Quantum Matter in Low Dimensions

NORDITA

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UNIVERSAL RESULTS FOR 2D PERCOLATION FROM QUANTUM FIELD THEORY

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Based on :

GD, NPB 818 (2009) 196 [arXiv:0902.3339]

GD, J. Viti, J. Cardy, J.Phys.A 43 (2010) 152001 [arXiv:1001.5424]

GD, J. Viti, NPB, in press [arXiv:1006.2301]

GD, J. Viti, arXiv:1009.1314

Outline

– Introduction

– Random percolation

– Universal results for cluster connectivities

– Universal ratios of critical amplitudes

– Correlated percolation

– Conclusion

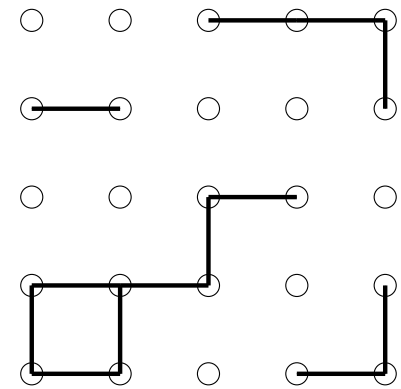
Percolation

Phenomenology: coffee machines, oil fields, gelation, forest fires, epidemics, galaxy clustering, ...

Theory: study of cluster properties on regular lattices

Universality: close to criticality some properties do not depend on lattice details $\longrightarrow$ field theory (non-conventional)

Random percolation



Take an infinite regular lattice

a bond is present with probability p

$W = p^{n_b}(1-p)^{N-n_b}$ weight of a config with n_b bonds

cluster $\equiv$ connected set of bonds

$P \equiv$ probability that a site belongs to an infinite cluster

percolation threshold p_c: $P > 0$ for $p > p_c$

n-point connectivity $P_n(x_1, \ldots, x_n) =$ probability that $x_1, \ldots, x_n$ belong to the same finite cluster

connectivity length ξ: $P_2(x_1, x_2) \sim e^{-|x_1 - x_2|/\xi}, \quad |x_1 - x_2| \to \infty$

mean cluster size $S = \sum_x P_2(x, 0)$

Scaling and universality

$P \simeq B(p-p_c)^\beta, \quad \xi \simeq f_\pm |p-p_c|^{-\nu}, \quad S \simeq \Gamma_\pm |p-p_c|^{-\gamma}, \qquad p \to p_c^\pm$

β, γ, ν, $\frac{f_+}{f_-}$, $\frac{\Gamma_+}{\Gamma_-}$, $\frac{B^2 f_+^2}{\Gamma_+}$, ... are universal

Percolation as the Potts model with $q \to 1$ states

$$\mathcal{H} = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}\,, \qquad s(x) = 1, \ldots, q \qquad\qquad S_q \text{ invariance}$$

$$\begin{aligned} Z &= \sum_{\{s(x)\}} e^{-\mathcal{H}} = \quad \text{(Kasteleyn, Fortuin, '69)} \\ &= \sum_{\text{bond configs}} p^{n_b}(1-p)^{N-n_b} q^{N_c}, \qquad\qquad p = 1 - e^{-J} \end{aligned}$$

$N_c =$ number of clusters

The KF representation shows that q can be taken real. 2nd order transition at J_c for $q \leq q_c$ ($q_c = 4$ in 2D, $q_c \in (2,3)$ in 3D)

The percolative transition is the $q \to 1$ limit of the ferromagnetic Potts transition

$P = \lim_{q \to 1} \frac{\langle \sigma_\alpha(x) \rangle}{q-1}$ $\qquad\qquad \sigma_\alpha(x) = \delta_{s(x),\alpha} - 1/q, \qquad \alpha = 1, \ldots, q$

$P_2(x_1, x_2) = \lim_{q \to 1} \frac{q^2}{q-1} \langle \sigma_\alpha(x_1) \sigma_\alpha(x_2) \rangle_{conn}$

- Scaling and universality for percolation follow from those for Potts

- Scaling ($J \to J_c$) Potts model = Potts field theory = simplest S_q invariant field theory

Three-point connectivity (GD, Viti, '10)

