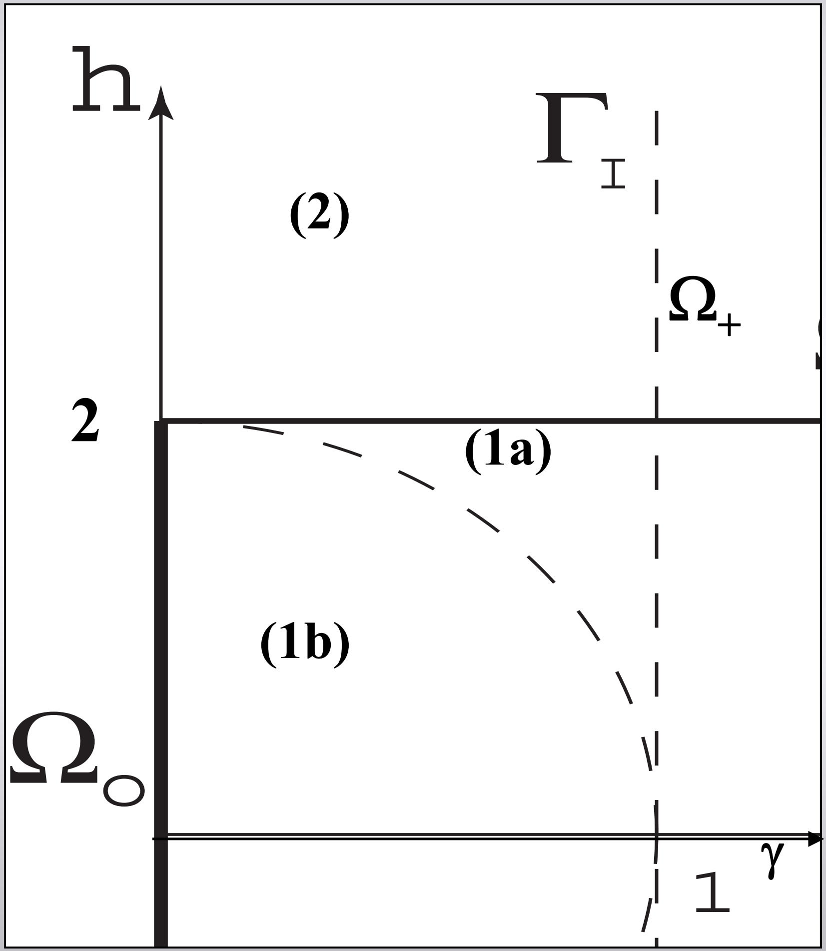
We consider only $|GS>_+$



Phase Diagram of the XY Model
(only γ>0 shown)

Asymptotic Entropy of the XY model

- We have completely analytical expressions for the asymptotic entropy
- But less discuss physics first and math later!
- (Exact formulae will come, have faith)



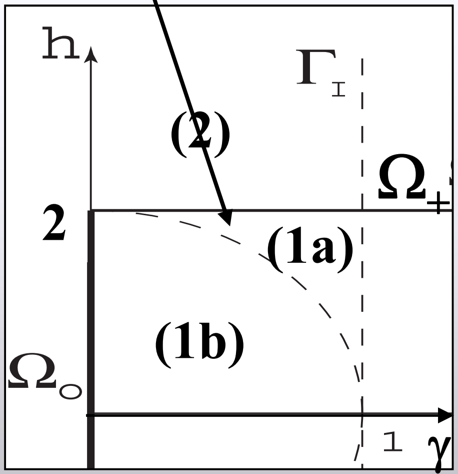
Minima of the Entropy

- Absolute minimum at $\mathbf{h} \to \infty$ or $\gamma \to 0$ $(\mathbf{h} > 2)$: $\mathbf{S}_\infty \to \mathbf{0}$
  as the ground state becomes ferromagnetic ($\uparrow \ldots \uparrow$)
- Local minimum $\mathbf{S}_\infty = \ln 2$
  at the boundary between cases **1a** and **1b** ($h = 2\sqrt{1-\gamma^2}$)
- The ground states are factorized:
  (each state is factorized and has no entropy)

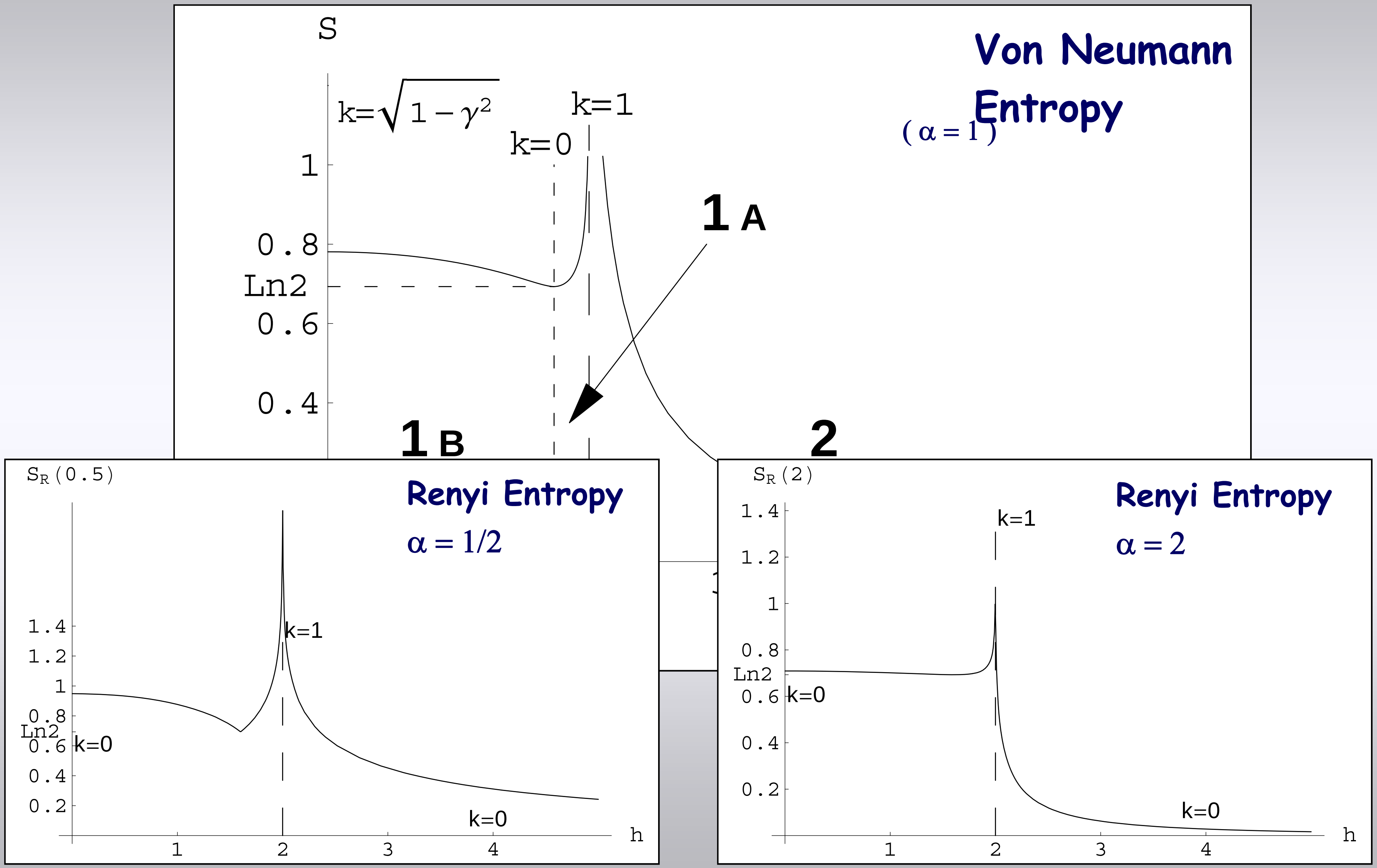
$$|GS_1\rangle = \prod_{n\in\text{lattice}} \left[\cos(\theta)|\uparrow_n\rangle + \sin(\theta)|\downarrow_n\rangle\right]$$

