Probing non-Abelian statistics with quantum Hall interferometry

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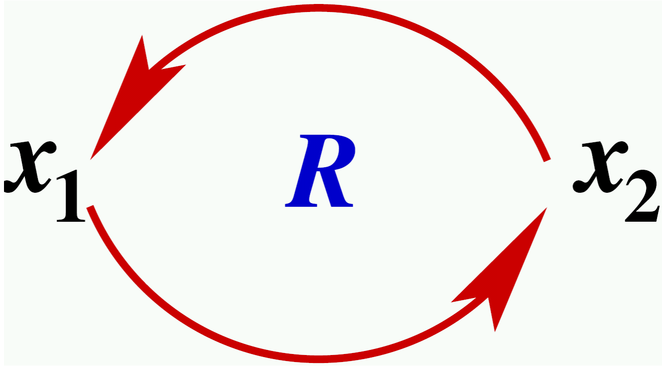
Quantum exchange statistics

- Two quantum-mechanical particles are located at x_1 & x_2

  What happens to their wavefunction when the particles are exchanged?

W. Pauli



$$\hat{R}\Psi(\mathsf{x}_1, \mathsf{x}_2) = \Psi(\mathsf{x}_2, \mathsf{x}_1)$$

$$\Psi(\mathsf{x}_2, \mathsf{x}_1) = \pm\Psi(\mathsf{x}_1, \mathsf{x}_2)$$

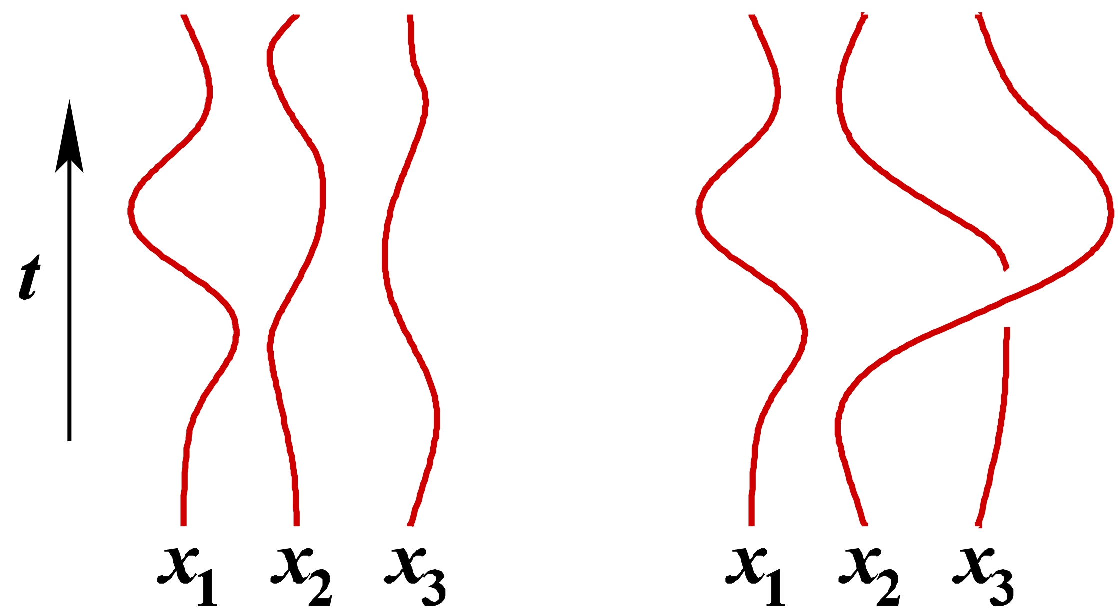
(bosons/fermions)

Quantum exchange statistics

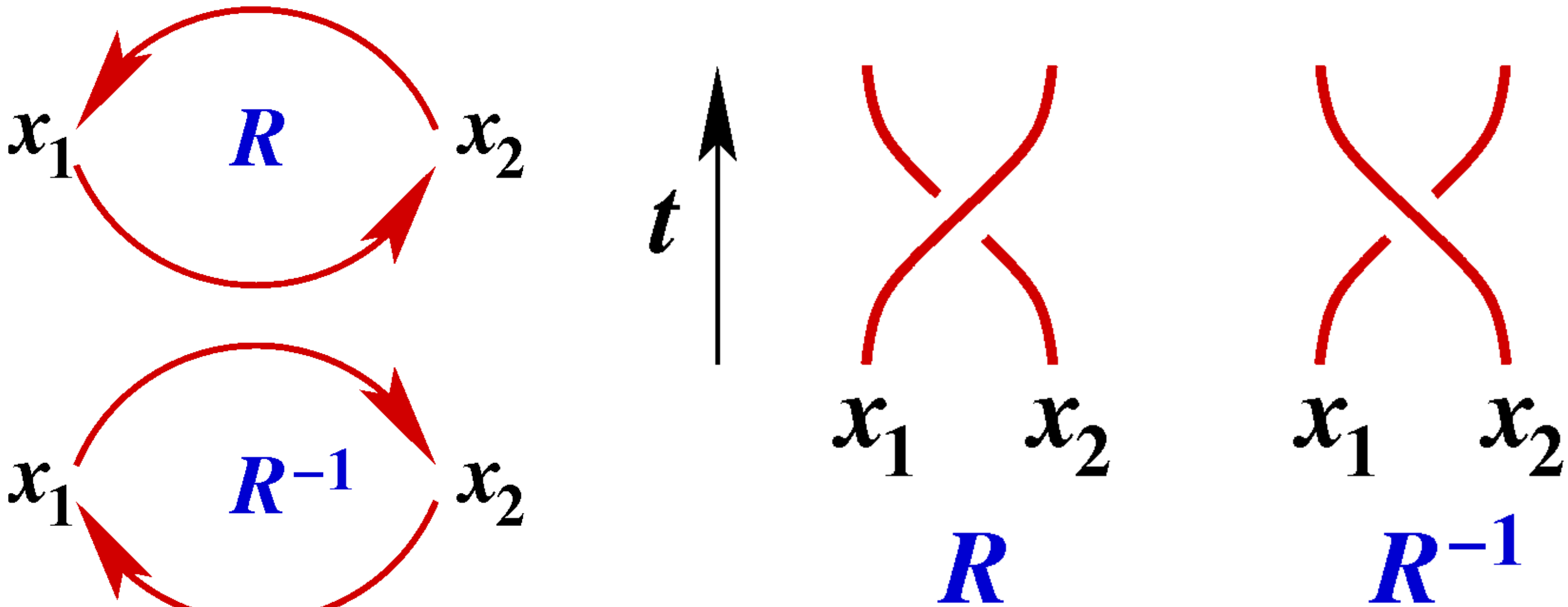
- Quantum-mechanical amplitude for particles at $x_1, x_2, \ldots, x_n$ at time t_0 to return to these coordinates at time t.

Feynman: Sum over all trajectories, weighting each one by e^{iS}.

Exchange statistics: What are the relative amplitudes for these trajectories?



Exchange statistics in (3+1)D vs (2+1)D



- In (3+1)D, $R^{-1} = R$
- If $R^{-1} = R$ then $R^2 = 1$, and the only types of particles are bosons and fermions.
- In (2+1)D, $R^{-1} \neq R$ ⇨ More types of particles!

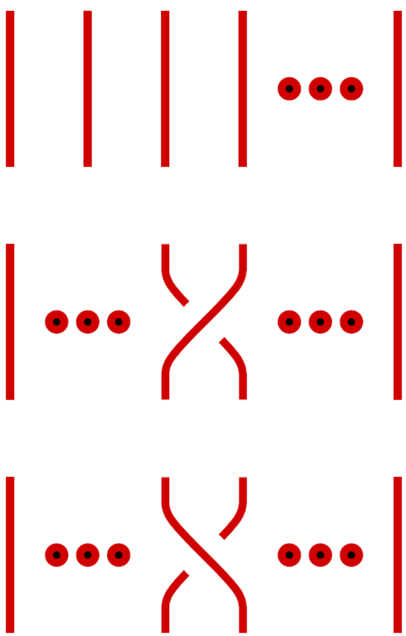
Exchange statistics in (2+1)D: The Braid Group

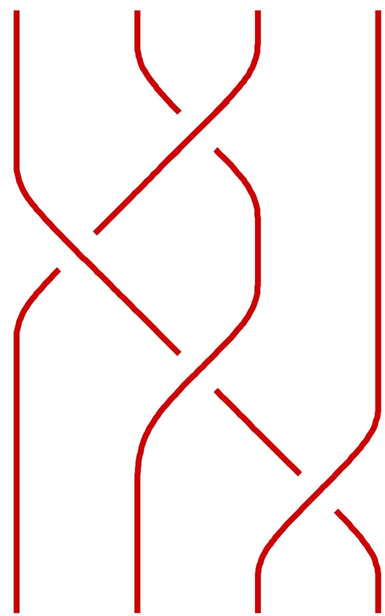
In (2+1)D, the particle statistics correspond to representations of the *braid group*:

I

R_i

R_i^{-1}

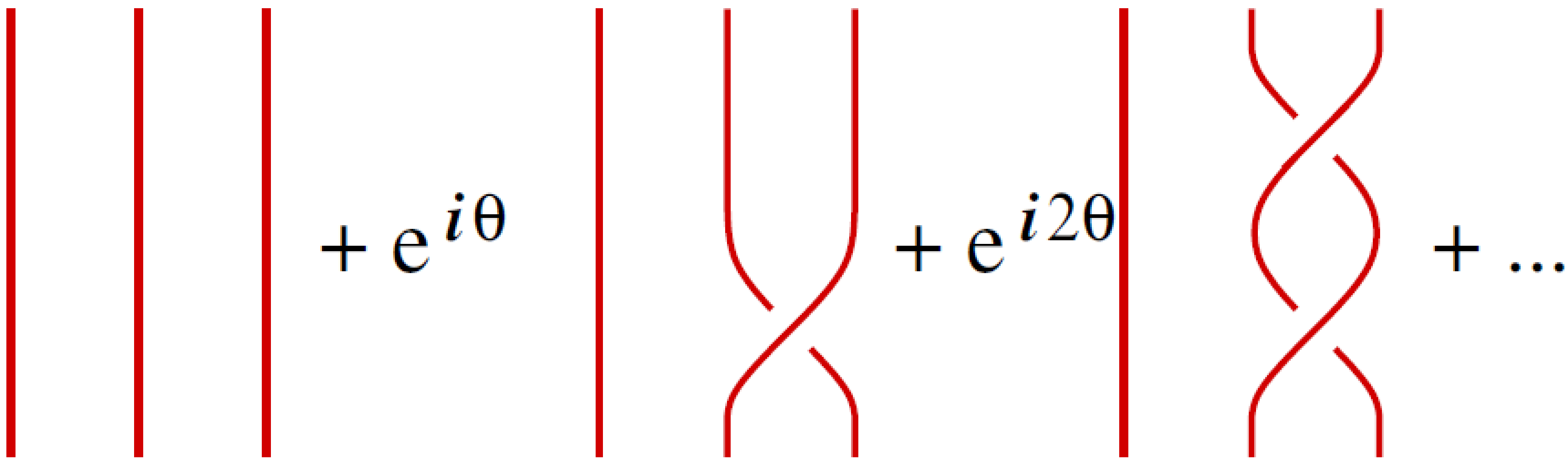




$R_2 R_1^{-1} R_2 R_3$

Abelian Anyons

- One possibility in (2+1)D: weight topologically distinct classes of trajectories in space-time by different overall phase factors (Leinaas and Myrrheim, Wilczek).

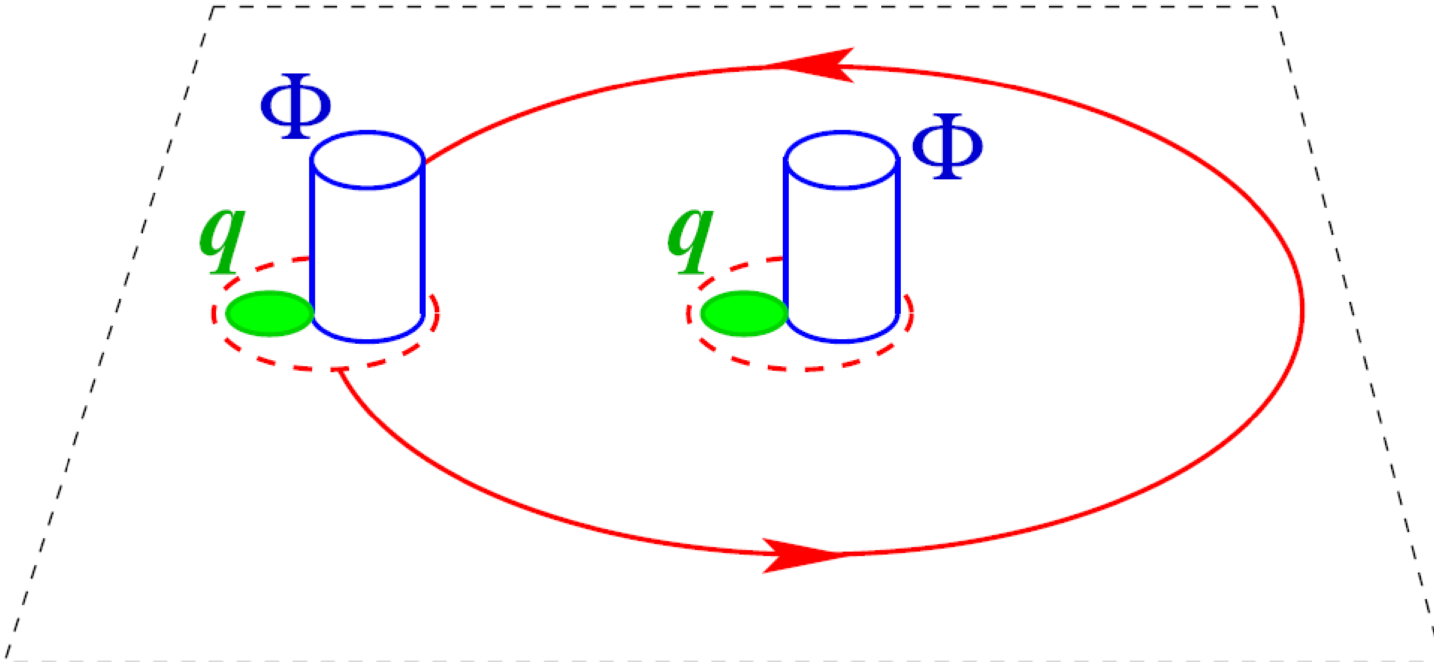


- θ – statistical angle
- For the mathematically inclined: These phase factors realise an Abelian representation of the braid group.
- Nontrivial consequence: GS degeneracy (Wen & Niu 1990)

  – *topological order*

Example: Toy model of Abelian anyons

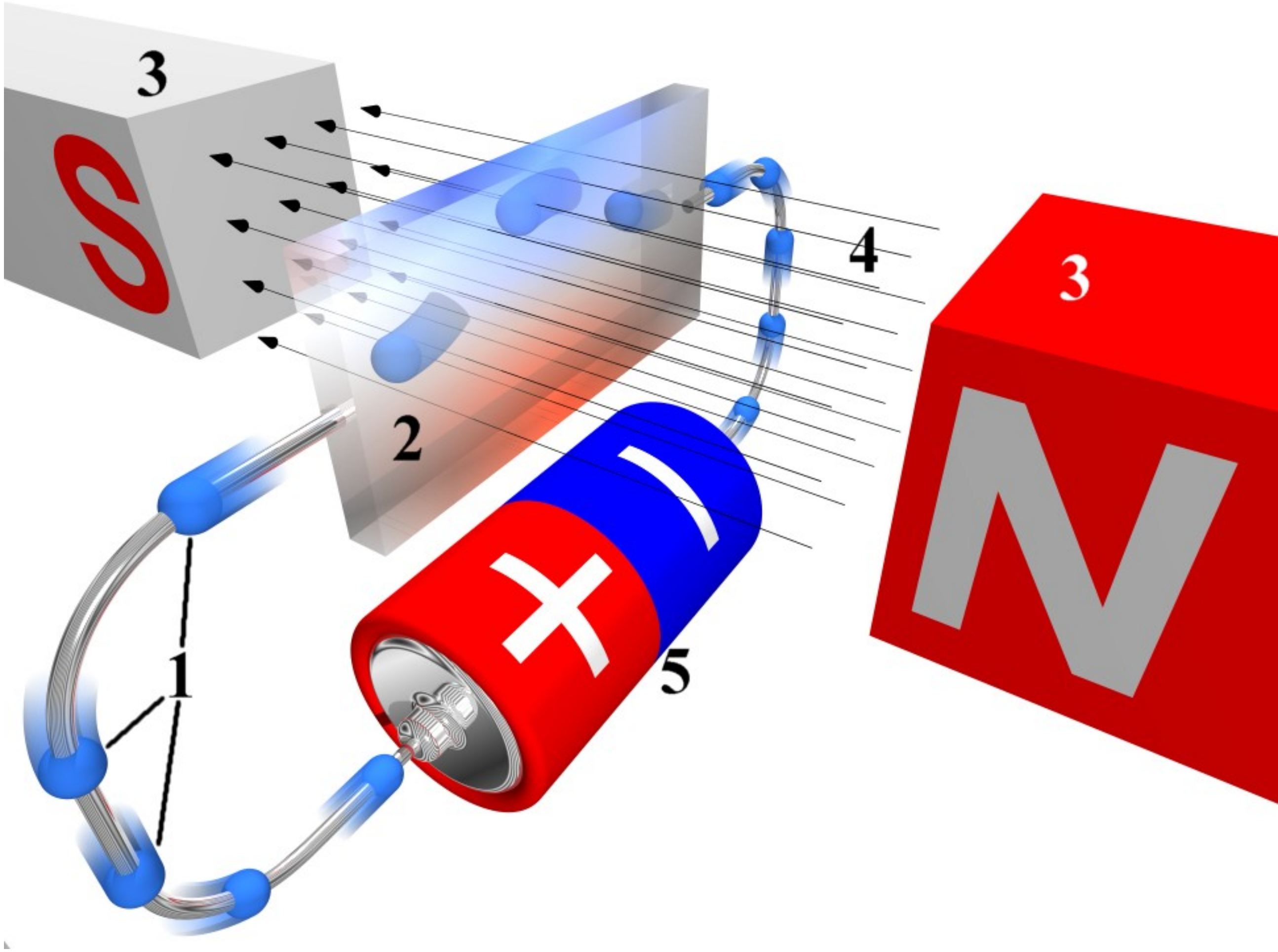
Charge q - flux Φ composites (Wilczek '82)



The Aharonov-Bohm phase $2\theta = q\Phi + q\Phi = 2q\Phi$
(in the units $\hbar = c = 1$)

E.g., for $q = \dfrac{e}{n}$, $\Phi = \Phi_0 = \dfrac{2\pi}{e}$, the statistical angle $\theta = \dfrac{2\pi}{n}$.