Consider percolation at $p \to p_c^-$ $(P=0)$, $|x_i - x_j| \equiv r_{ij} \gg$ lattice spacing:

$$P_2(x_1,x_2) = \lim_{q\to 1} \frac{q^2}{q-1} \langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\rangle$$
$$P_3(x_1,x_2,x_3) = \lim_{q\to 1} \frac{q^3}{(q-1)(q-2)} \langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\sigma_\alpha(x_3)\rangle$$

$$R(x_1,x_2,x_3) = \frac{P_3(x_1,x_2,x_3)}{\sqrt{P_2(x_1,x_2)P_2(x_1,x_3)P_2(x_2,x_3)}}$$ is an universal function of r_{ij}/ξ

p=p$_c$

In general, conformal invariance implies

$$\langle A_1(x_1)A_2(x_2)\rangle = \delta_{X_1,X_2} C_{12}\, r_{12}^{-2X_1}$$
$$\langle A_1(x_1)A_2(x_2)A_3(x_3)\rangle = C_{123}\, r_{12}^{X_3-X_1-X_2} r_{13}^{X_2-X_1-X_3} r_{23}^{X_1-X_2-X_3}$$

$$A_i(x_1)A_j(x_2) = C_{ij}^k\, r_{12}^{X_k-X_i-X_j}\, A_k(x_2) + \ldots, \qquad C_{ijk} = C_{ij}^k \quad \text{if} \quad C_{ij} = \delta_{ij}$$

Then $P_2 \propto r_{12}^{-2X_\sigma}$, $P_3 \propto (r_{12}r_{13}r_{23})^{-X_\sigma}$,

$$R(x_1,x_2,x_3) \equiv R_c = \lim_{q\to 1} \frac{q^3}{(q-1)(q-2)} C_{\sigma_\alpha\sigma_\alpha\sigma_\alpha} \qquad \text{with} \quad C_{\sigma_\alpha\sigma_\alpha} = \frac{q-1}{q^2}$$

2D from now on. The critical Potts field theory has central charge

$$c = 1 - \frac{6}{t(t+1)}, \qquad \sqrt{q} = 2\sin\frac{\pi(t-1)}{2(t+1)} \qquad \text{(Dotsenko, Fateev, '84)}$$

The spin fields σ_α (multiplicity $q-1$) and the energy field ε (multiplicity 1) have scaling dimensions

$$X_\sigma = X_{(t+1)/2,(t+1)/2}, \qquad X_\varepsilon = X_{2,1} \qquad \text{(Nienhuis)}$$

within the parameterization $$X_{m,n} = \frac{[(t+1)m - tn]^2 - 1}{2t(t+1)}$$

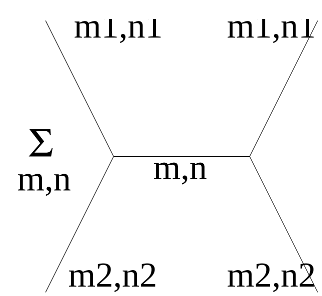
Minimal models (BPZ '84): correspond to discrete values of t (including $t = 3, 4, 5, ..$), possess a finite number of primary fields with dimensions $X_{m,n}$ (m, n positive integers) appearing with integer multiplicity; correlators satisfy linear differential equations

Potts is minimal only at $q = 2, 3$ ($t = 3, 5$); $X_\sigma|_{q=1} = X_{3/2,3/2}|_{t=2} = 5/48$