$$|GS_2\rangle = \prod_{n\in\text{lattice}} \left[\cos(\theta)|\uparrow_n\rangle - \sin(\theta)|\downarrow_n\rangle\right]$$

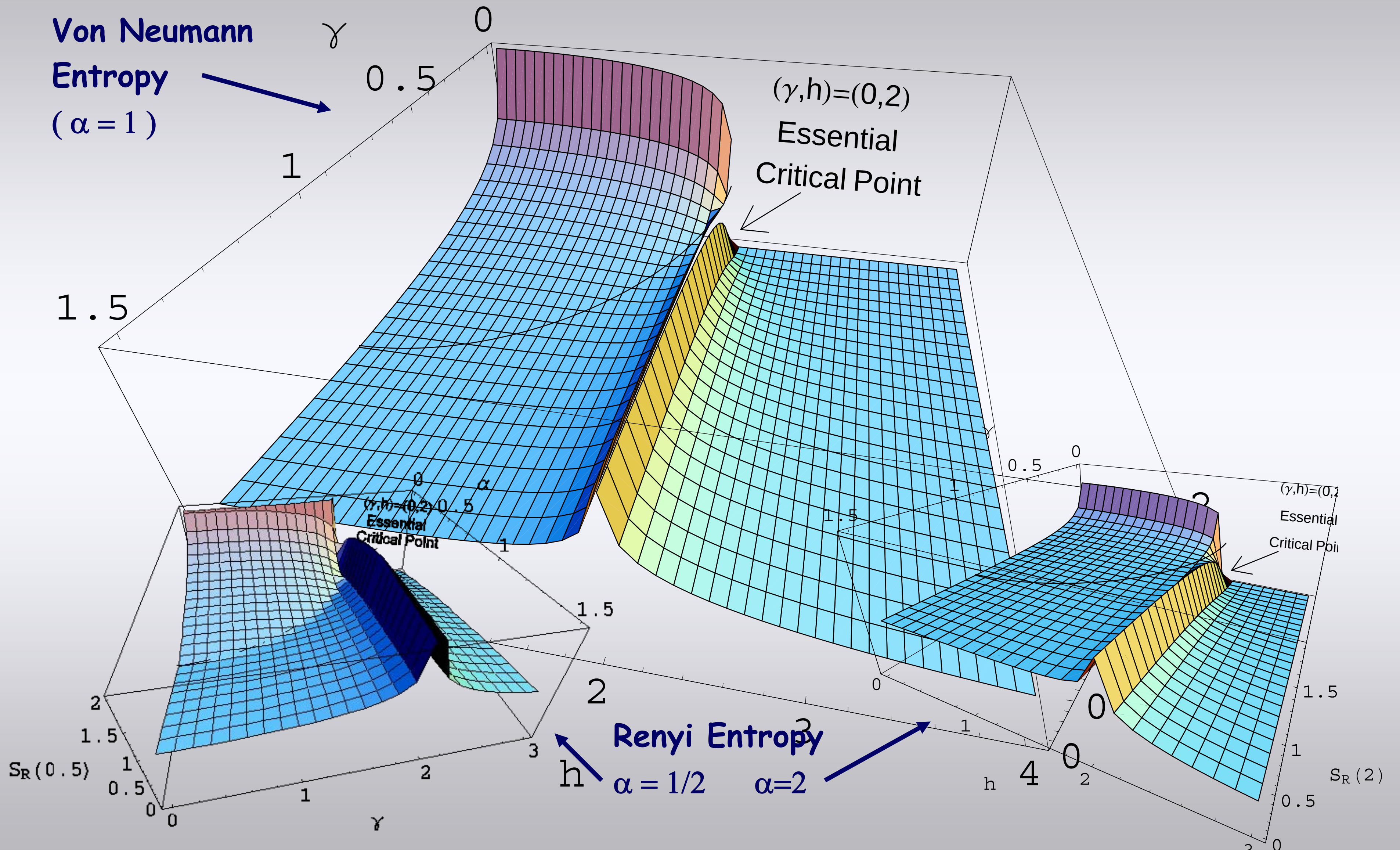
$$\cos^2(2\theta) = (1-\gamma)/(1+\gamma)$$



Entropy at fixed γ



3-D plot of the Entropy



The Essential Critical Point (ECP)