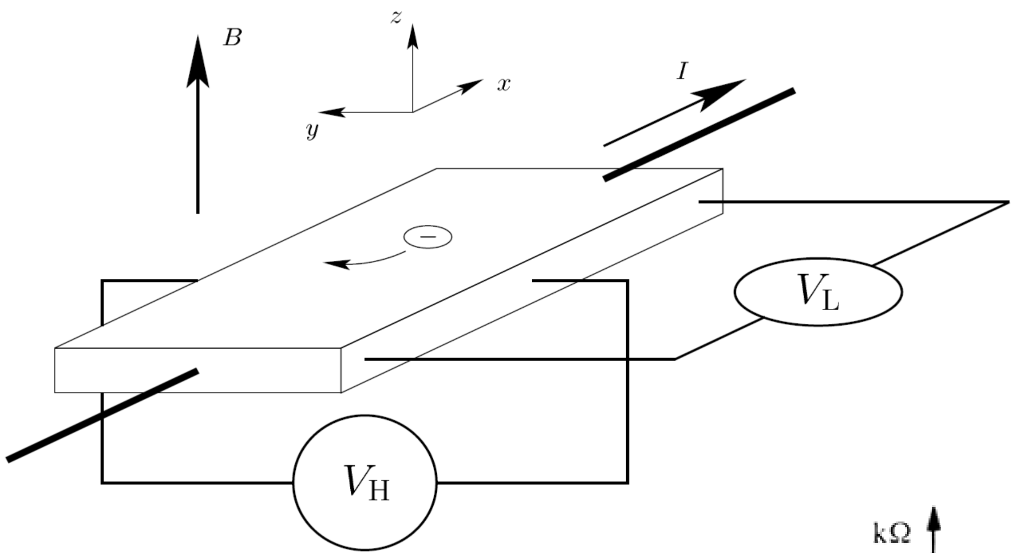
Hall Effect



$$E_y = v_x B$$

$$\rho_H = \frac{E_y}{j_x} = \frac{B}{ne}$$

Integer Quantum Hall Effect



Expect: $$\rho_H = \frac{B}{ne} = \frac{h}{\nu e^2}$$

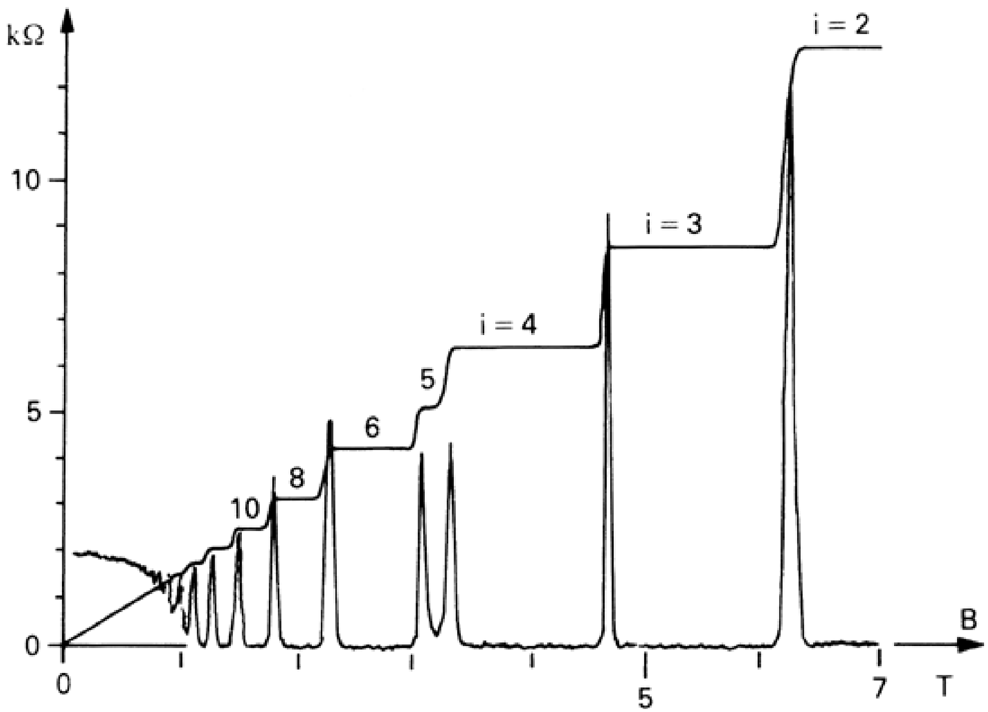
where $$\nu = \frac{N_e}{N_\Phi}$$ – *Filling factor*

$$N_\Phi = BA/\Phi_0$$

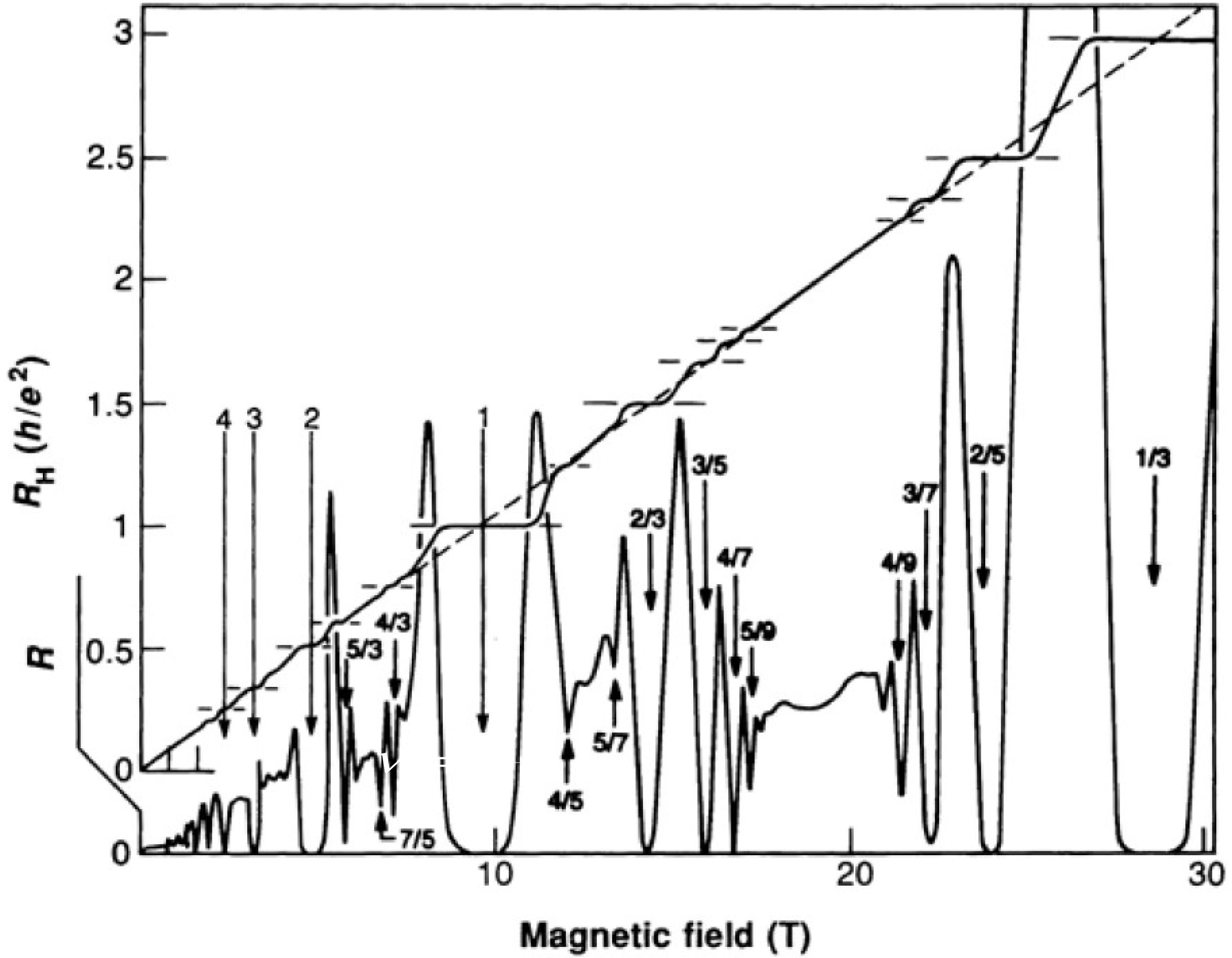
Experimentally:

$$\rho_H = \frac{h}{me^2}$$

where m is an integer



Fractional Quantum Hall Effect

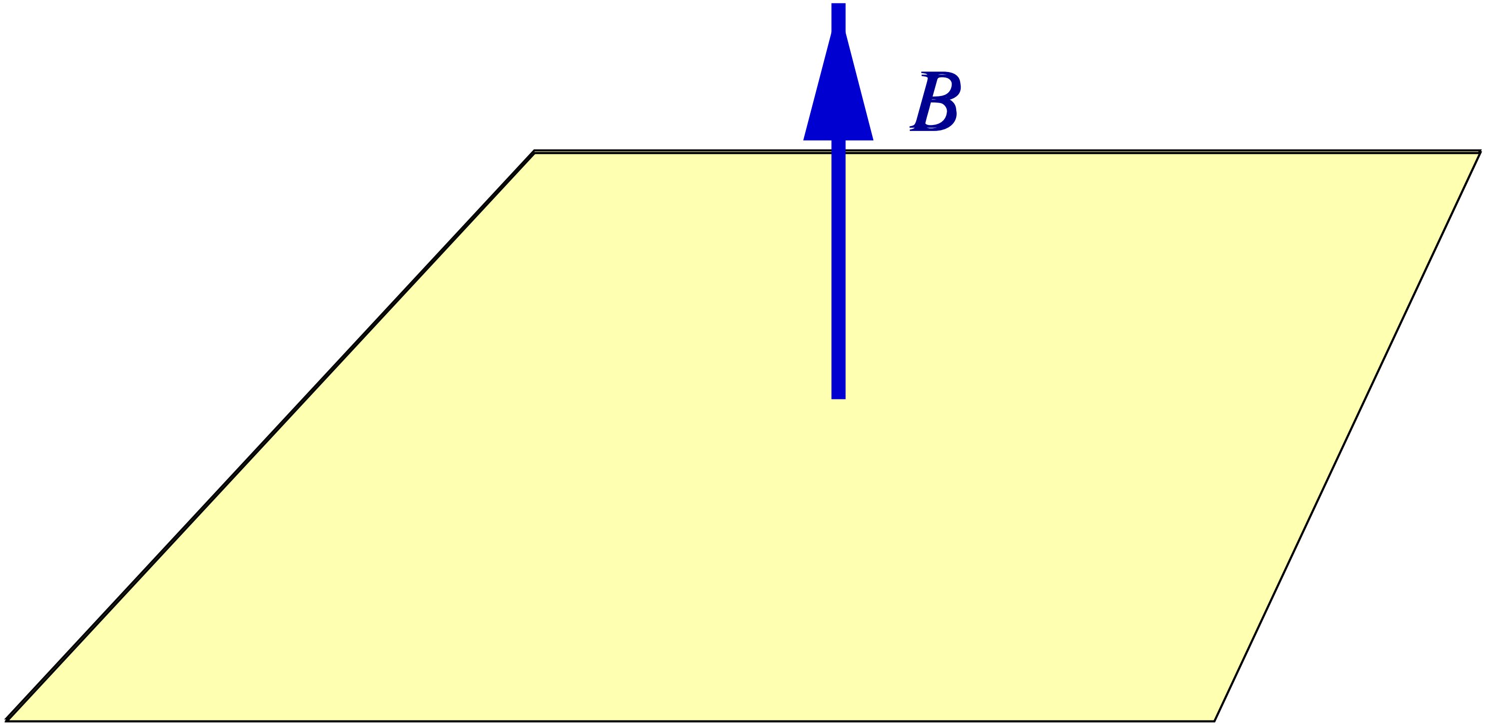


Eisenstein, Stormer, Science 248, 1990

Fractional Charge

Adiabatically insert one flux quantum Φ_0

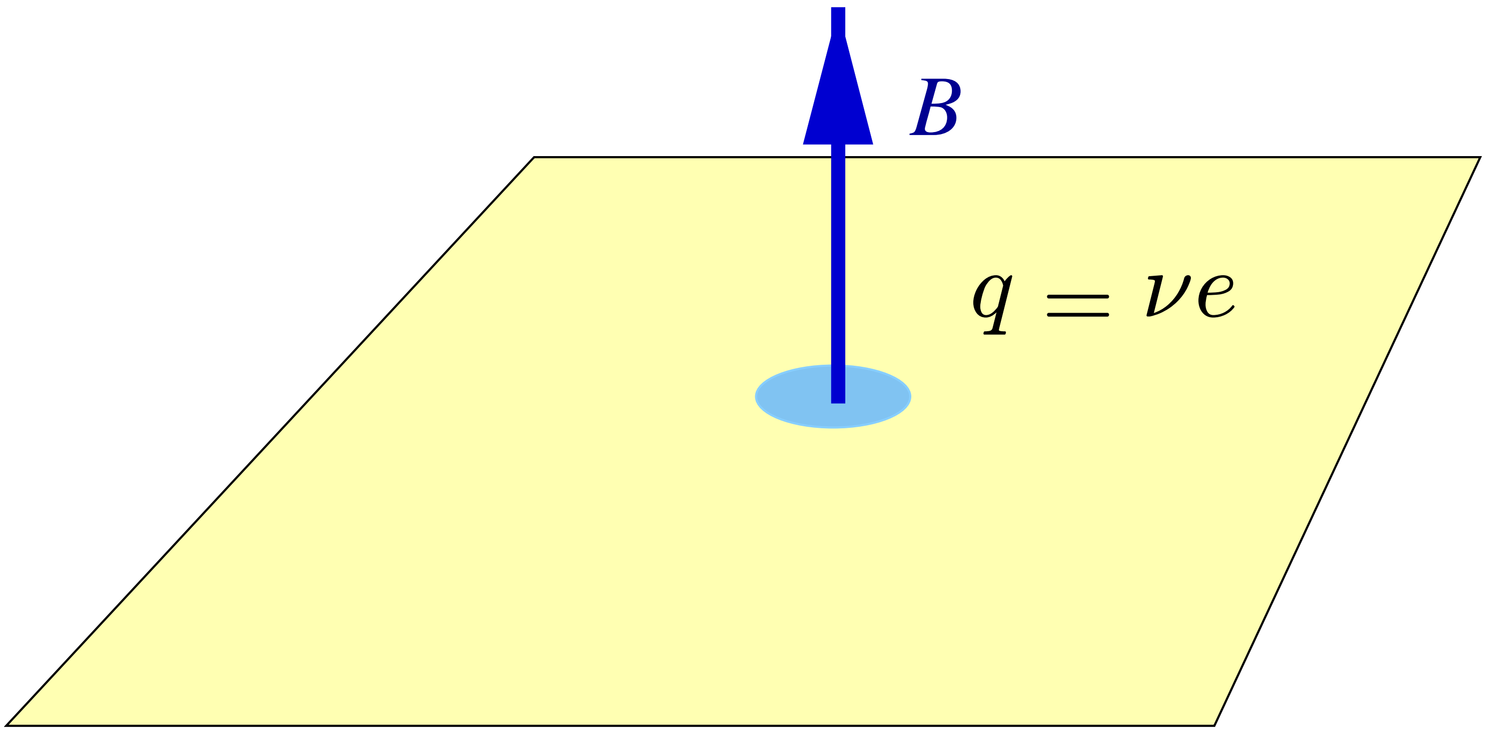
Maxwell: $\nabla \times \mathsf{E} = -\frac{\partial \mathsf{B}}{\partial t}$