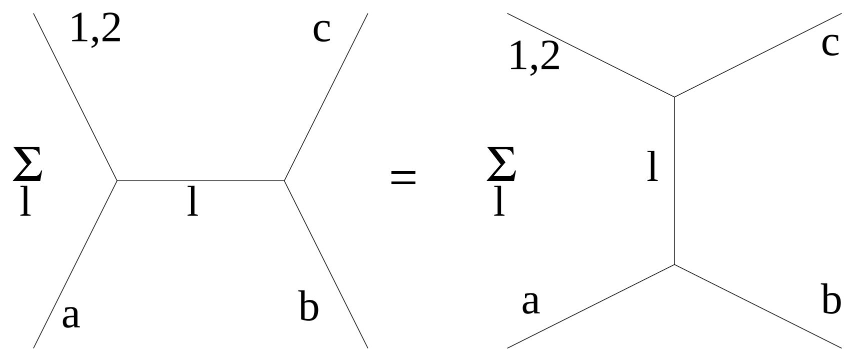
Structure constants for minimal models

1. Dotsenko, Fateev, '85: Coulomb gas

The formulae can be evaluated only at minimal vertices



2. Al. Zamolodchikov, '05: conformal bootstrap



$$C_{X_1,X_2}^{X_3} = C_{X_1,X_2,X_3} = \frac{A\,\Upsilon(2\beta-\beta^{-1}+a_1+a_2+a_3)}{[\Upsilon(2a_1+\beta)\Upsilon(2a_1+2\beta-\beta^{-1})]^{\frac{1}{2}}}\times$$

$$\times \quad \frac{\Upsilon(a_1+a_2-a_3+\beta)\Upsilon(a_2+a_3-a_1+\beta)\Upsilon(a_3+a_1-a_2+\beta)}{[\Upsilon(2a_2+\beta)\Upsilon(2a_2+2\beta-\beta^{-1})\Upsilon(2a_3+\beta)\Upsilon(2a_3+2\beta-\beta^{-1})]^{\frac{1}{2}}}$$

$$\beta=\sqrt{t/(t+1)}\,, \qquad X_i = 2a_i(a_i+\beta-\beta^{-1})$$

$$A = \frac{\beta^{\beta^{-2}-\beta^2-1}[\gamma(\beta^2)\gamma(\beta^{-2}-1)]^{1/2}}{\Upsilon(\beta)}\,, \qquad \gamma(x)\equiv\frac{\Gamma(x)}{\Gamma(1-x)}$$

$$\Upsilon(x) = \exp\left\{\int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left[\left(\frac{Q}{2}-x\right)\frac{t}{2}\right]}{\sinh\frac{\beta t}{2}\sinh\frac{t}{2\beta}}\right]\right\}, \qquad Q=\beta+\beta^{-1}$$

- reproduces the minimal structure constants (when these do not vanish! E.g. $C_{X_\sigma,X_\sigma}^{X_\sigma}|_{q=2}\neq 0$)

- mathematically, can be evaluated for any c and X_i

Potts duality: $\exists$ disorder fields $\mu_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, q$, $\alpha \neq \beta$, $X_\mu = X_\sigma$

$$\frac{q^2}{q-1} \langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\rangle_{J\leq J_c} = \langle \mu_{\alpha\beta}(x_1)\mu_{\beta\alpha}(x_2)\rangle_{J^*\geq J_c}$$

$$\frac{q^3}{(q-1)(q-2)} \langle \sigma_\alpha(x_1)\sigma_\alpha(x_2)\sigma_\alpha(x_3)\rangle_{J\leq J_c} = \langle \mu_{\alpha\beta}(x_1)\mu_{\beta\gamma}(x_2)\mu_{\gamma\alpha}(x_3)\rangle_{J^*\geq J_c}$$

OPE: $\mu_{\alpha\beta}\,\mu_{\beta\gamma} = \delta_{\alpha\gamma}(I + C_\varepsilon\,\varepsilon + \ldots) + (1-\delta_{\alpha\gamma})(C_\mu\,\mu_{\alpha\gamma} + \ldots)$

$$R_c = \lim_{q\to 1} \frac{q^3}{(q-1)(q-2)}\, C_{\sigma_\alpha\sigma_\alpha\sigma_\alpha} = C_\mu|_{q=1}$$

The OPE has a two-channel structure ($\alpha = \gamma$, $\alpha \neq \gamma$) equivalent to

$$\begin{aligned} \mu\,\bar{\mu} &= I + C_\varepsilon\,\varepsilon + \ldots \\ \mu\,\mu + \bar{\mu}\,\bar{\mu} &= C_\mu(\mu + \bar{\mu}) + \ldots \end{aligned}$$

$\phi \equiv \dfrac{\mu + \bar{\mu}}{\sqrt{2}}$ then satisfies $\phi\,\phi = I + C_\varepsilon\,\varepsilon + \dfrac{C_\mu}{\sqrt{2}}\,\phi + \ldots$