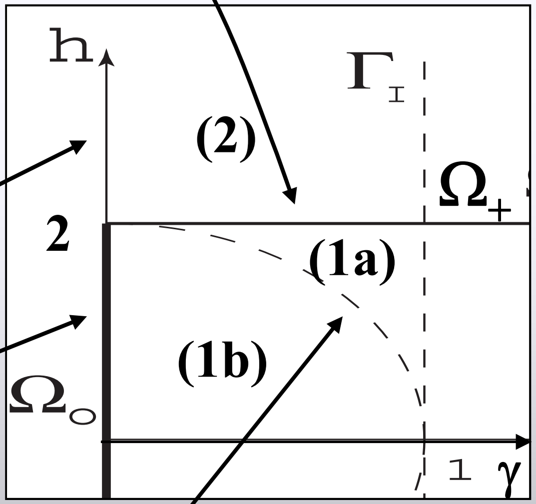
- Point $(\gamma,h)=(0,2)$ is a bi-critical point:
- Theory is critical, but not conformal (quadratic spectrum)
- Let's study the entropy close to this point:

  – Approaching it along $h=2,\ \gamma>0$: $S_\infty = \infty$

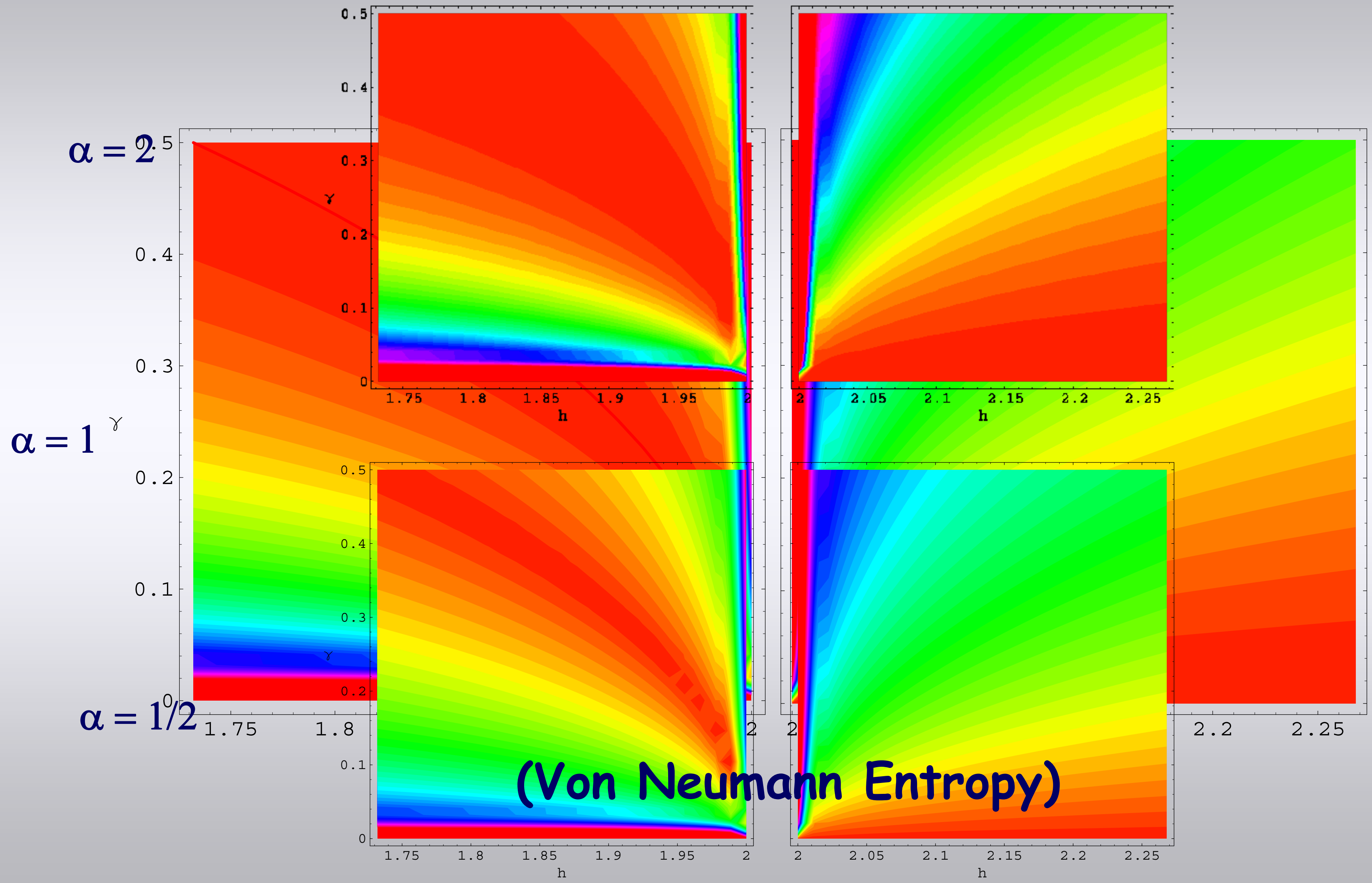
  – Approaching it along $\gamma=0,\ h>2$: $S_\infty = 0$