Fractional Charge

Adiabatically insert one flux quantum Φ_0

Maxwell: $\nabla \times \mathsf{E} = -\dfrac{\partial \mathsf{B}}{\partial t}$ Hall: $j_x = E_y / \rho_H$

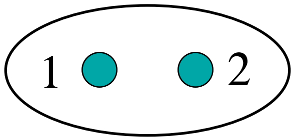


$$\rho_{\mathrm{H}} = \frac{h}{\nu e^2}$$

- $\theta = \dfrac{2\pi}{m}$ for a Fractional Quantum Hall state at a filling factor $\nu = \dfrac{1}{m}$

Non-Abelian Anyons

For fixed particle positions, we can have more than one wavefunction describing their combined state: $\psi_a(\mathsf{x}_1, \mathsf{x}_2), \psi_b(\mathsf{x}_1, \mathsf{x}_2) \ldots$

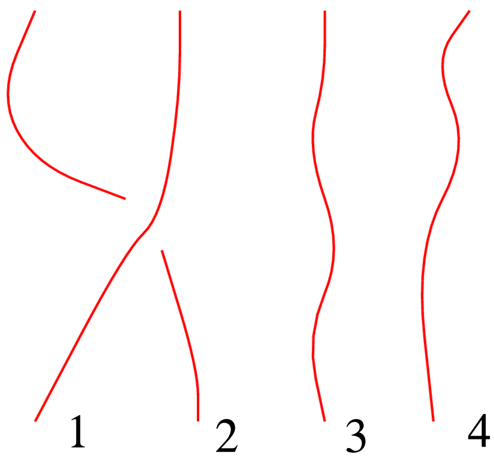


Exchanging particles positions may mix these wavefunctions:

$$\psi_a(1,2) \rightarrow \psi_a(2,1) = \sum_b R_{ab}\psi_b(1,2)$$

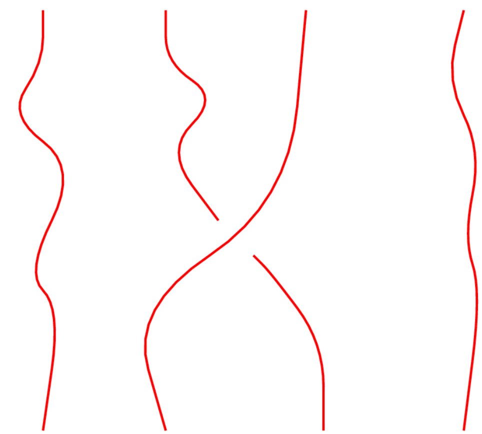
Exchanging
1 and 2:

$$\vec{\Psi} \rightarrow \mathsf{R}^{12} \cdot \vec{\Psi}$$



Exchanging
2 and 3:

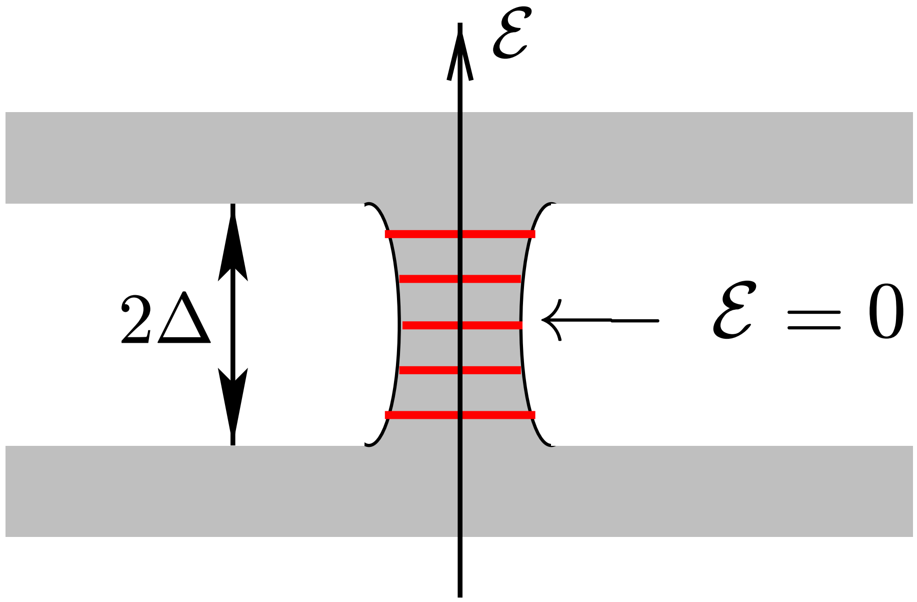
$$\vec{\Psi} \rightarrow \mathsf{R}^{23} \cdot \vec{\Psi}$$



- Matrices $\mathbf{R}^{12}$ and $\mathbf{R}^{23}$ need not commute, hence **Non-Abelian Statistics**.
- Matrices $\mathbf{R}$ form a higher-dimensional representation of the braid-group.
- For fixed particle positions, we have a non-trivial multi-dimensional Hilbert space where we can store information

Non-Abelian Anyons

p+*ip* superconductor: $\Delta(\mathsf{k}) \sim (k_x \pm i k_y)$



(Volovik ’99, Read & Green ’99)

States inside a vortex core:

$$\gamma(\mathcal{E}) = \gamma^\dagger(-\mathcal{E})$$

$$\gamma = \gamma^\dagger$$

– Majorana fermion!

$$\gamma^2 = (\gamma^\dagger)^2 = 1$$

It takes two Majorana fermions to create a “normal” fermion:

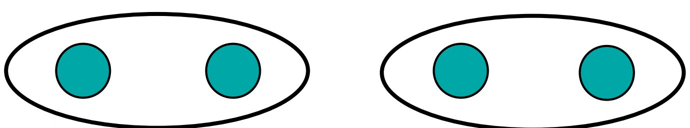
$$\Psi = (\gamma_1 + i\gamma_2)/2, \quad \Psi^2 = (\Psi^\dagger)^2 = 0$$

Non-Abelian Anyons

- The combined state of two Majorana fermions can be described by the occupation number of the "normal" fermion:

$$|0\rangle : \ \Psi^\dagger \Psi |0\rangle = 0 \quad \text{and} \quad |1\rangle \equiv \Psi^\dagger |0\rangle \ \ (\Psi = (\gamma_1 + i\gamma_2)/2)$$

- To see their non-Abelian nature, look at 4 such Majorana modes:



$$|0\rangle, |1\rangle \qquad\qquad |0\rangle, |1\rangle$$

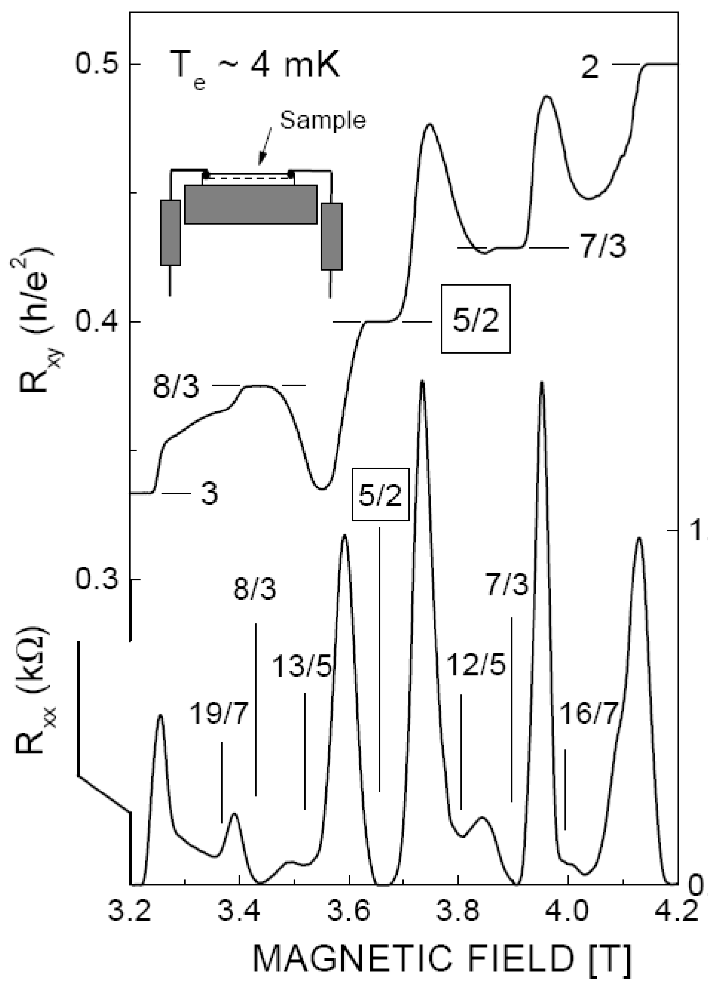
Braiding matrices in the $|00\rangle$, $|10\rangle$, $|01\rangle$, $|11\rangle$ basis

$$R_{12}, R_{34} = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{i\pi/4} & & \\ & & e^{\mp i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}$$

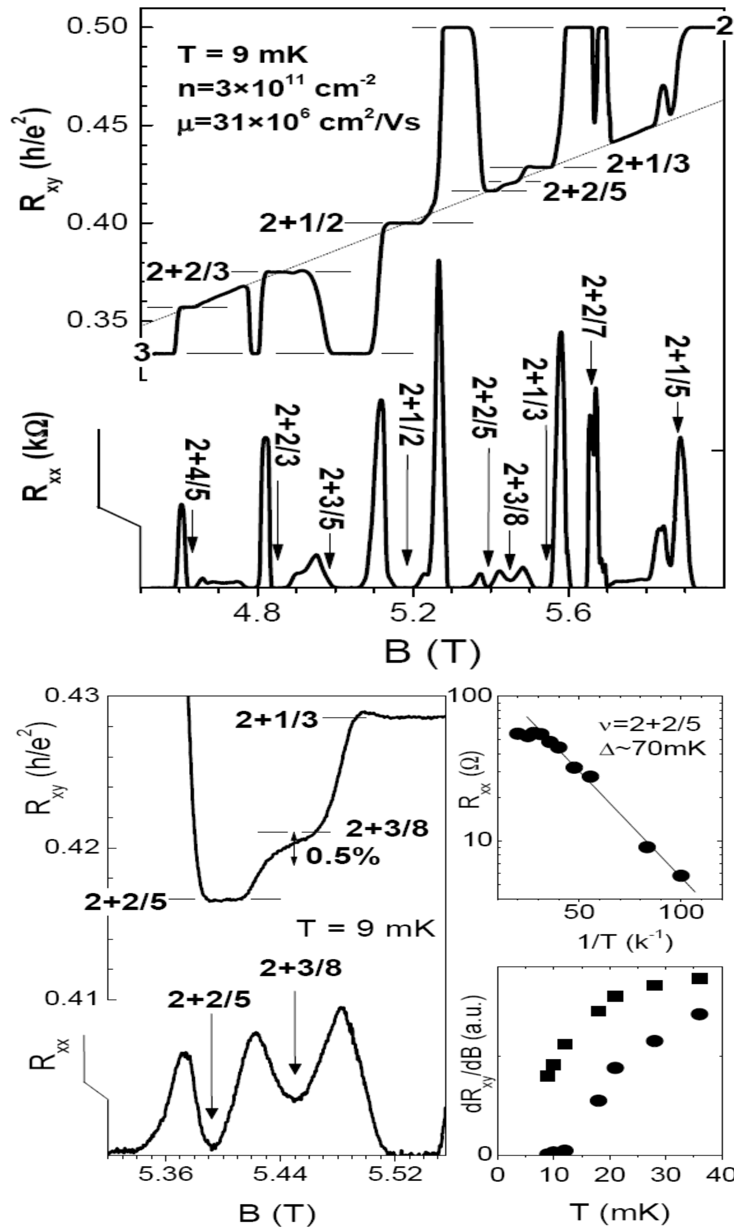
$$R_{23} = \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$$

(Ivanov, 2001; Stern, Mariani and von Oppen, 2004)

“Unusual” FQHE states



Pan et al. PRL 83,1999
Gap at 5/2 is 0.11 K

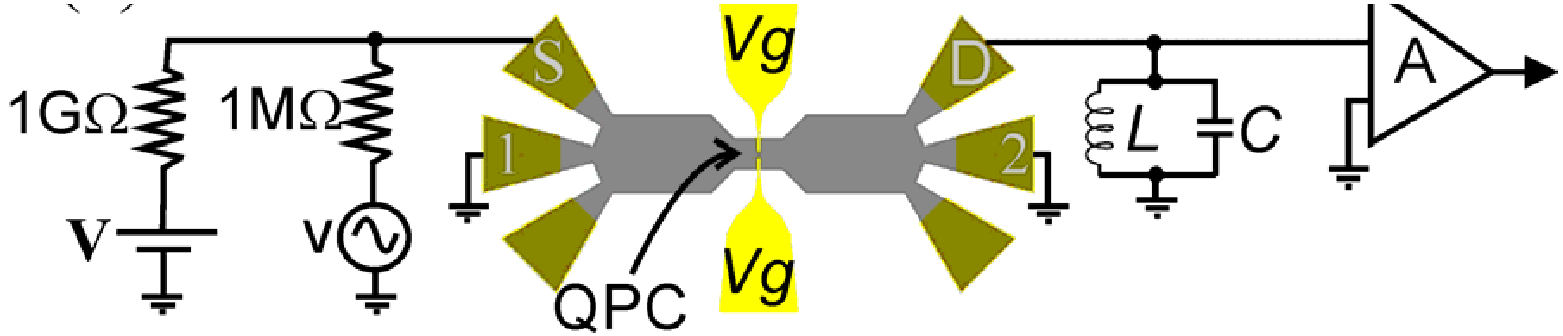


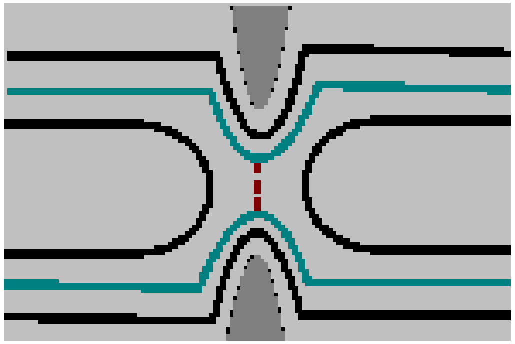
Xia et al. PRL 93, 2004
Gap at 5/2 is 0.5K, at 12/5: 0.07K

Non-Abelian QH state?

First thing first: *Can we detect the charge of the quasiparticles?*
For a FQHE "superconductor" at the charge should be e/4, not e/2 as a simple argument based on the fraction ν=5/2 would suggest.

Shot Noise: Dolev, Heiblum *et al*. Nature '08





Idea: for independent tunnelling events, the current and the noise power are not independent:

$$e^* = \frac{S_I(\omega=0)}{2I}$$

Kane & Fisher '94; Fendley, Ludwig & Saleur '96

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For a FQHE "superconductor" at the charge should be e/4, not e/2 as a simple argument based on the fraction ν=5/2 would suggest.