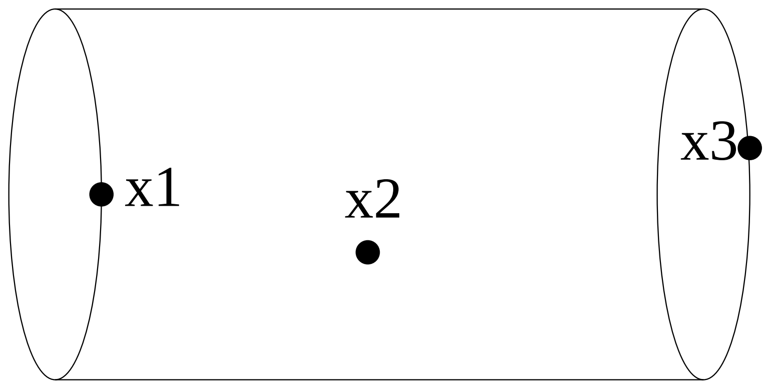
Neutral OPE (like Zamolodchikov's), C_μ never needs to vanish (ok for Ising)

We are led to take $C_\mu = \sqrt{2}\, C^{X_\sigma}_{X_\sigma, X_\sigma} \implies R_c = 1.0220..$

Kleban, Simmons and Ziff (2008) numerically determined

$$R(x_1, x_2, x_3) \approx 1.022$$

at p_c for the configuration in the figure, with x_2 far away from x_1 and x_3



Potts structure constants

$$\frac{q^3}{(q-1)(q-2)} C_{\sigma_\alpha \sigma_\alpha \sigma_\alpha} = C_\mu = \sqrt{2}\, C^{X_\sigma}_{X_\sigma, X_\sigma}$$

$$C_{\sigma_\alpha \sigma_\beta} = \frac{q\delta_{\alpha\beta} - 1}{q^2}, \qquad C_{\sigma_1\sigma_1\sigma_1} = (1-q) C_{\sigma_1\sigma_1\sigma_2} = \frac{1}{2}(q-1)(q-2) C_{\sigma_1\sigma_2\sigma_3}$$

$$C_\varepsilon = -\frac{q^2}{q-1} C^{\varepsilon}_{\sigma_\alpha \sigma_\alpha} = C^{X_\varepsilon}_{X_\sigma, X_\sigma} \qquad (C^{X_\varepsilon}_{X_\sigma, X_\sigma}|_{q=2} = -1/2)$$

The OPE $\sigma_\alpha\, \mu_{\beta\gamma}$ produces parafermions. We expect they have spin $X_{1,3}/2$

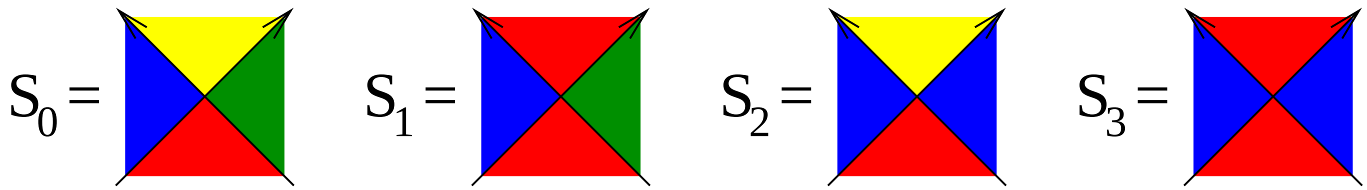
$\mathbf{p} \neq \mathbf{p}_c$

Potts field theory becomes massive, but it is integrable

$$\mathcal{A} = \mathcal{A}_{CFT} + (J - J_c) \int d^2x\, \varepsilon(x)$$

Exact S-matrix (Chim, A.Zamolodchikov, '92)

q degenerate vacua at $J > J_c$; elementary excitations are domain walls $K_{\alpha\beta}$