  – Approaching it along $\gamma=0,\ h<2$: $S_\infty = \infty$

  – Approaching it along $h = 2\sqrt{1-\gamma^2}$: $S_\infty = \ln 2$

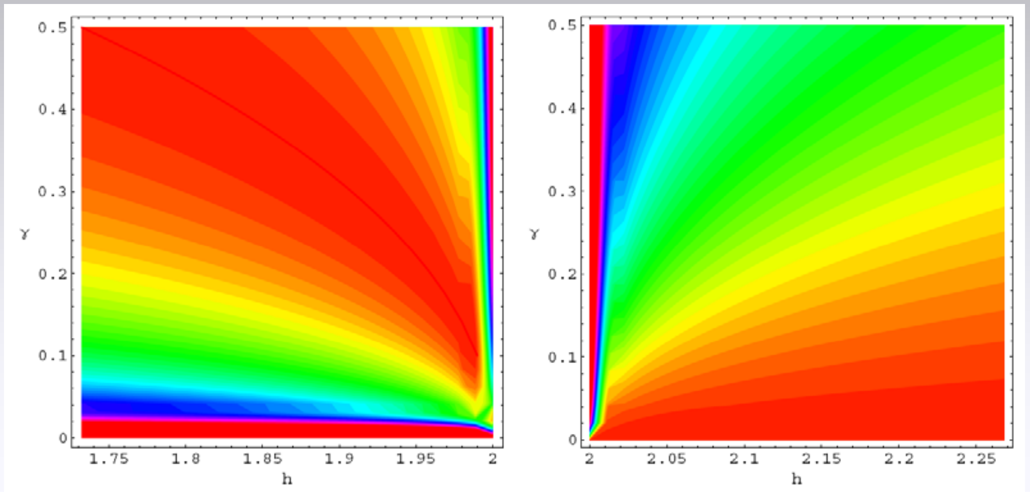


Entropy around the ECP



Curves of constant Entropy

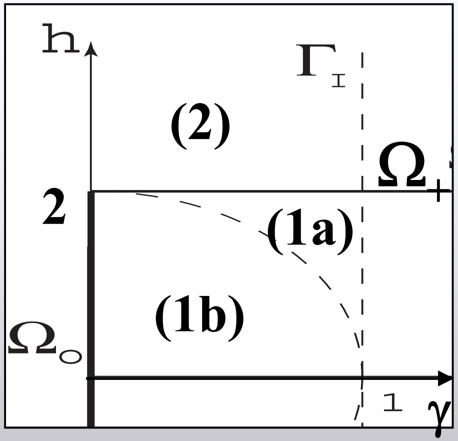
- Curves of constant entropy are simple Hyperbolae and Ellipses:



$$\text{Case } 2 \quad (h > 2): \qquad \left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad 0 \leq \kappa < \infty$$

$$\text{Case } 1a \quad (2\sqrt{1-\gamma^2} < h < 2): \qquad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa > 1$$

$$\text{Case } 1b \quad (h < 2\sqrt{1-\gamma^2}): \qquad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa < 1.$$

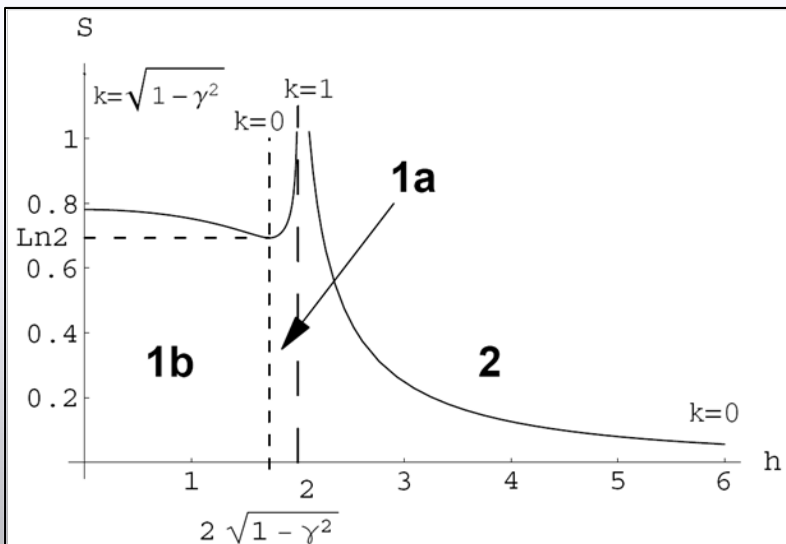


All these curves pass through the Essential Critical Point!

Importance of the Essential Critical Point

- From any point in the phase diagram one reaches the ECP following a curve of constant Entropy
- The range of the Entropy in the phase diagram is the positive real axis

⇒ Near the ECP the Entropy reaches every positive value!



- Small variations in the parameters change the Entropy dramatically
- ECP is an essential singularity for the entropy

Entropy on the critical phases

Phase transitions: as the gap closes $S_\infty \to +\infty$

$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\left(\frac{1}{12}\ln|h-2| + \frac{1}{3}\ln 4 + \frac{1}{6}\ln\gamma\right) + O\left(|h-2|\ln^2|h-2|\right)$$

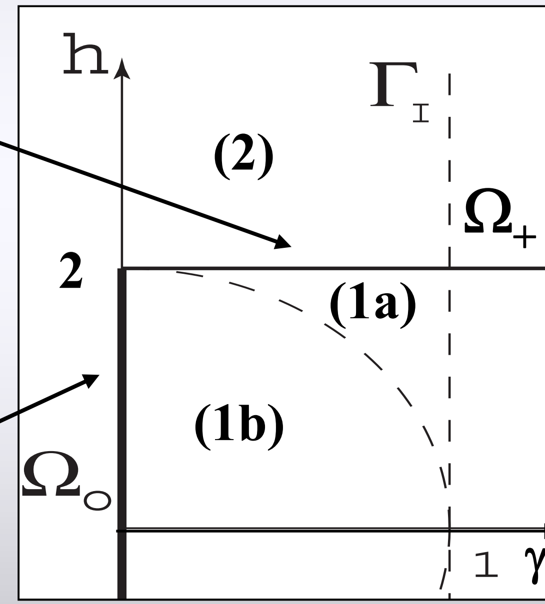
$$S_\infty \to -\frac{1}{6}\ln|2-h| + \frac{1}{3}\ln 4 + \frac{1}{6}\ln\gamma + O(h-2)$$

$h \to 2$ and $\gamma \neq 0$

Critical Magnetic Field:
(Calabrese, Cardy, 2004)

Isotropic XY model (XX Model):
(Jin, Korepin 2003)

$\gamma \to 0$ and $0 < h < 2$



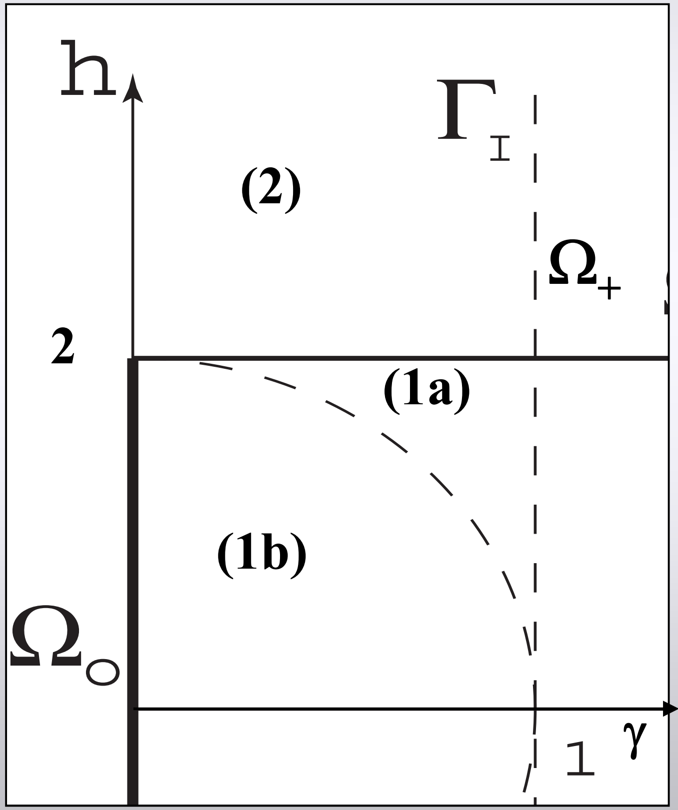
$$S_R(\alpha) = \frac{1+\alpha}{\alpha}\left(-\frac{1}{6}\ln\gamma + \frac{1}{12}\ln(4-h^2) + \frac{1}{6}\ln 2\right) + O(\gamma\ln^2\gamma)$$