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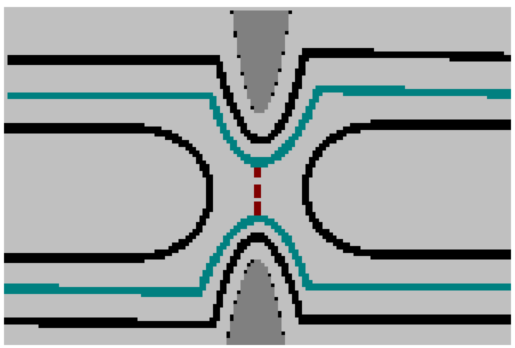
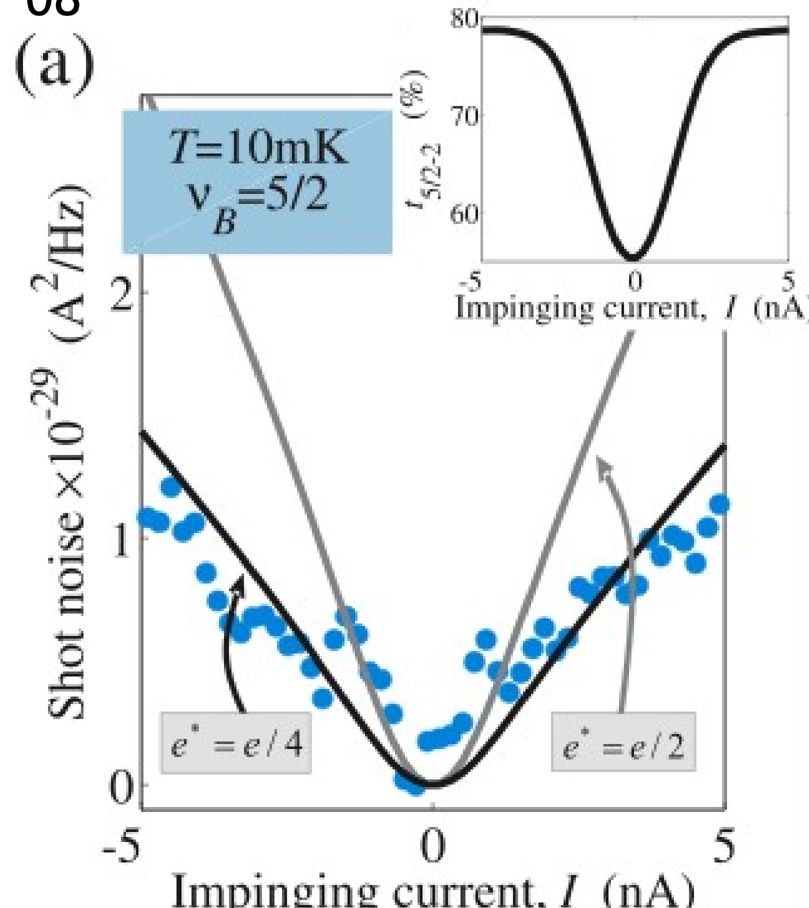




$$e^* = \frac{S_I(\omega = 0)}{2I}$$

Non-Abelian QH state?

Second check: Tunnelling conductance across a QPC:
According to the edge theory in the weak tunnelling regime, the differential tunnelling conductance:

$$G_t \propto T^{2(g-1)} F(g, e^* I_t R_{xy}/kT)$$

(Wen '92; Fendley, Ludwig & Saleur '95)

g is the scaling dimension of the quasiparticle propagator and can be evaluated for all candidate theories.
Unlike the charge, it is generally different for different paired states

$$\Psi_{\mathrm{MR}} = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\dots\right)\prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4}$$

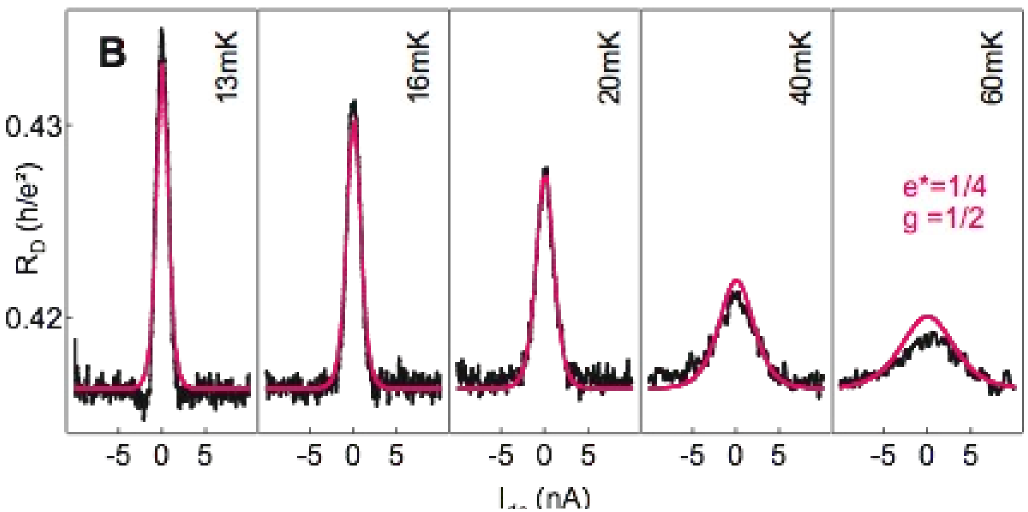
$$\Psi_{331} = \prod_{j<k}(z_j - z_k)^3 \prod_{j<k}(w_j - w_k)^3 \prod_{j<k}(z_j - w_k) \prod_j e^{-(|z_j|^2+|w_j|^2)/4}$$

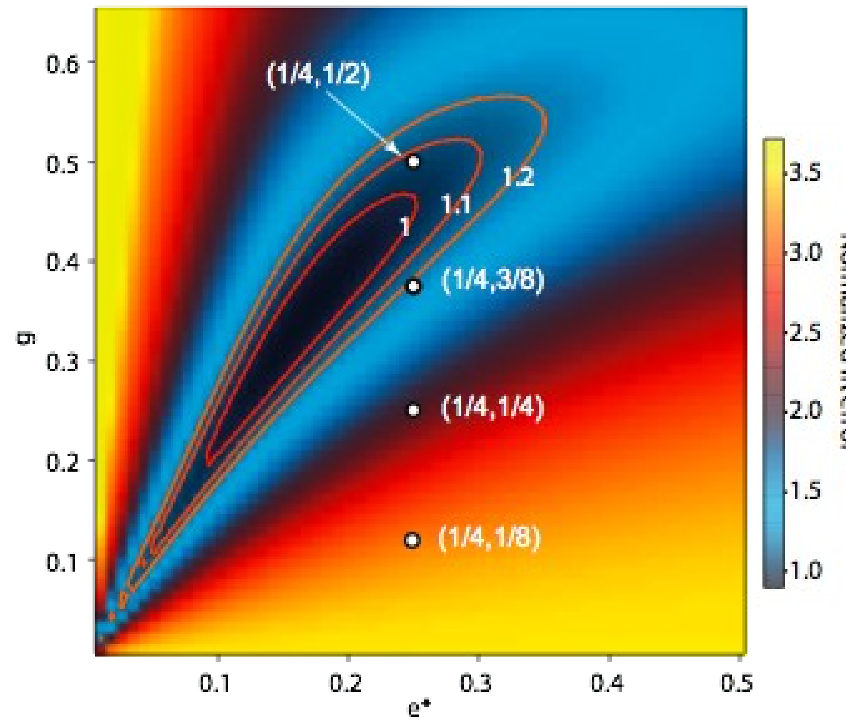
Non-Abelian QH state?

Tunnelling conductance across a QPC

$$G_t \propto T^{2(g-1)} F(g, e^* I_t R_{xy}/kT)$$

Radu, Miller, Marcus et al. Science '08:

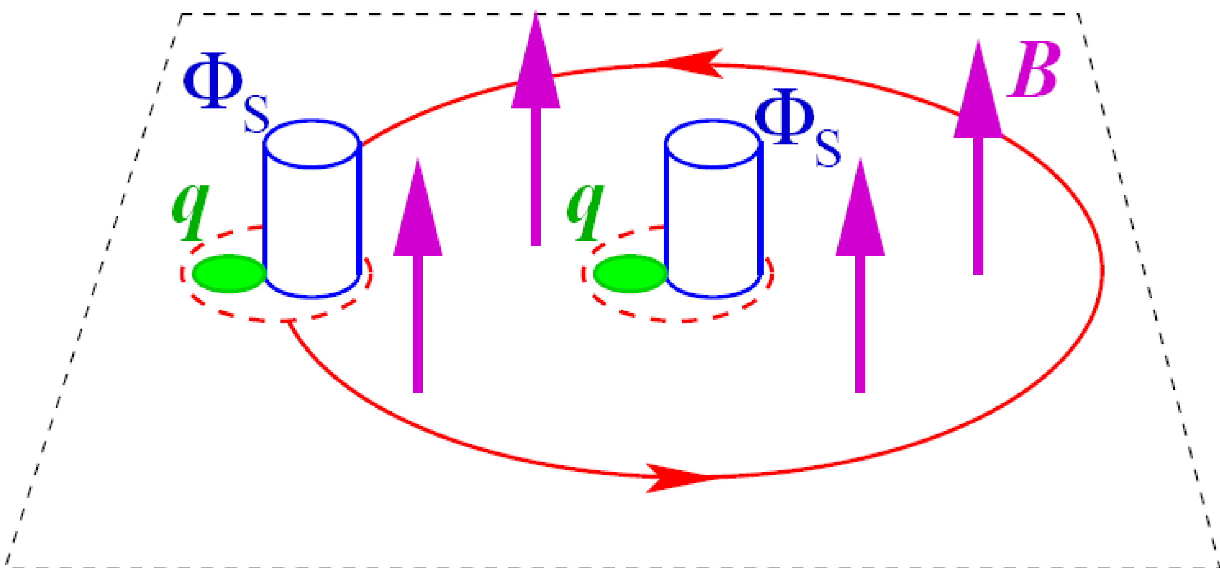




Consistent with the ‘anti-Pfaffian’ state
(Lee, Ryu, Nayak, Fisher ’08; Levin & Halperin ‘08).
But only a true probe of topological properties will be definitive!

Probing Abelian Statistics in FQHE

- Back to the toy model:

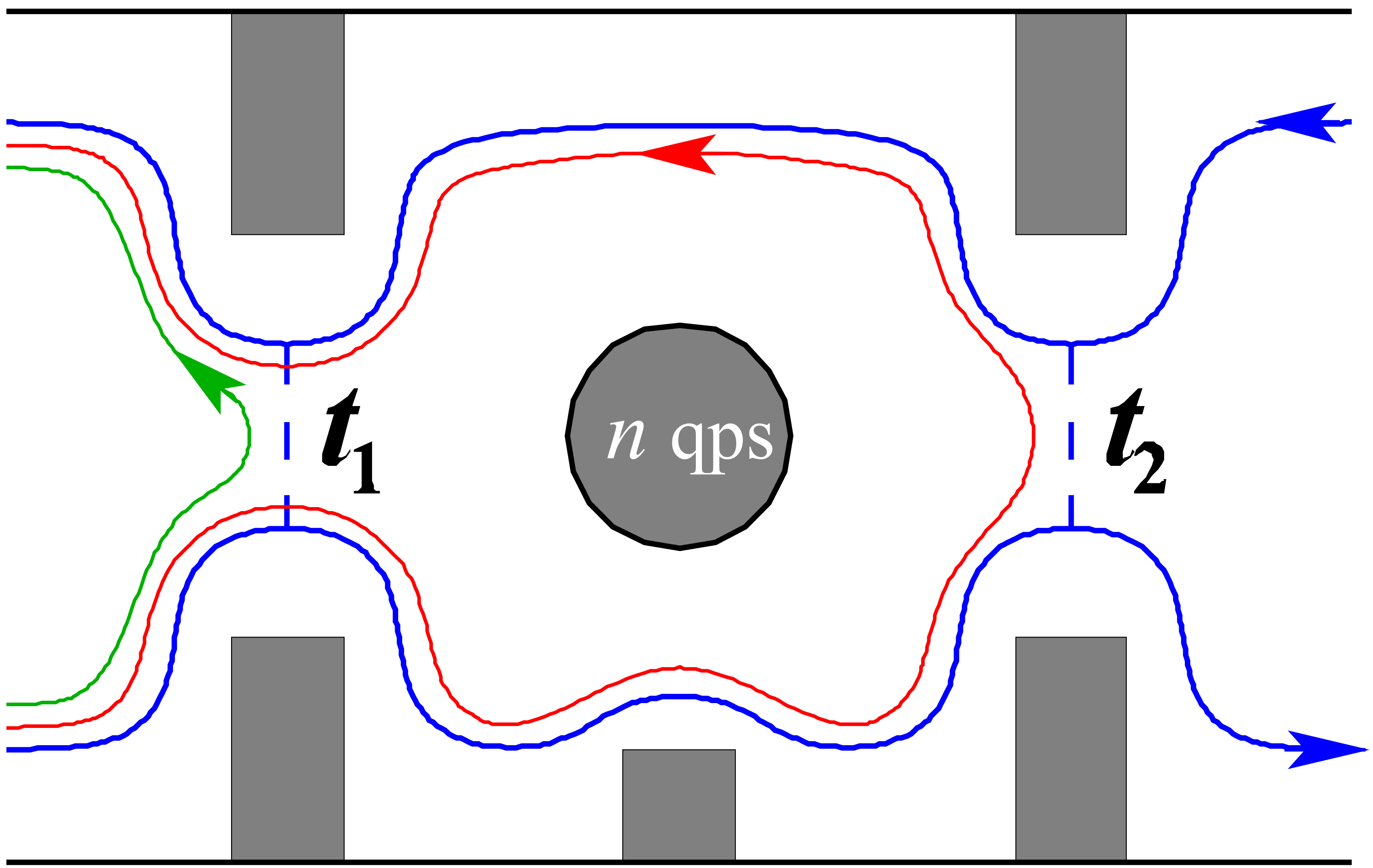


The total Aharonov-Bohm phase $2\theta = q\Phi_{\mathrm{S}} + q\Phi_{\mathrm{B}}$ where $\Phi_{\mathrm{B}} = BA$.

Problem: How do we tell these two contributions apart?

FQHE Quasiparticle interferometer

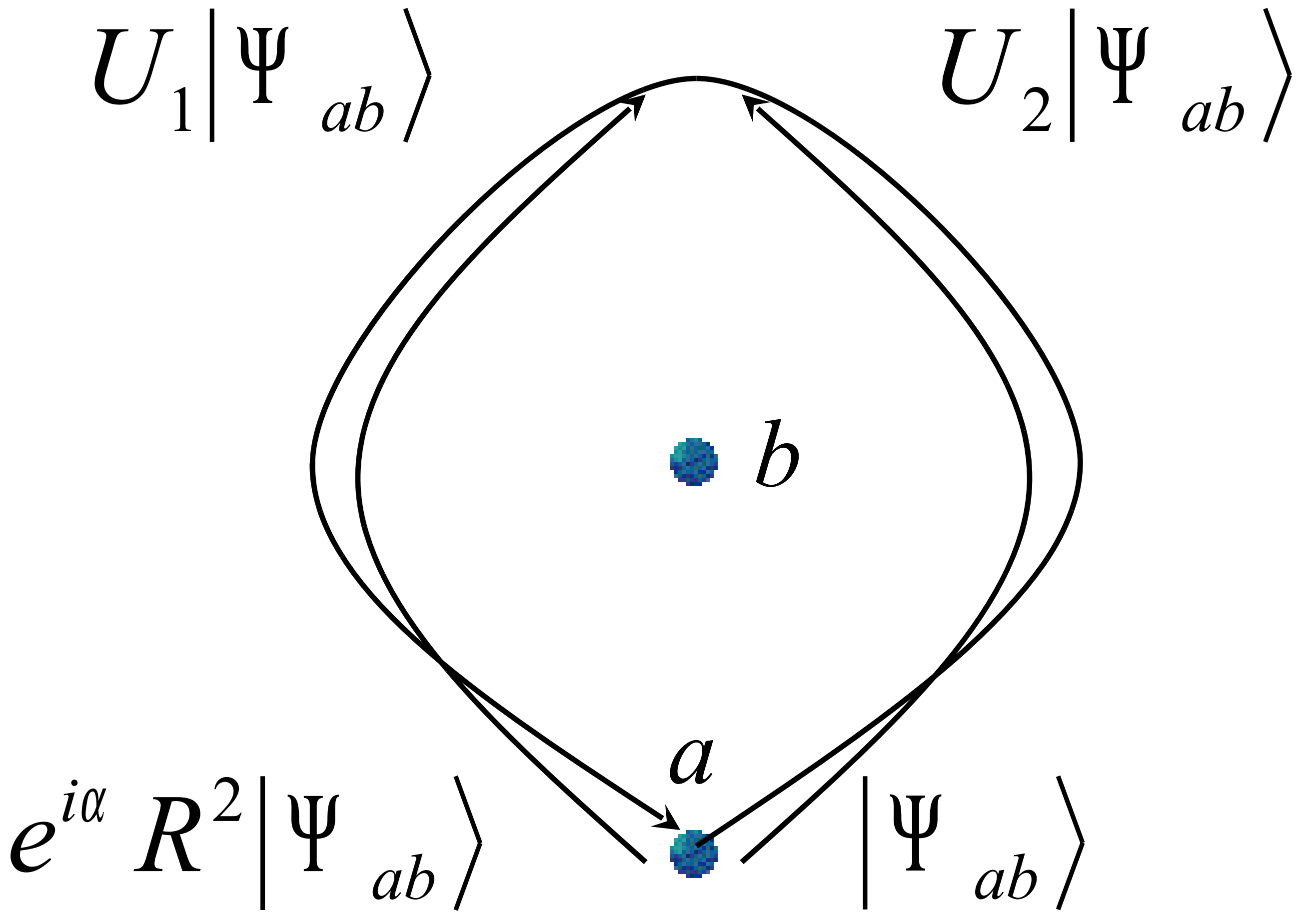
Measure σ_{xx} due to the quasiparticles tunnelling between the edges