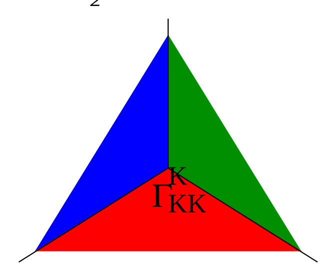


$$S_0(\theta) = \frac{\sinh \lambda\theta \, \sinh \lambda(\theta - i\pi)}{\sinh \lambda\left(\theta - \frac{2\pi i}{3}\right) \sinh \lambda\left(\theta - \frac{i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_1(\theta) = \frac{\sin \frac{2\pi\lambda}{3} \, \sinh \lambda(\theta - i\pi)}{\sin \frac{\pi\lambda}{3} \, \sinh \lambda\left(\theta - \frac{2i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$S_2(\theta) = \frac{\sin \frac{2\pi\lambda}{3} \, \sinh \lambda\theta}{\sin \frac{\pi\lambda}{3} \, \sinh \lambda\left(\theta - \frac{i\pi}{3}\right)} \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_3(\theta) = \frac{\sin \lambda\pi}{\sin \frac{\pi\lambda}{3}} \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$\sqrt{q} = 2 \sin \frac{\pi\lambda}{3} \qquad \Pi\left(\frac{\lambda\theta}{i\pi}\right) = \frac{\sinh \lambda\left(\theta + i\frac{\pi}{3}\right)}{\sinh \lambda(\theta - i\pi)} \exp\left(\int_0^\infty \frac{dx}{x} \frac{\sinh \frac{x}{2}\left(1 - \frac{1}{\lambda}\right) - \sinh \frac{x}{2}\left(\frac{1}{\lambda} - \frac{5}{3}\right)}{\sinh \frac{x}{2\lambda} \cosh \frac{x}{2}} \sinh \frac{x\theta}{i\pi}\right)$$

$$\Gamma^K_{KK} = \sqrt{-i \, \mathrm{Res}_{\theta = 2i\pi/3} S_0(\theta)}$$



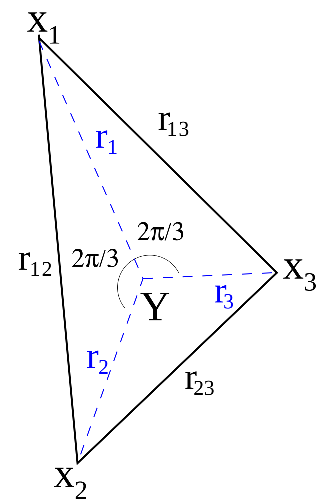
Spectral series for correlation functions (GD, Cardy, '98)

For $p < p_c$:

$$P_2(x_1, x_2) = \lim_{q\to 1} \langle \mu_{\alpha\beta}(x_1)\mu_{\beta\alpha}(x_2)\rangle_{J>J_c} = \frac{F_\mu^2}{\pi} K_0(r_{12}/\xi) + O(e^{-2r_{12}/\xi})$$

$$P_3(x_1, x_2, x_3) = \lim_{q\to 1} \langle \mu_{\alpha\beta}(x_1)\mu_{\beta\gamma}(x_2)\mu_{\gamma\alpha}(x_3)\rangle_{J>J_c} = \frac{F_\mu^3}{\pi} g\, K_0(r_Y/\xi) + O(e^{-\rho/\xi})$$

$$g \equiv \Gamma_{KK}^K|_{q=1} = 1.0450..$$



Triangle with internal angles $< 2\pi/3$

$$r_Y \equiv r_1 + r_2 + r_3 \quad < \rho \equiv \min\{r_{12} + r_{13},\, r_{12} + r_{23},\, r_{13} + r_{23}\}$$

$$R(x_1, x_2, x_3) \simeq \sqrt{\pi}\, g \frac{K_0(r_Y/\xi)}{\sqrt{K_0(r_{12}/\xi)K_0(r_{13}/\xi)K_0(r_{23}/\xi)}}\,, \qquad r_{ij} \gg \xi, \quad p \to p_c^-$$

In the opposite limit $r_{ij} \ll \xi$ one recovers R_c

Universal amplitude ratios (GD, Viti, Cardy, '10)

$$S \simeq \Gamma^{\pm}|p-p_c|^{-\gamma}, \qquad \xi \simeq f^{\pm}|p-p_c|^{-\nu}$$

$$P \simeq B(p-p_c)^{\beta}, \qquad \frac{\langle N_c\rangle}{N} \simeq A^{\pm}|p-p_c|^{2-\alpha}$$