$$S_\infty \Longrightarrow -\frac{1}{3}\ln\gamma + \frac{1}{6}\ln(4-h^2) + \frac{1}{3}\ln 2 + O(\gamma\ln^2\gamma)$$

XY Model Recap

$$H = -\sum_i \left[(1+\gamma)\sigma_i^x \sigma_{i+1}^x + (1-\gamma)\sigma_i^y \sigma_{i+1}^y + h\,\sigma_i^z \right]$$

- Saturates in gapped phases
- Correct logarithmic scaling close to conformal points
- Essential Singularity close to non-conformal bi-critical point



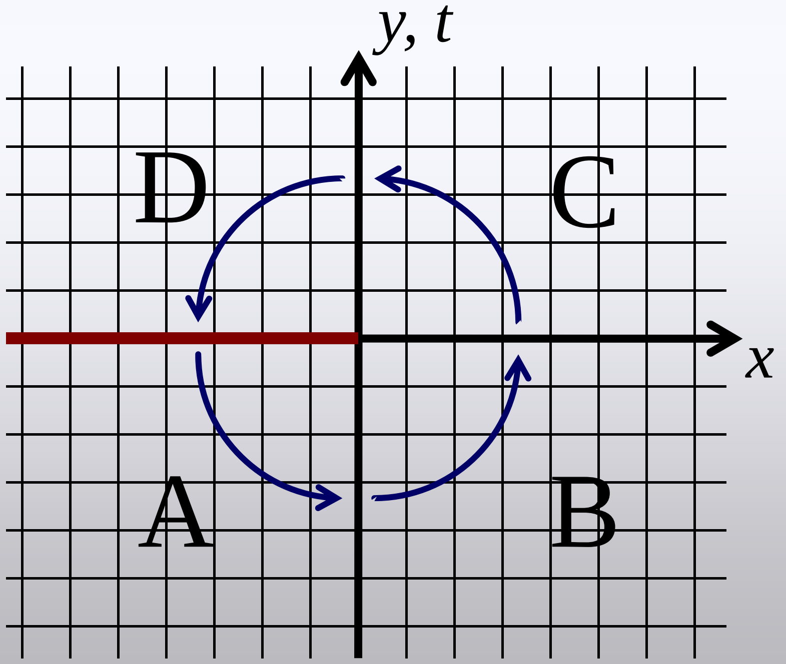
XYZ Spin Chain

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

- Commutes with transfer matrices of 8-vertex model
- Use of Baxter's Corner Transfer Matrices (CTM)

$$\mathcal{Z} = \mathrm{tr}\,(ABCD)$$

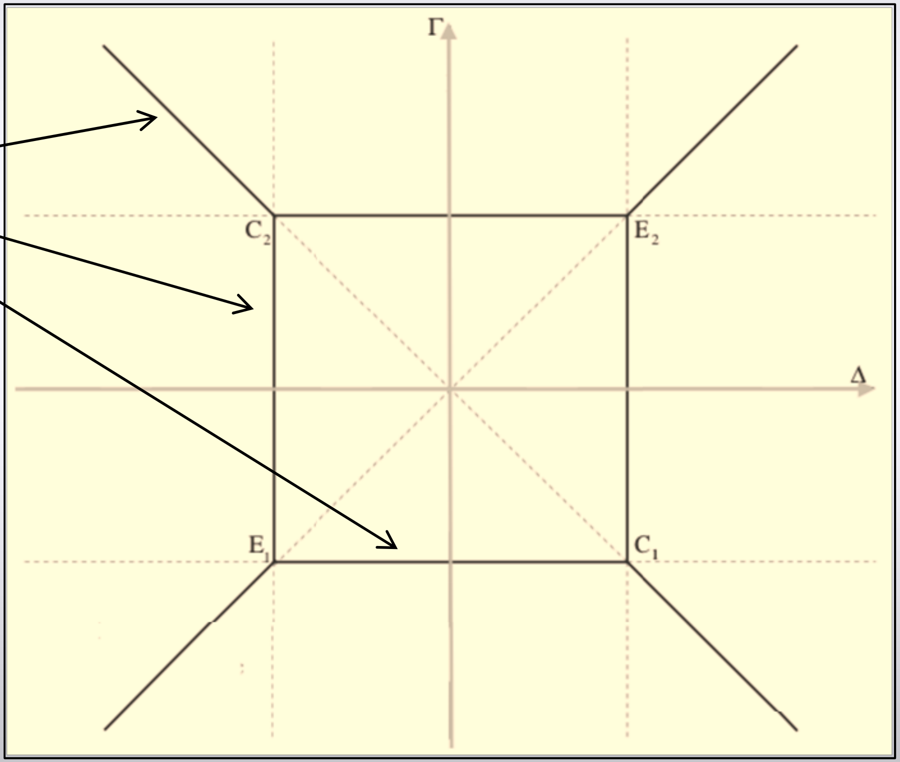
$$\rho_{\sigma,\sigma'} = (ABCD)_{\sigma,\sigma'}$$