After Chamon, Freed, Kivelson, Sondhi, Wen 1997,
Fradkin, Nayak, Tsvelik, Wilczek 1998

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2|t_1||t_2||M_{ab}|\cos(\beta + \theta_{ab})$$

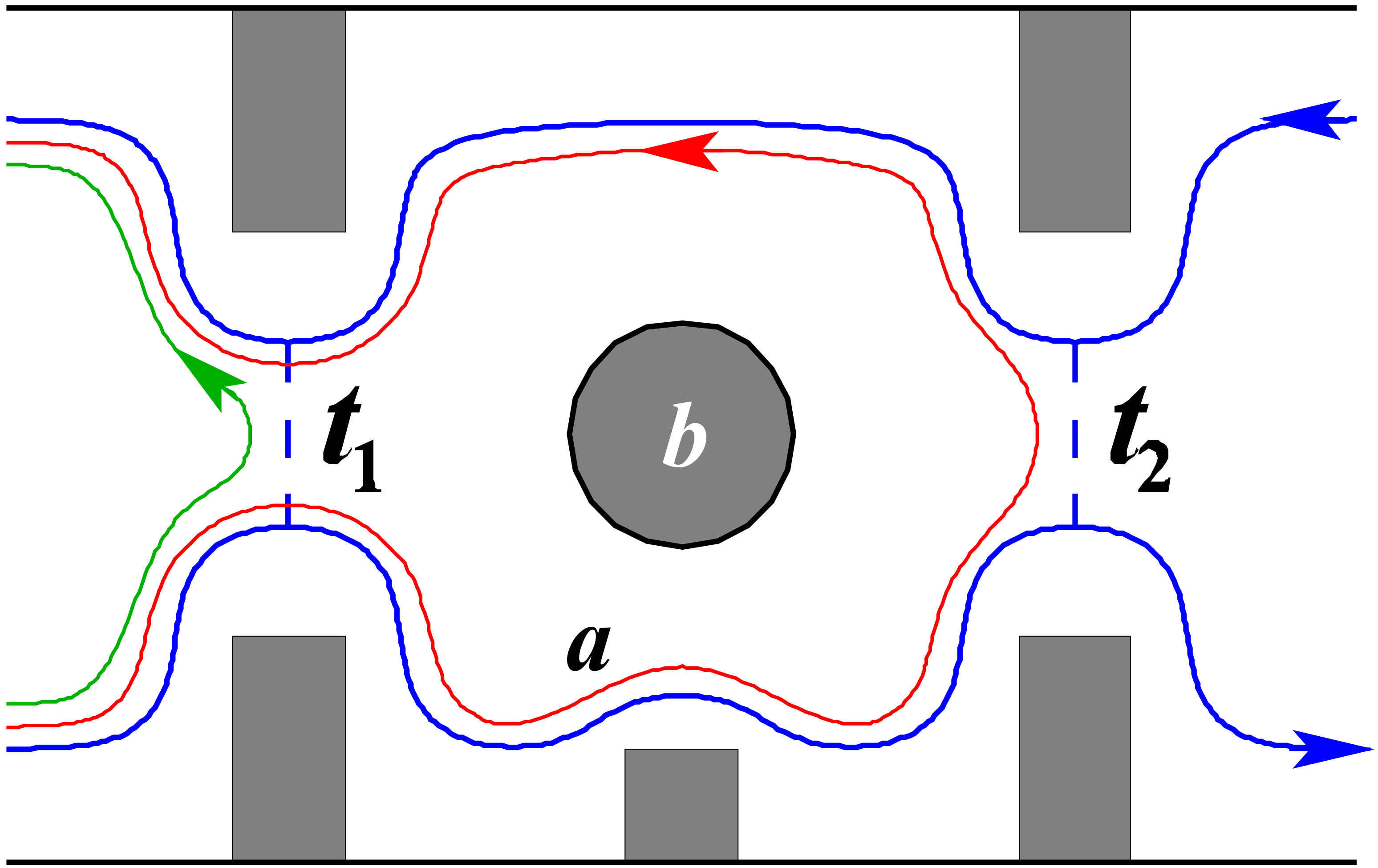
$$\left| U_1|\Psi_{ab}\rangle + U_2|\Psi_{ab}\rangle \right|^2 = 2 + 2\,\mathrm{Re}\left\{ \langle \Psi_{ab}| U_1^{-1} U_2 |\Psi_{ab}\rangle \right\}$$



$$M_{ab} = \langle \Psi_{ab}|R^2|\Psi_{ab}\rangle = |M_{ab}|e^{i\theta_{ab}}$$

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2|t_1 t_2||M_{ab}|\cos(\beta + \theta_{ab})$$

$\beta = \alpha + \arg(t_2/t_1)$ is a parameter that can be experimentally varied.



$|M_{ab}| < 1$ is a smoking gun that indicates non-Abelian braiding.

A digression: General Anyon Models

(unitary braided tensor categories)

Describes two dimensional systems with an energy gap.
Allows for multiple particle types and variable particle number.

1. A finite set of particle types or anyonic “charges.”

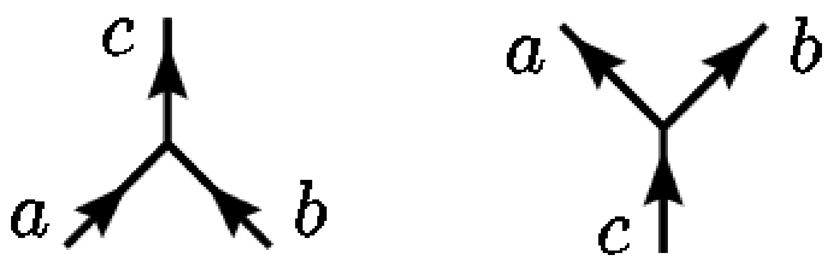
2. Fusion rules (specifying how charges can combine or split).

3. Braiding rules (specifying behavior under particle exchange).

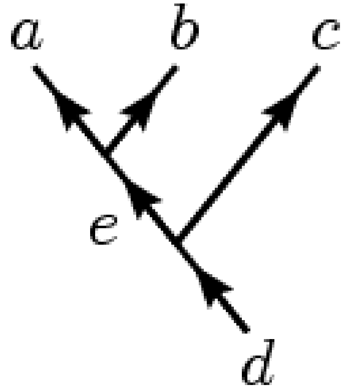
There is a particle type I corresponding to "vacuum."

Each particle type a has a "charge conjugate" $\bar{a}$ that is the unique particle type which may fuse with a to give I.

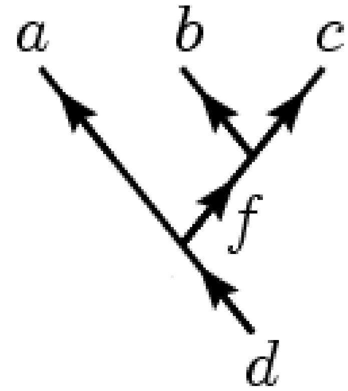
The fusion rules are specified as: $a \times b = \sum_c N_{ab}^c c$ where the integer N_{ab}^c is the number of channels for fusing a and b into c. Diagrammatically, this gives a set of allowed trivalent vertices:



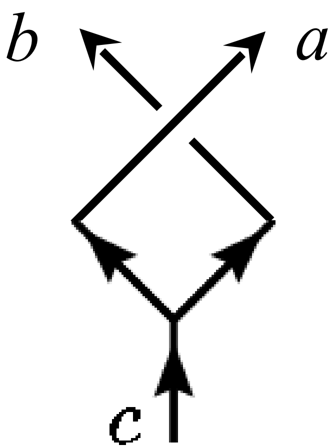
Associativity relations for fusion:



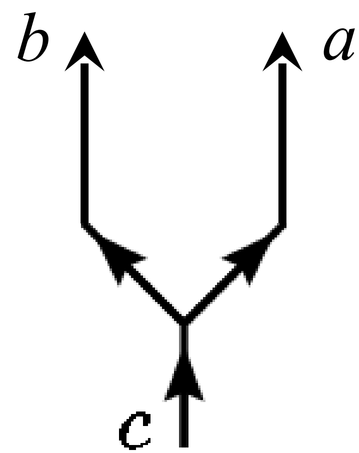
$$= \sum_f \left[F^{abc}_d\right]_{ef}$$



Braiding rules:



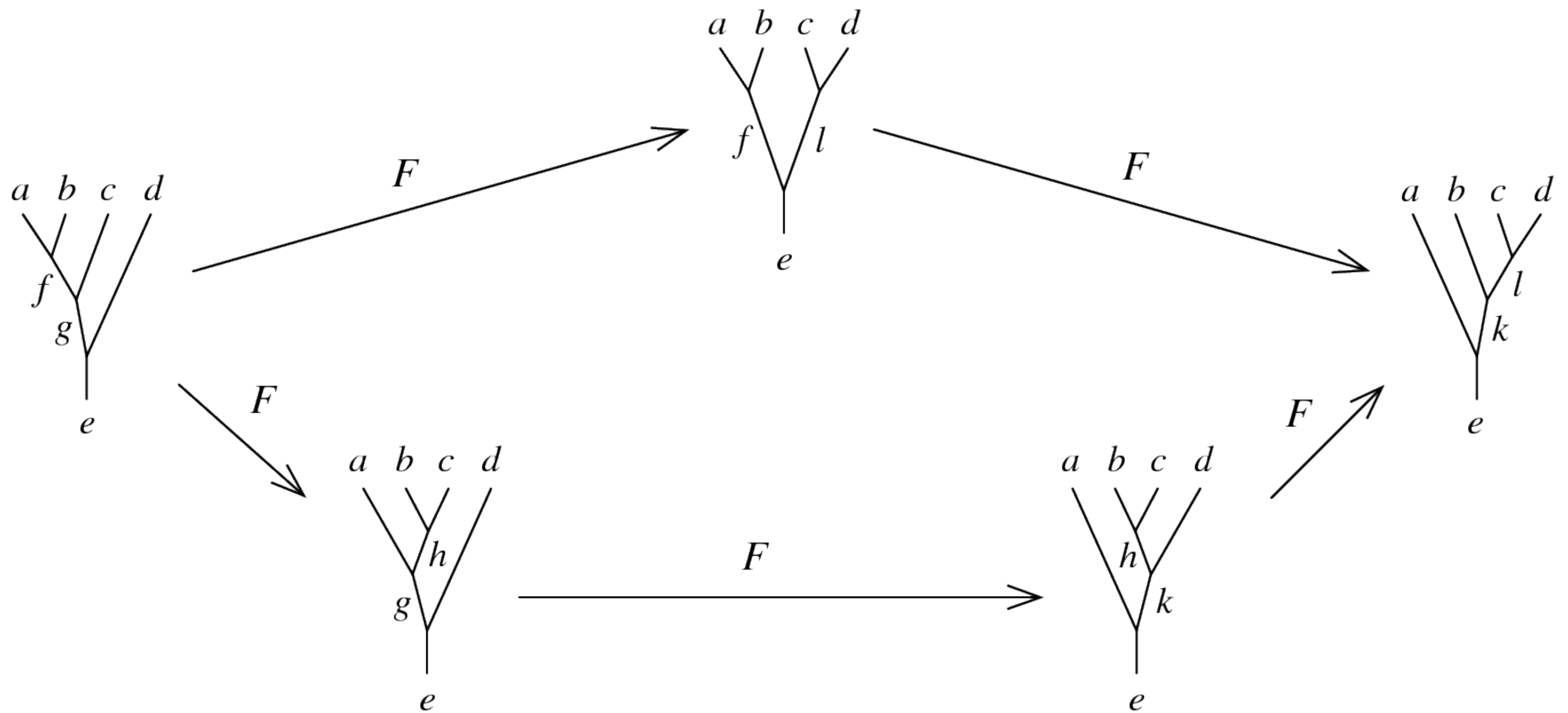
$$= R^{ab}_c$$



* These are all subject to consistency conditions.

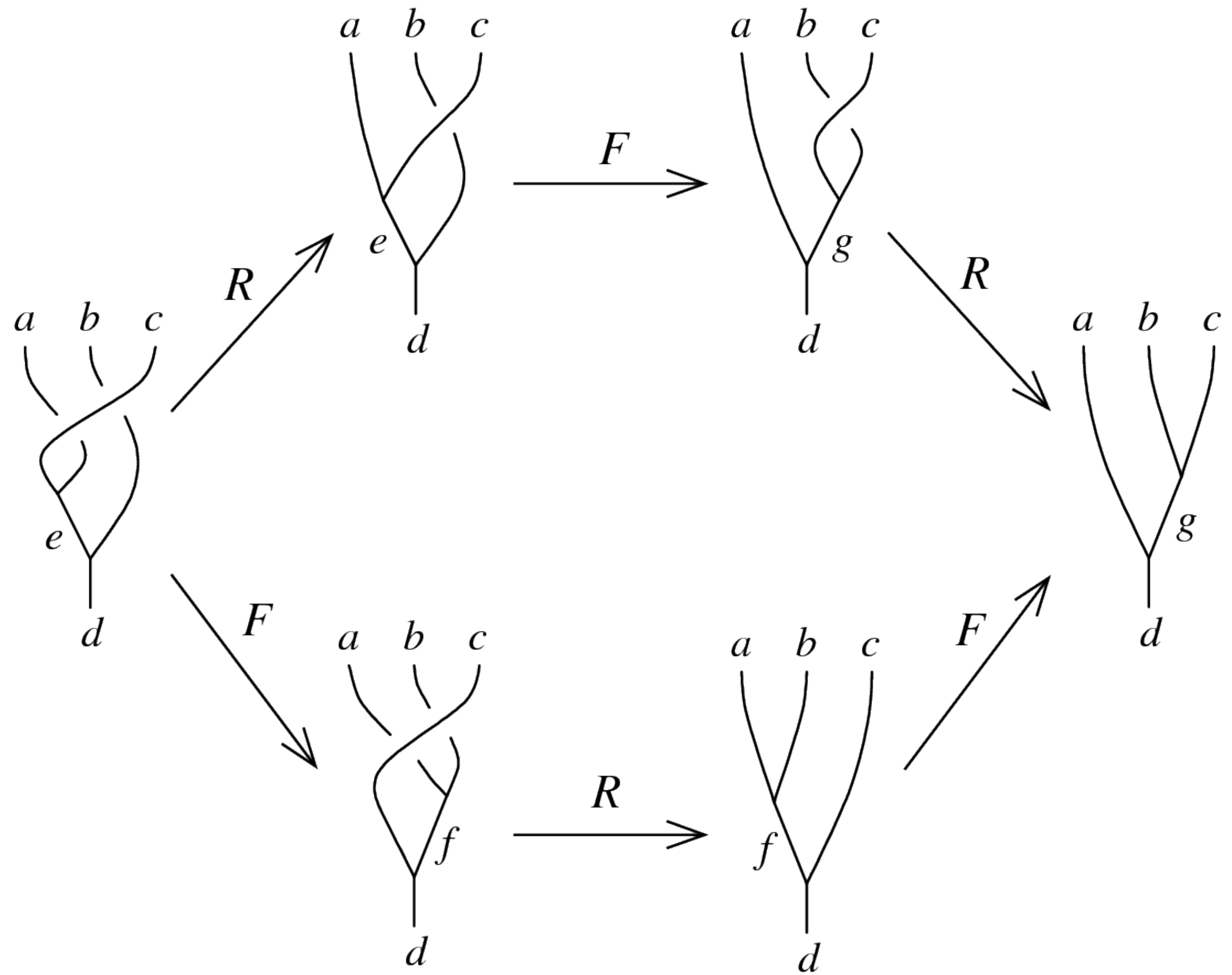
Consistency conditions

A. Pentagon equation:



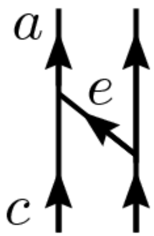
Consistency conditions

B. Hexagon equation:

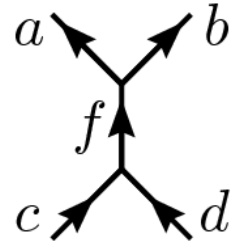


Doing calculations in this language:

- Some useful tools:



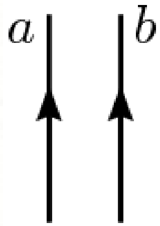
$$= \sum_f \left[F^{ab}_{cd}\right]_{(e,f)}$$



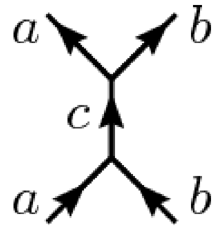
Where

$$\left[F^{ab}_{cd}\right]_{(e,f)} = \sqrt{\frac{d_e d_f}{d_a d_d}} \left[F^{ceb}_{f}\right]^*_{(a,d)}$$

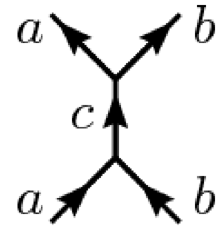
- A “resolution of the identity”:



$$= \sum_c [F^{ab}_{ab}]_{(I,c)}$$

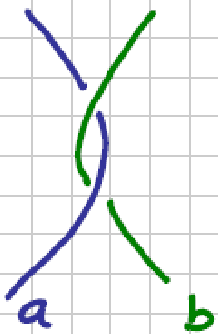


$$= \sum_c \sqrt{\frac{d_c}{d_a d_b}}$$

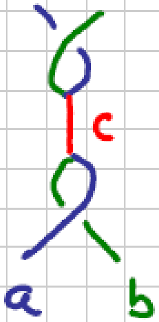


Doing calculations in this language:

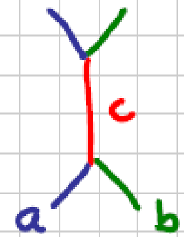
- Undoing braids:



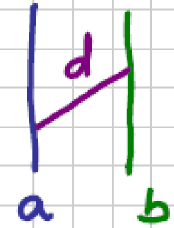
$$= \sum_c \left[F^{ba}_{ba}\right]_{1,c}$$



$$= \sum_c \left[F^{ba}_{ba}\right]_{1,c} R^{ab}_c \, R^{ba}_c$$

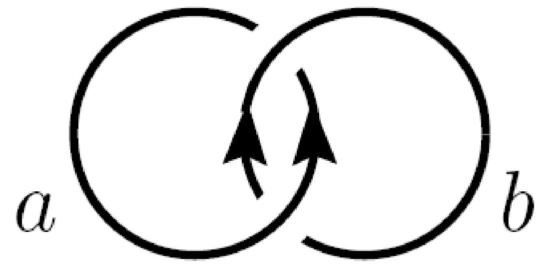


$$= \sum_{c,d} \left[F^{ba}_{ba}\right]_{1,c} R^{ab}_c \, R^{ba}_c \left[F^{ab}_{ab}\right]^{-1}_{(cd)}$$

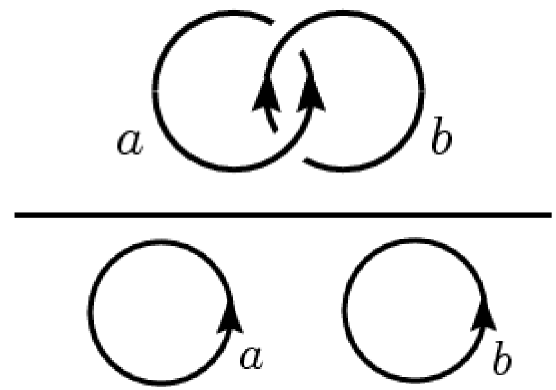


Topological S-matrix

$$S_{ab} = \frac{1}{D}$$



$$M_{ab} = \frac{S_{ab}S_{II}}{S_{Ia}S_{Ib}} =$$



$|M_{ab}| = 1$ corresponds to Abelian braiding and

$|M_{ab}| < 1$ iff the braiding is non - Abelian.

$\nu = 5/2$ is believed to be an "Ising" theory

Ising particle types: I, σ, ψ

(a "boson," a "Majorana mode" and a "fermion")

Fusion rules: $I \times I = I, \quad I \times \sigma = \sigma, \quad I \times \psi = \psi,$

$$\sigma \times \sigma = I + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = I$$

Monodromy: $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

quasiholes carry anyonic charge: $(e/4, \sigma)$

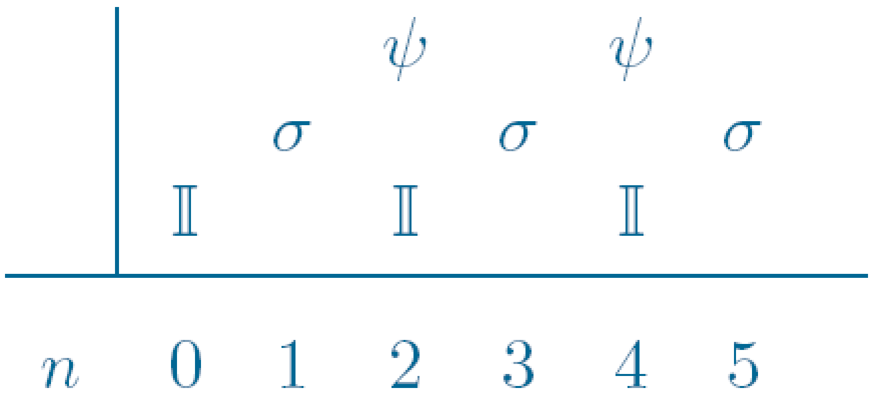
electrons carry anyonic charge: $(-e, \psi)$

n quasiholes carry anyonic charge: $(ne/4, \sigma)$ for n odd

$(ne/4, I \text{ or } \psi)$ for n even

Combined anyonic state of the antidot

Adding non-Abelian "anyonic charges" for the Moore-Read state:



Even-Odd effect

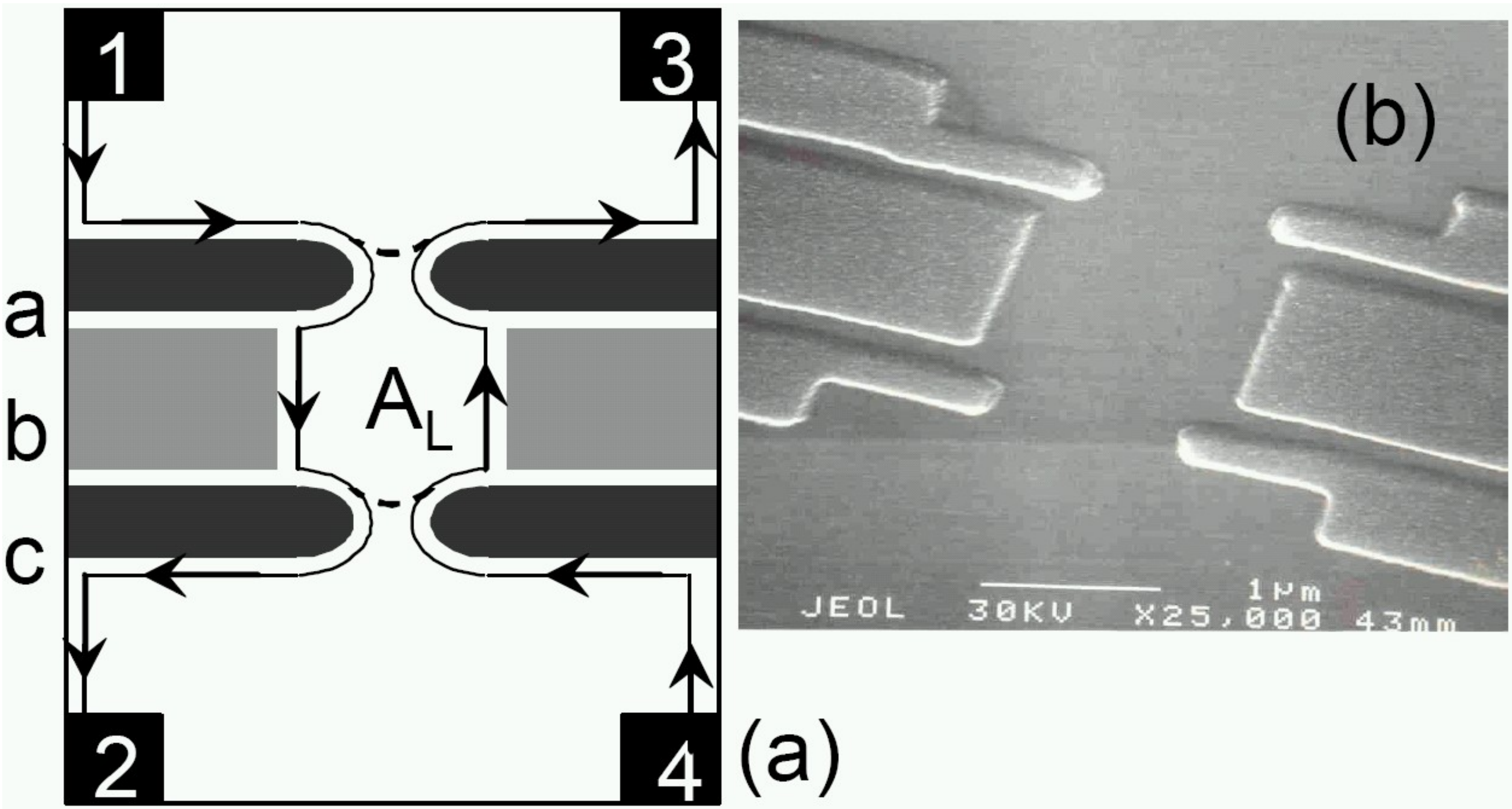
(Stern & Halperin; Bonderson, Kitaev & KS, PRL 2006):

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 , \qquad n \text{ odd}$$

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + (-1)^{N_\psi} 2 |t_1| |t_2| \cos\left(\beta + n\frac{\pi}{4}\right) , \qquad n \text{ even}$$

Current experimental status

Lucent group: Willett et al. PNAS 2009; arXiv:0911.0345



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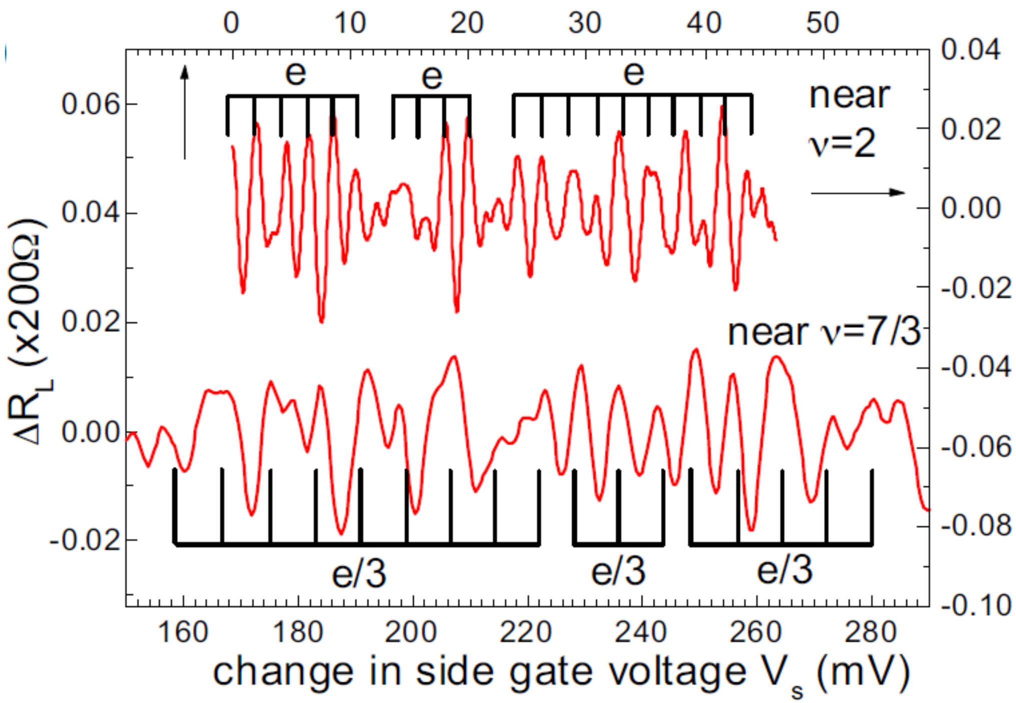

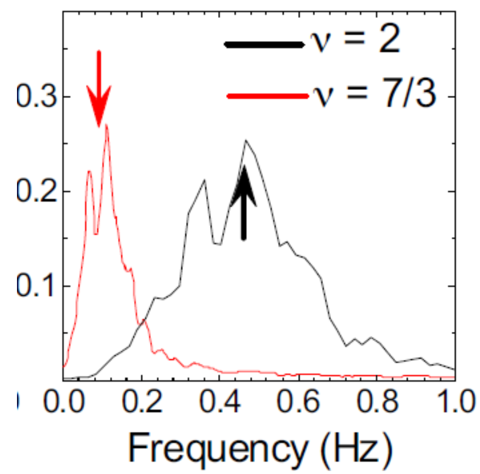

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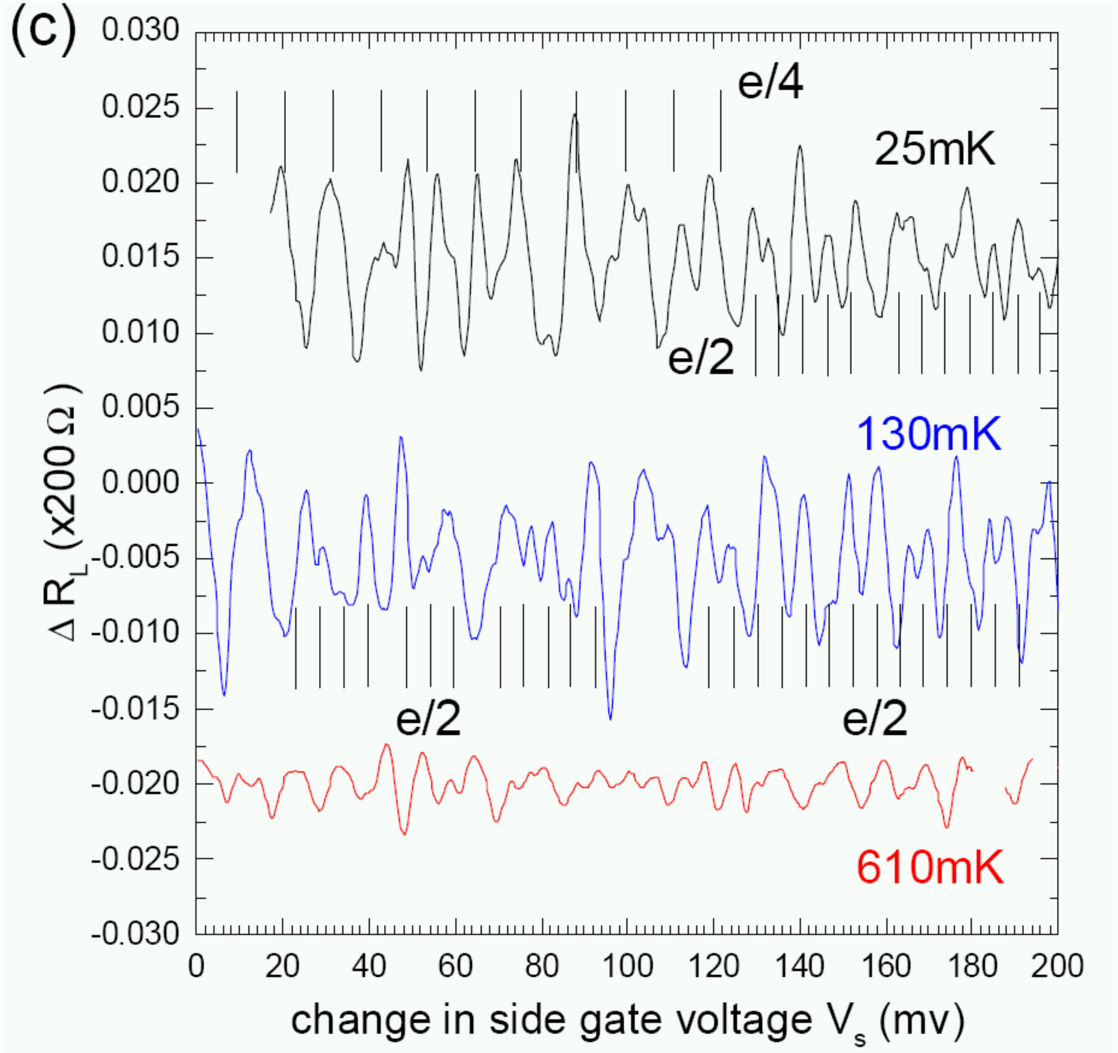