	Field Theory	Lattice
A^-/A^+	1	1^a
f^-/f^+	2	-
f^-_{2nd}/f^-	1.001	-
f^-_{2nd}/f^+_{2nd}	3.73	4.0 ± 0.5^c
$4B^2(f^-_{2nd})^2/\Gamma^-$	2.22	2.23 ± 0.10^d
$(-80/27\,A^-)^{1/2} f^-_{2nd}$	0.926	$\approx 0.93^{a+b}$
Γ^-/Γ^+	160.2	162.5 ± 2^e

a) Domb, Pearce, J. Phys. A 9 (1976) L137

b) Aharony, Stauffer, J. Phys. A 30 (1997) L301

c) Corsten, Jan, Jerrard, Physica A 156 (1989) 781

d) Daboul, Aharony, Stauffer, J. Phys. A 33 (2000) 1113

e) Jensen, Ziff, Phys. Rev. E 74 (2006) 20101

30 years of efforts by the lattice community, Γ^-/Γ^+ most elusive

$$\frac{\Gamma_-}{\Gamma_+} = \frac{\int d^2x\, P_2(x,0)|_{p<p_c}}{\int d^2x\, P_2(x,0)|_{p>p_c}}$$

$$P_2(x_1,x_2)|_{p<p_c} = \frac{F_\mu^2}{\pi} K_0(r_{12}/\xi) + O(e^{-2r_{12}/\xi})$$

$$P_2(x_1,x_2)|_{p>p_c} = \frac{F_\sigma}{\pi^2}\int_0^\infty d\theta\, |f(2\theta)\Omega(2\theta)|^2 K_0\left(2\frac{r_{12}}{\xi}\cosh\theta\right) + O(e^{-3r_{12}/\xi})$$

$$f(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{-2\int_0^\infty \frac{dx}{x}\,\frac{\sinh\frac{x}{3}\cosh\frac{x}{6}}{\sinh^2 x\cosh\frac{x}{2}}\,\sin^2\frac{(i\pi-\theta)x}{2\pi}\right\}$$

$$\Omega(\theta) = \mathcal{C}\int_{-\infty}^{+\infty}\frac{dx}{2\pi}\,W\left(-x-\frac{\theta}{2}+i\pi\right)W\left(-x+\frac{\theta}{2}+i\pi\right)e^{-x/6}$$

$$\mathcal{C} = \frac{1}{4}\exp\left[\int_0^\infty \frac{dt}{t}\,\frac{4\sinh^2\frac{t}{4}\sinh\frac{t}{2}}{\sinh^2 t}\right], \qquad W(\theta) = -\frac{2}{\cosh\theta}\,\exp\left[\int_0^\infty \frac{dt}{t}\,\frac{2\sinh\frac{t}{2}}{\sinh^2 t}\sin^2\left(\frac{t}{2\pi}(i\pi-\theta)\right)\right]$$

$$F_\mu^2 = F_\sigma \lim_{\theta\to\infty}|f(\theta)\Omega(\theta)|$$

The structure above p_c is produced by the non-locality between Potts spin and domain wall excitations

Correlated percolation

Study cluster properties in presence of interaction among the sites

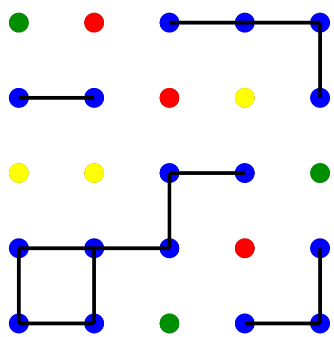
Potts interaction

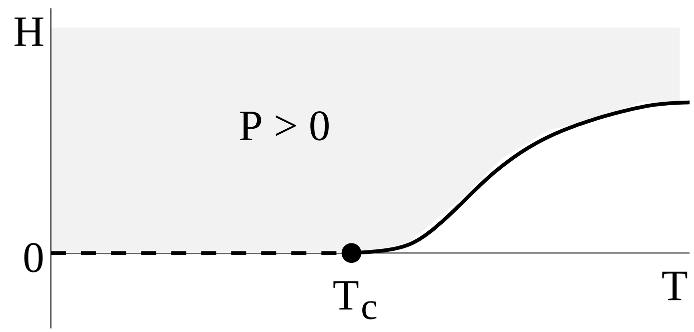
$$-\mathcal{H}(T,H) = \frac{1}{T}\sum_{\langle x,y\rangle}(\delta_{\tau(x),\tau(y)} - 1) + H\sum_{x}(\delta_{\tau(x),blue} - 1), \qquad \tau(x) \text{ takes } q \text{ colors}$$