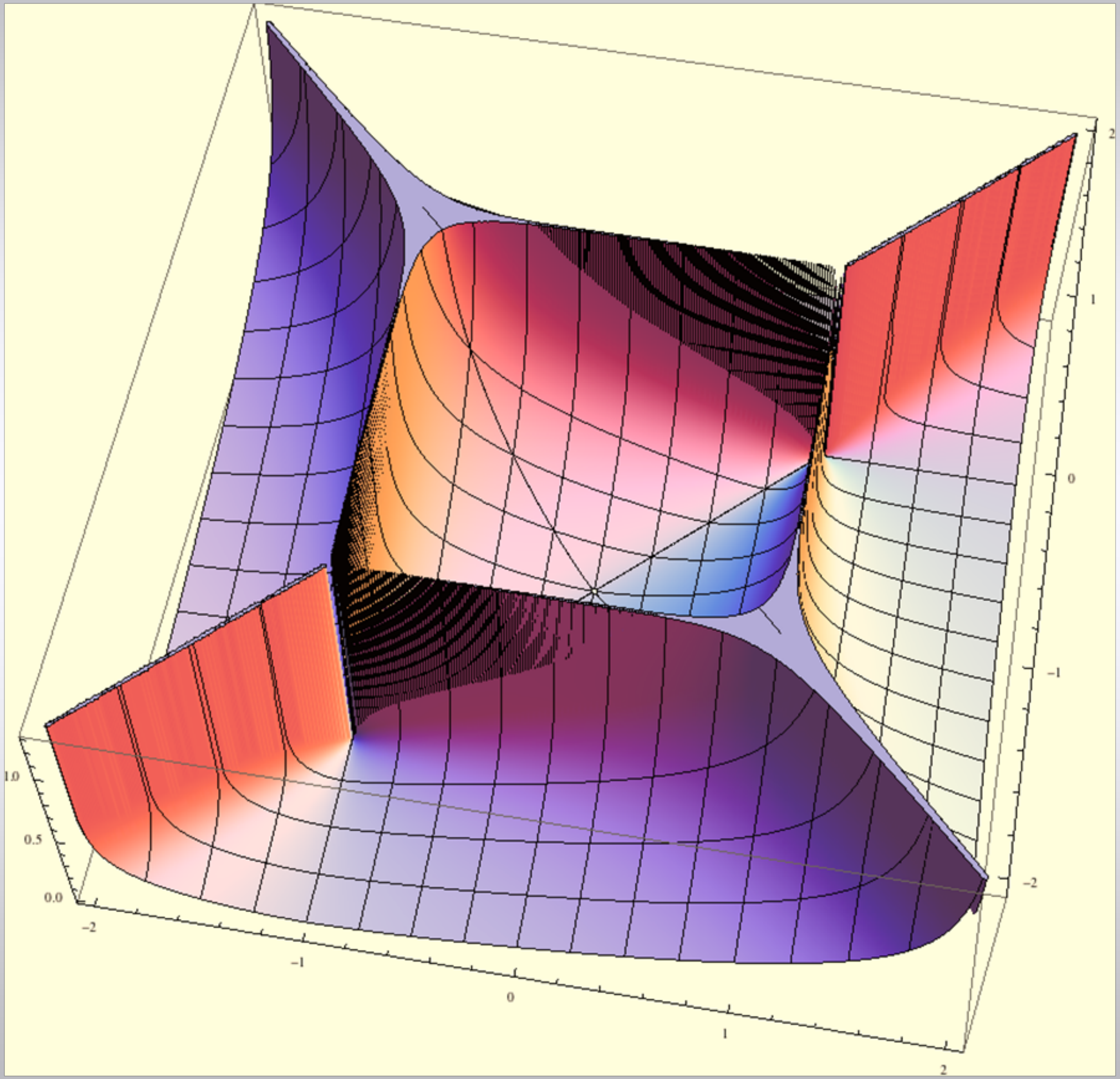
Phase Diagram of XYZ model

$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

- Gapped in bulk of plane
- Critical on dark lines (rotated XXZ paramagnetic phases)
- 4 "tri-critical" points:
  $C_{1,2}$ conformal
  $E_{1,2}$ quadratic spectrum