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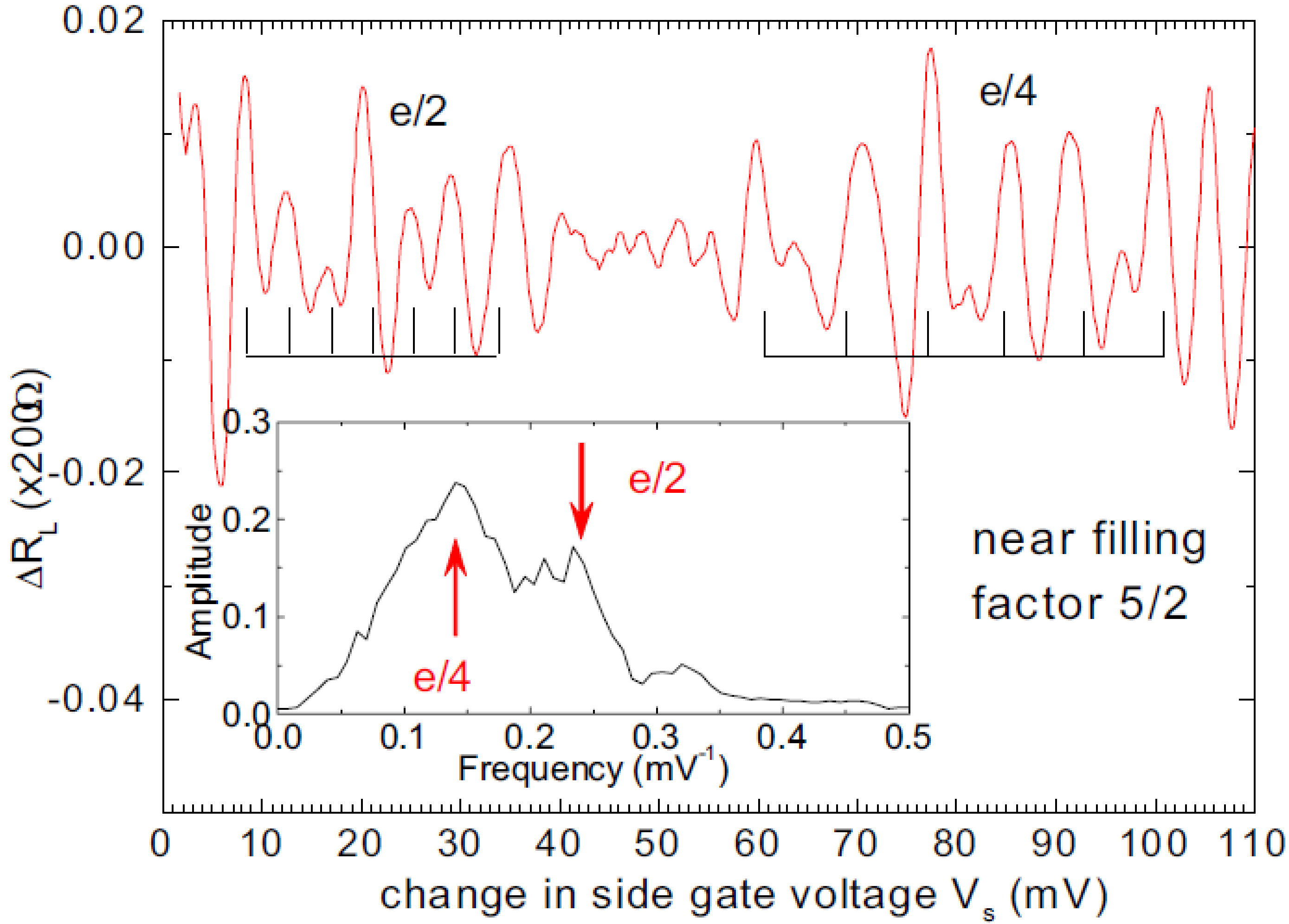

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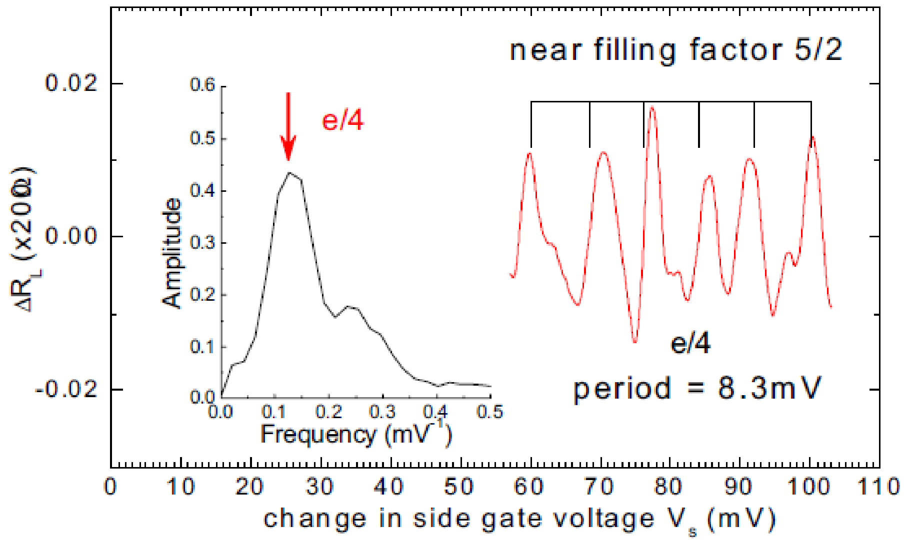
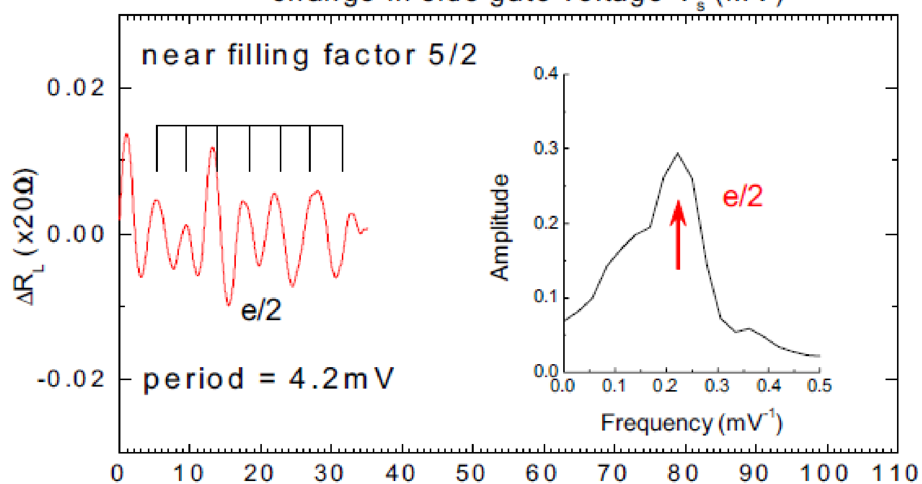


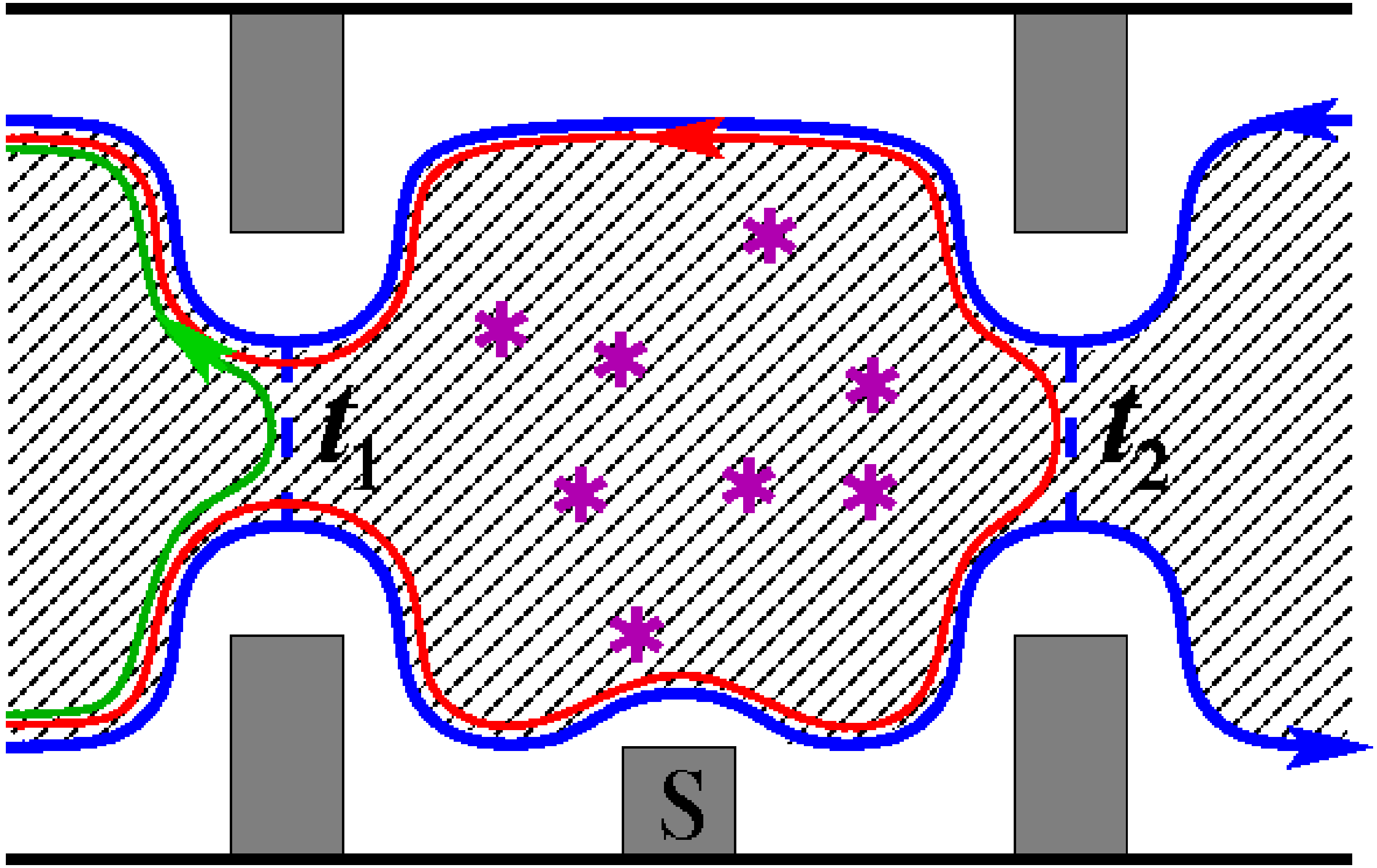


Analysis:
Bishara, Bonderson, KS, Nayak & Slingerland
arXiv:0903.3108,
PRB **80**, 155303 (2009)

Viewpoint: J. Moore
Physics 2, 82 (2009)

What's going on?

A somewhat plausible picture:



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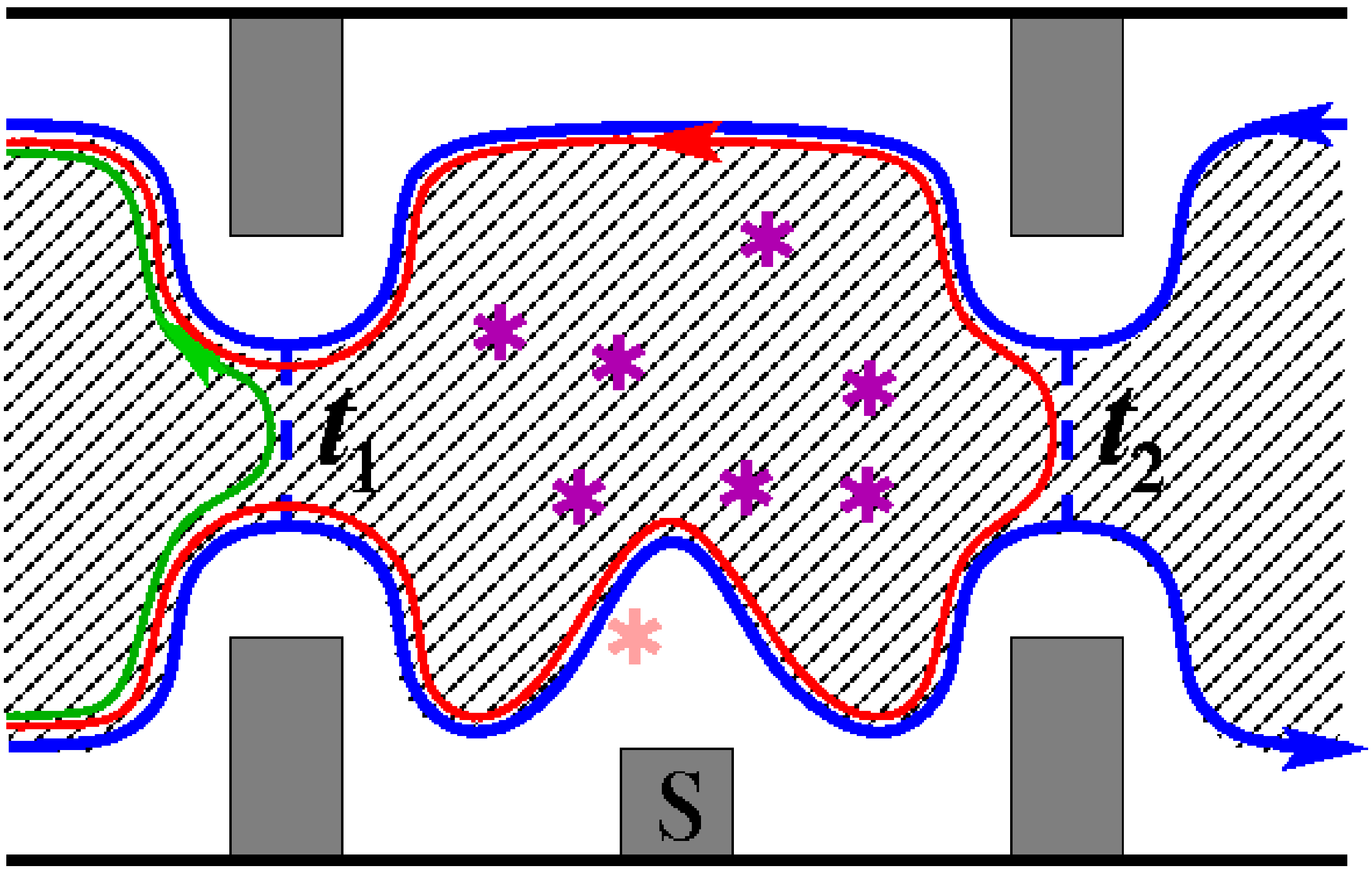
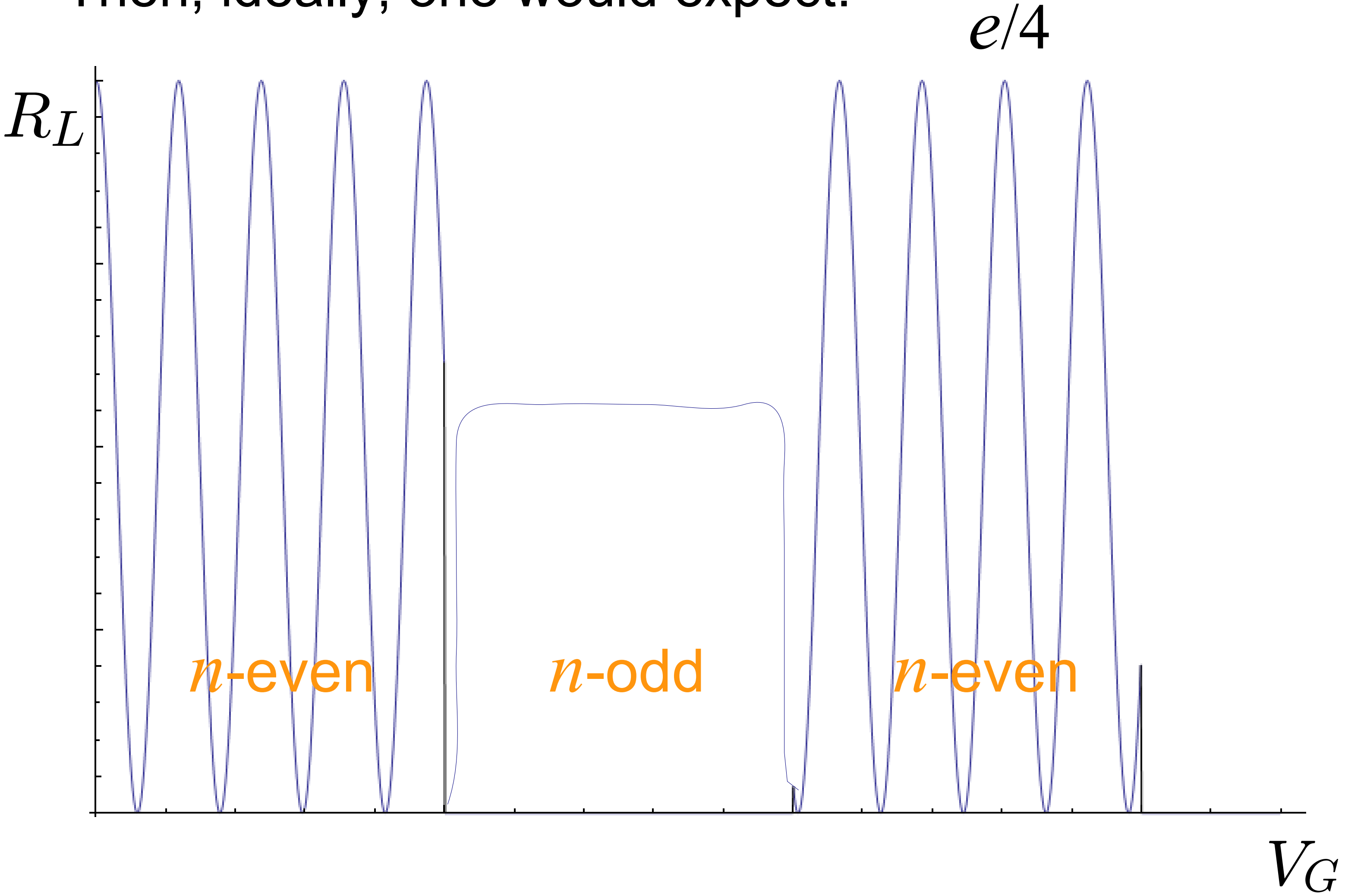