Put a bond with probability p between blue nearest neighbors

RG analysis (Coniglio, Peruggi, '82): two fixed points for p, i.e. two universality classes

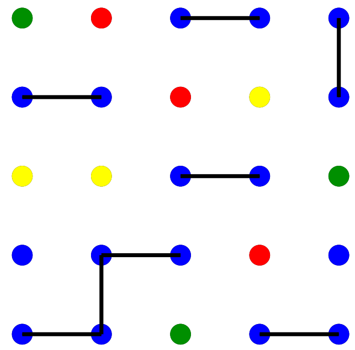
a) p=1: **spin clusters**

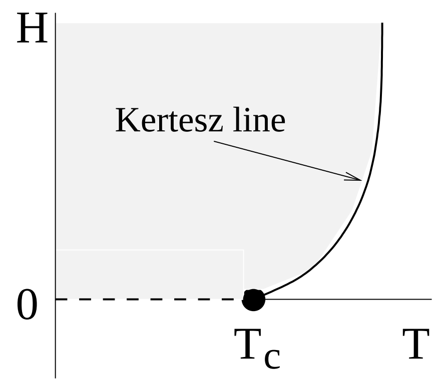




- perc. exponents at T_c determined by $X_s = X_{t/2,t/2}$ (Vanderzande, '92)
- for q=2 (Ising) the transition above T_c is integrable in the scaling limit; deviations from $H = 0$ are entirely due to corrections to scaling (GD, '09)

b) p=1-e$^{-1/T}$: **KF clusters** (or "droplets")





- q=1 (all sites are blue) is again random percolation
- percolative exponents at T_c coincide with the magnetic ones

Three-point connectivity for Potts clusters (GD, Viti, '10)

q	1	2	3	4
R_c^{KF}	1.0220..	1.0524..	1.0923..	1.1892..
R_c^{spin}	$-$	1.3767..	1.3107..	1.1892..
$\left(\Gamma_{KK}^{K}\right)^{KF}$	1.0450..	1.1547..	1.3160..	1.8612..

$R(x_1, x_2, x_3)$ for Ising spin clusters is expected to interpolate from $R_c^{spin}|_{q=2}$ at short distances to $R_c^{KF}|_{q=1}$ at large distances

Universal amplitude ratios for Ising clusters (GD, Viti, '10)

	spin clusters	KF clusters
Γ_a/Γ_b^+	non-universal	40.3
$f_{2nd,a}/f_a$	"	0.99959..
f_a/f_b^+	"	2
$f_a/\hat{f}_a$	"	1
$A_{k,a}/A_{k,b}^+; k=0,-1$	"	1
Γ_b^+/Γ_b^-	-	1
f_b^+/f_b^-	1/2	1
$f_{2nd,b}^-/f_b^-$	0.6799	0.61
$f_{2nd,b}^+/f_{2nd,b}^-$	-	1
$f_b^+/\hat{f}_b^\pm$	1/2	1
U_b	24.72	15.2
$A_{k,b}^+/A_{k,b}^-; k=0,-1$	1	1
$A_{0,b}^\pm/A_{-1,b}^\pm$	$-\gamma-\ln\pi=-1.7219..$	$-\gamma-\ln\pi=-1.7219..$
r_b	$\frac{3\sqrt{3}(\gamma+\ln\pi)}{64\pi^2}=0.014165..$	$-\frac{\gamma+\ln\pi}{12\pi^2}=-0.014539..$
f_c^+/f_c^-	1/2	-
$f_{2nd,c}^-/f_c^-$	1.002	-
$f_c^+/\hat{f}_c^\pm$	$\sin\frac{\pi}{5}=0.58778..$	-
$A_{k,c}^+/A_{k,c}^-; k=0,1$	1	-
$A_{0,c}^\pm/A_{1,c}^\pm$	$-0.42883..$	-
r_c	$-3.7624..\times 10^{-3}$	-

$\gamma = 0.5772..$ Euler-Mascheroni constant

Conclusion

– Recent progresses in field theory yield new universal results for 2D percolation

– Percolation guides us in making progesses with less understood sectors of quantum field theory