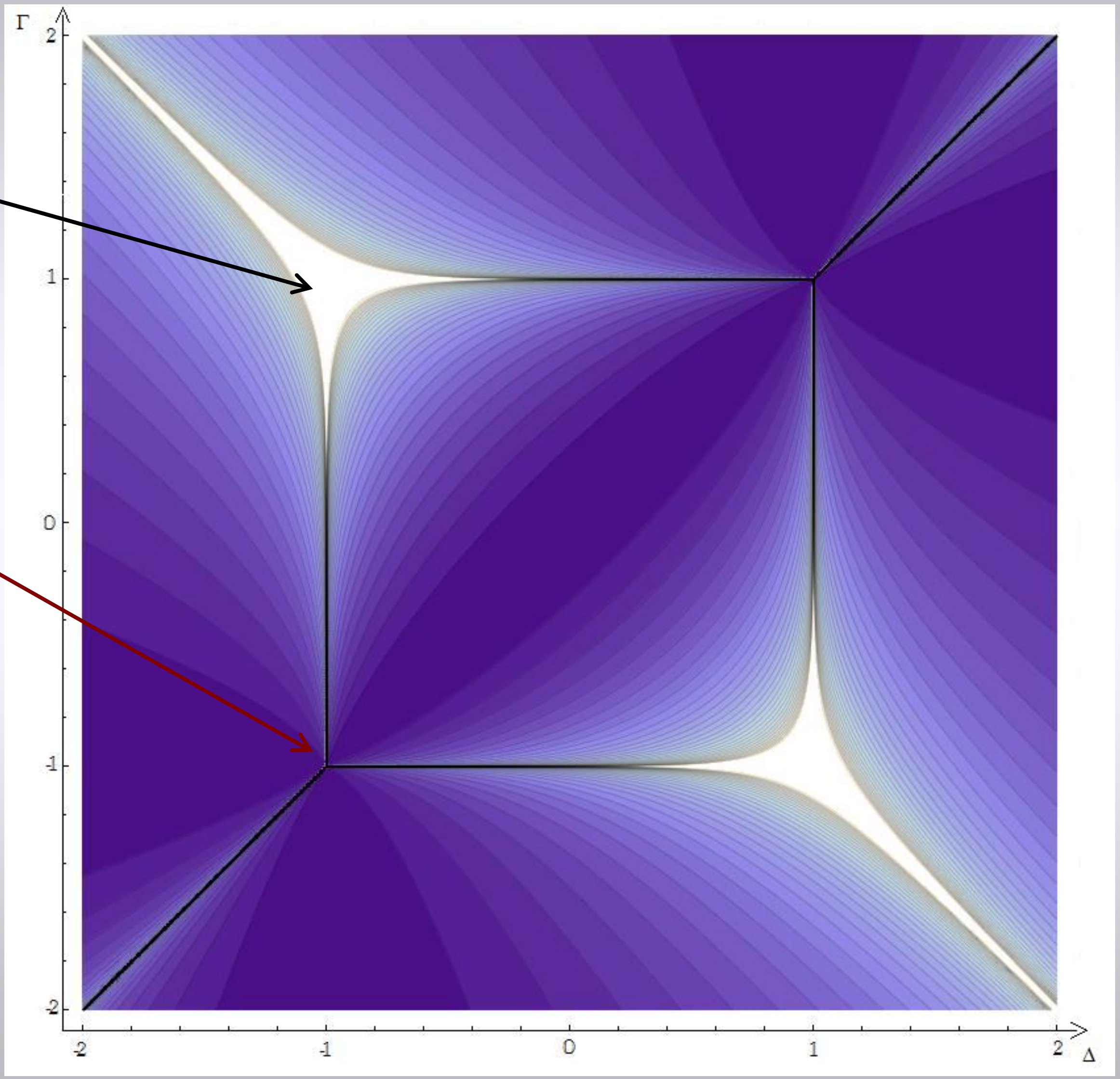


3-D plot of entropy



Iso-Entropy lines

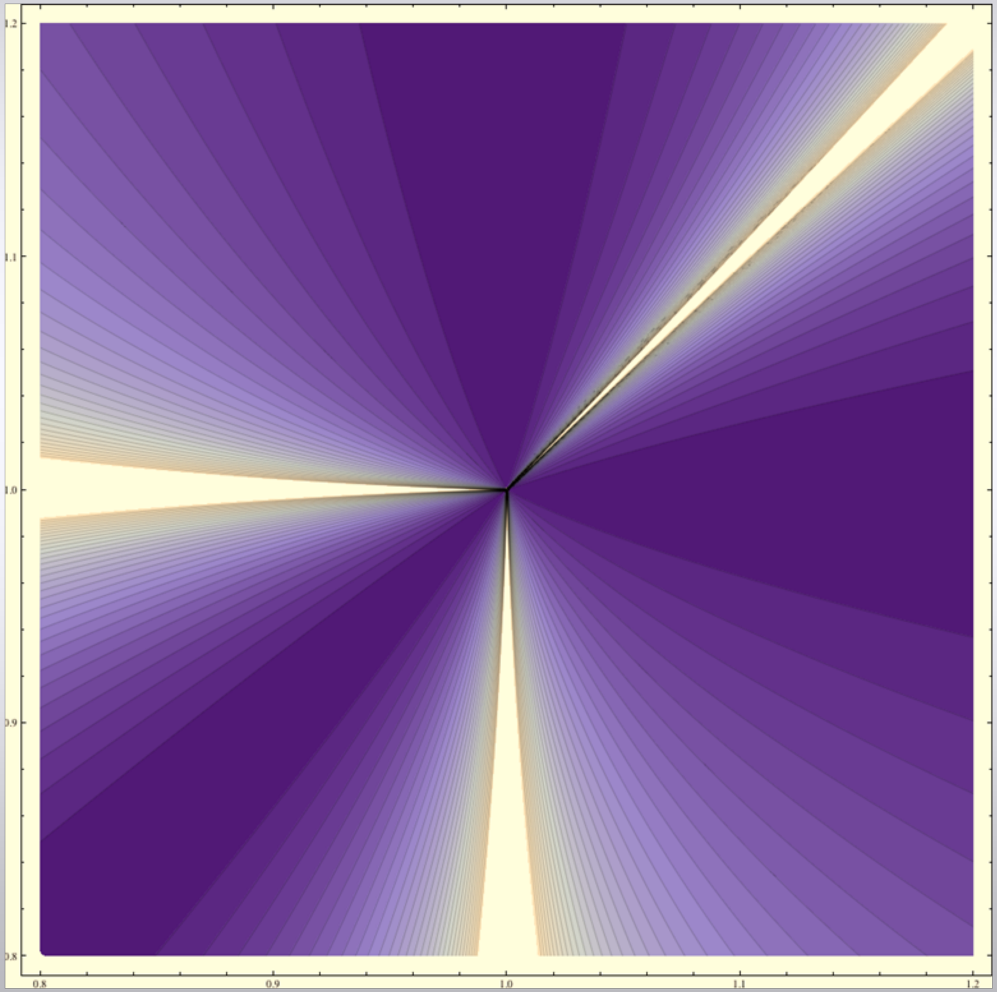
- Conformal point: entropy diverges close to it
- Non-conformal point (ECP): entropy goes from 0 to ∞ arbitrarily close to it (depending on direction)



Close-up to non-conformal point

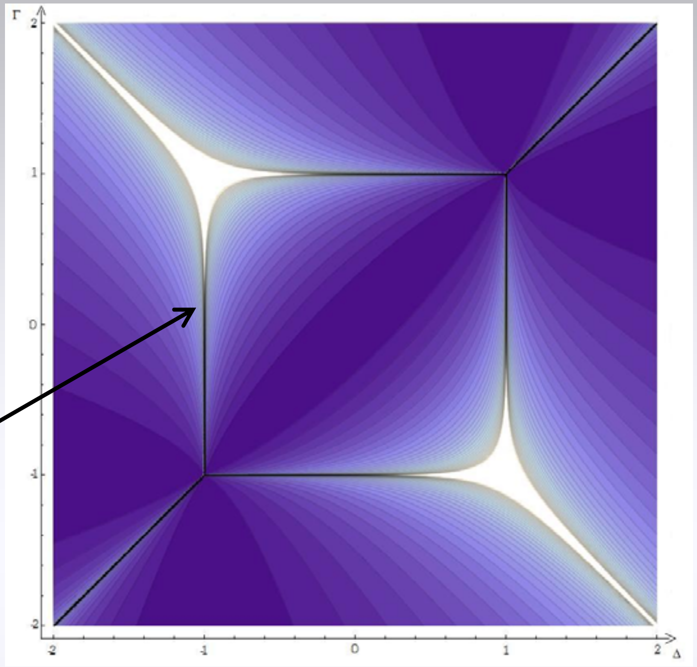
$$H_{XYZ} = -\sum_j \left(\sigma_j^x \sigma_{j+1}^x + \Gamma \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right)$$

- Isotropic Ferromagnetic Heisenberg: quadratic spectrum
- Curves of constant entropy pass through it
- Same physics as XY model