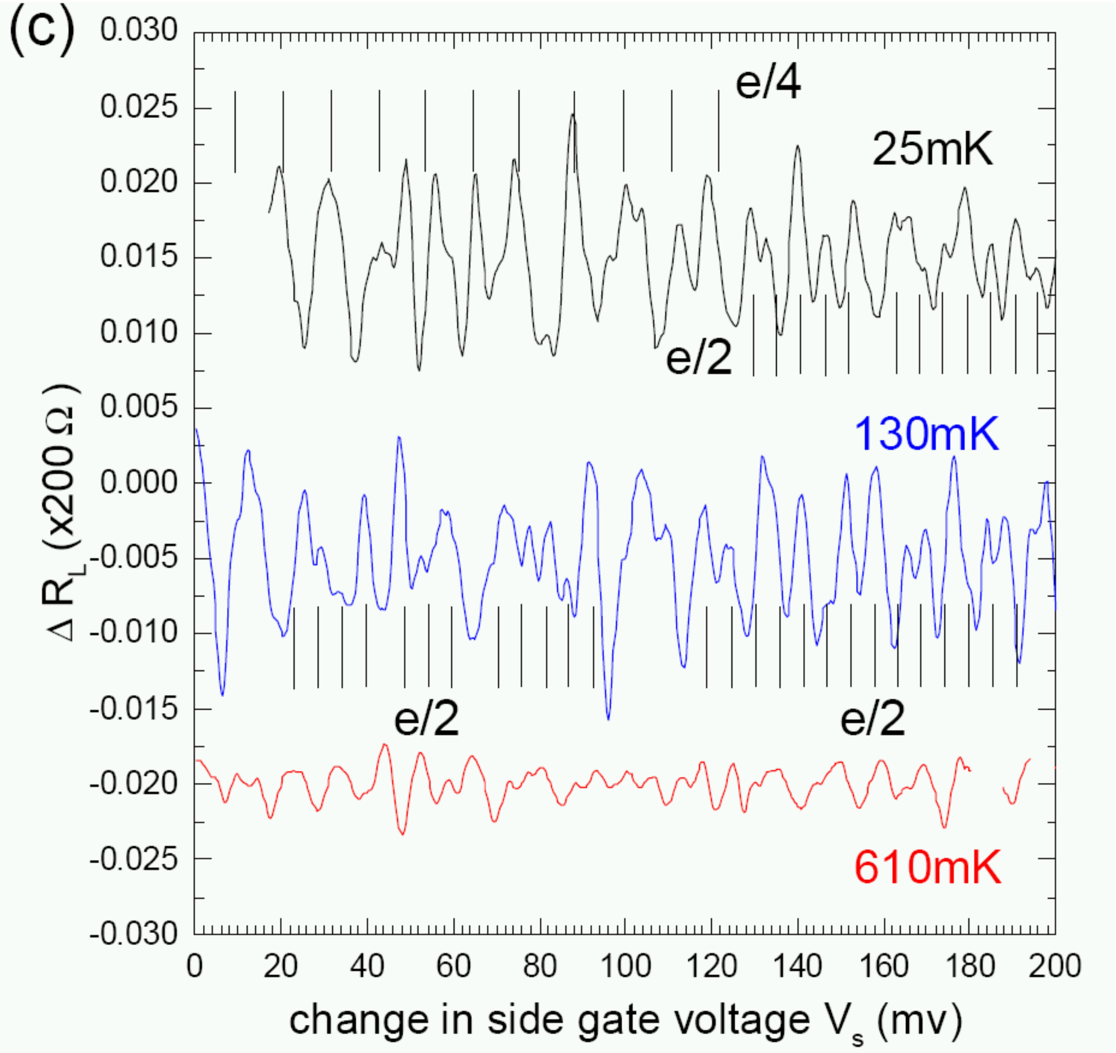
What's going on?

Then, ideally, one would expect:



What's actually seen:

Lucent group: Willett et al. PNAS 2009; arXiv:0911.0345



A possible scenario

$$\Psi_{\mathrm{MR}} = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\right)\prod_{j<k}(z_j - z_k)^2\prod_j e^{-|z_j|^2/4}$$

Laughlin quasiholes will have charge *e*/2

Edge theory: $L = \int \mathrm{d}x \, \left\{\frac{1}{2\pi}\partial_x\phi(\partial_t + v_c\partial_x)\phi + i\psi(\partial_t + v_n\partial_x)\psi\right\}$

where ϕ is a chiral boson, ψ is a chiral Majorana fermion

(Milovanović & Read '96)

Important fields:

$$\Phi_{\mathrm{el}} = \psi e^{i\sqrt{2}\phi} \qquad \Phi_{1/2} = e^{i\phi/\sqrt{2}} \qquad \Phi_{1/4} = \sigma e^{i\phi/2\sqrt{2}}$$

does not involve a neutral mode!

A possible scenario

- Besides e/4, there are also e/2 particles – think "Cooper pairs"
- Somewhat technical: e/2 does not involve a neutral mode, e/4 does. Generally, charged modes are faster: $v_c > v_n$

This is important because of the dephasing length:

$$L_\phi = \frac{1}{2\pi T}\left(\frac{g_c}{v_c} + \frac{g_n}{v_n}\right)^{-1}$$

Bishara & Nayak '08
Kim & Ardonne '08

$g_c = g_n = 1/8$ for e/4 while $g_c = 1/2$ for e/2 in the MR
$g_n = 3/8$ for the $\overline{\text{Pf}}$

Estimates of X. Wan, Z. X. Hu, E. H. Rezayi and K. Yang (PRB '08):

for experiment at 25 mK (Willett *et al*.) (e/4): $L_\phi \sim 1.5\ \mu\text{m}$
(e/2): $L_\phi \sim 5\ \mu\text{m}$

Willett et al (2009): $A \sim 0.2\ \mu\text{m}^2$

A possible scenario

- Besides e/4, there are also e/2 particles – think “Cooper pairs”
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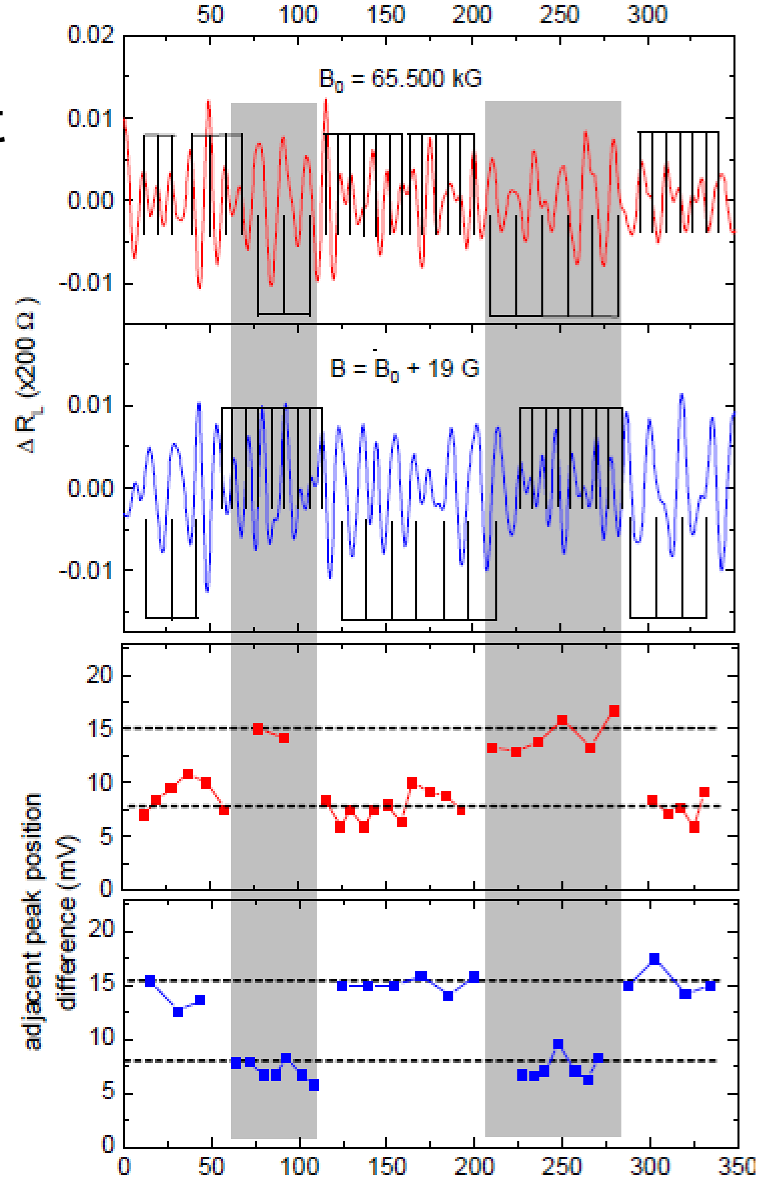
Estimates from Bishara, Bonderson, KS, Nayak & Slingerland arXiv:0903.3108, PRB **80**, 155303 (2009)

$e/4$	MR	$\overline{\text{Pf}}/\text{SU}(2)_2$	K=8	(3,3,1)	e/2
L_ϕ in μm	1.4	0.5	19	0.7	4.8
T^* in mK	36	13	484	19	121

A possible scenario

Perhaps the most convincing argument (Willett, Pfeiffer, West arXiv:0911.0345):
An addition of an extra magnetic flux of $\Phi_0/10$ flips the pattern!

(Increasing the total flux by $\Phi_0/10$ adds one e/4 quasihole.)



A drop of skepticism?

$$I_{12}^{(e/4)} \propto \begin{cases} \cos\left(\frac{e\Phi}{4} \mp \frac{n_q \pi}{4} + N_\psi \pi\right) & \text{for } n_q \text{ even} \\ 0 & \text{for } n_q \text{ odd} \end{cases}$$

$$+$$

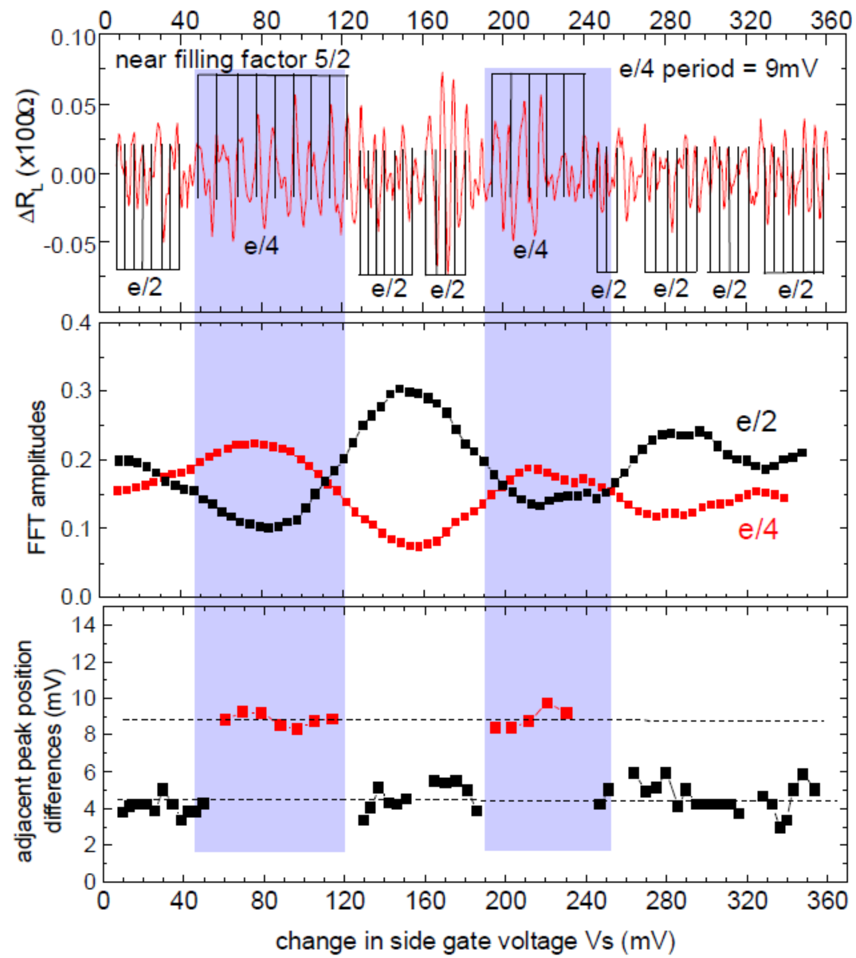
$$I_{12}^{(e/2)} \propto \cos\left(\frac{e\Phi}{2} - \frac{n_q \pi}{2}\right)$$

- Tunneling processes can change $N\psi$, causing uncontrolled π phase shifts in the e/4 oscillations
- e/2 bulk quasiparticles or pairs of e/4 bulk qps entering or leaving the interferometer cause:
  - $\pi/2$ phase shifts in the e/4 oscillations
  - π phase shifts in the e/2 oscillations

A drop of skepticism?

Problem:
Why doesn't the topological charge fluctuate rapidly (due to tunnelling of ψ to/from the edge), suppressing the amplitude of e/4 oscillations even if numbers of localised quasiparticles is *even*?

An estimate based on the numerics of Baraban et al. '09 leads to $\omega \sim 1$GHz.
What gives?



(Willett, Pfeiffer, West arXiv:0911.0345)

Conclusions

- We might have seen the first experimental signature of particles with non-Abelean statistics!
- The preliminary results look very promising, but more (and better) experimental data are clearly needed.

If we had bacon, we could have bacon and eggs, if we had eggs...

- Once the existence of non-Abelean anyons is confirmed, one can try using them for quantum computing
- Looking beyond ν=5/2: What other systems may support non-Abelian anyons?
  - ν=12/5 FQHE?
  - Topological insulator/SC composite structures?