Conformal check

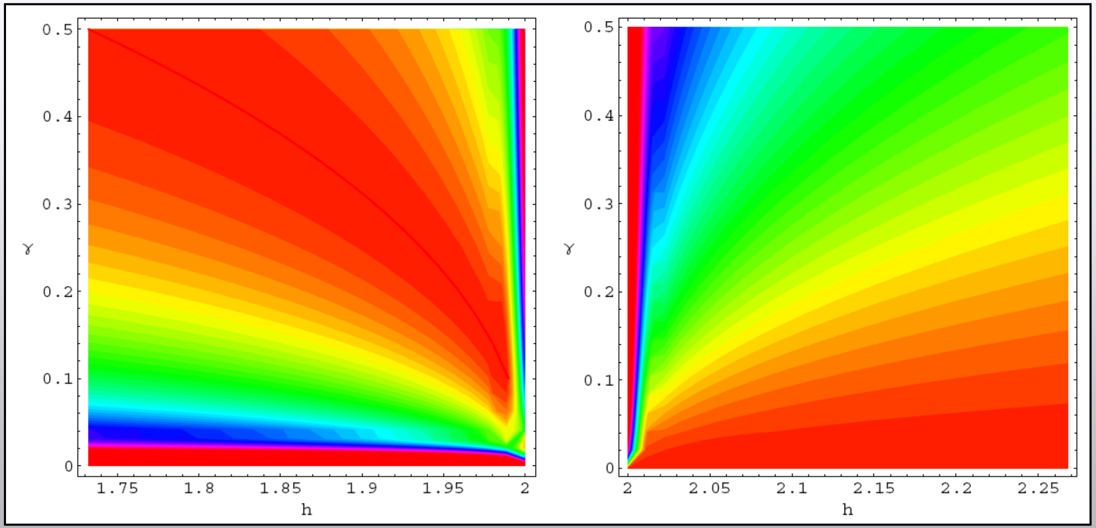
- Expansion close to conformal point agree with expectations:

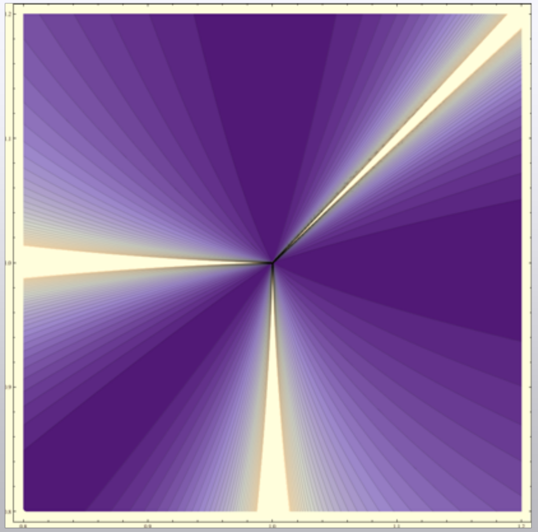


$$S_\alpha = \frac{1}{12}\left(1+\frac{1}{\alpha}\right)\ln(\xi) - \frac{1}{24}\left(11-\frac{1}{\alpha}\right)\ln 2 + \frac{\alpha}{6(1-\alpha)}\left[\frac{\xi^{-2}}{4}+\frac{\xi^{-4}}{256}+\mathrm{Ord}(\xi^{-6})\right] - \frac{1}{6(1-\alpha)}\left[4(4\xi)^{-2/\alpha}+(4\xi)^{-4/\alpha}+\mathrm{Ord}\left(\xi^{-6/\alpha}\right)\right]$$

Physics round-up

- Gapped phase saturates
- Close to conformal points: logarithmic divergence
- Close to non-conformal points: essential singularity





The Reduced Density Matrix

- All spin 1/2 integrable chain systems have the same structure for ρ (Baxter's Book, Peschel et al 2009, …):

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x_3 \end{pmatrix} \otimes \ldots$$

$$x_j = \begin{cases} \mathrm{e}^{-2j\epsilon}, & \text{ordered} \\ \mathrm{e}^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

where ϵ is characteristic of the model

Entanglement Spectrum

$$\rho = \bigotimes_{j=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & x_j \end{pmatrix} \qquad x_j = \begin{cases} \mathrm{e}^{-2j\epsilon}, & \text{ordered} \\ \mathrm{e}^{-(2j-1)\epsilon}, & \text{disordered} \end{cases}$$

- Eigenvalues form a geometric series
- Degeneracies from partitions of integers

  (Okunishi et al. 1999; Franchini et al. 2010)

- All these models have the same entanglement spectrum

$$\rho = \mathrm{e}^{-\mathcal{H}_{\text{Entanglement}}}$$

$\to \mathcal{H}_{\text{Entanglement}}$: free fermions with spectrum ϵ

Renyi Entropy

$$\mathcal{S}_\alpha = \frac{\alpha}{\alpha-1}\sum_{j=1}^{\infty}\ln\left(1+q^{2j}\right) + \frac{1}{1-\alpha}\sum_{j=1}^{\infty}\ln\left(1+q^{2j\alpha}\right)$$

$$q \equiv e^{-\epsilon}$$

- Series can be summed into elliptic theta functions:

$$\begin{aligned}\mathcal{S}_\alpha(\epsilon) &= \frac{\alpha}{6(1-\alpha)}\ln\frac{\theta_4(0,q)\theta_3(0,q)}{\theta_2^2(0,q)} \\ &\quad + \frac{1}{6(1-\alpha)}\ln\frac{\theta_2^2(0,q^\alpha)}{\theta_3(0,q^\alpha)\theta_4(0,q^\alpha)} - \frac{\ln(2)}{3}\ .\end{aligned}$$

- Analytical properties suitable for studying entropy as a function of ϵ and α (common to all integrable theories)
- ϵ dependence on microscopical parameter to study in the whole phase diagram

Conclusions

- We studied analytically the entropy (**Von Neumann** and **Renyi**) as a measure of bipartite entanglement for the XY and XYZ model
- Entropy diverges at critical phases, saturates to constant in gapped
- Near conformal points, entropy diverges logarithmically
- Near (multi-critical) non-conformal points, entropy reaches every positive value : essential singularity (ECP)

→ Universality close to ECPs?

- Entanglement spectrum common to all integrable models

→ Elliptic and modular properties