Confinement in Ising field theory and Ising spin chain: Bethe-Salpeter equation approach

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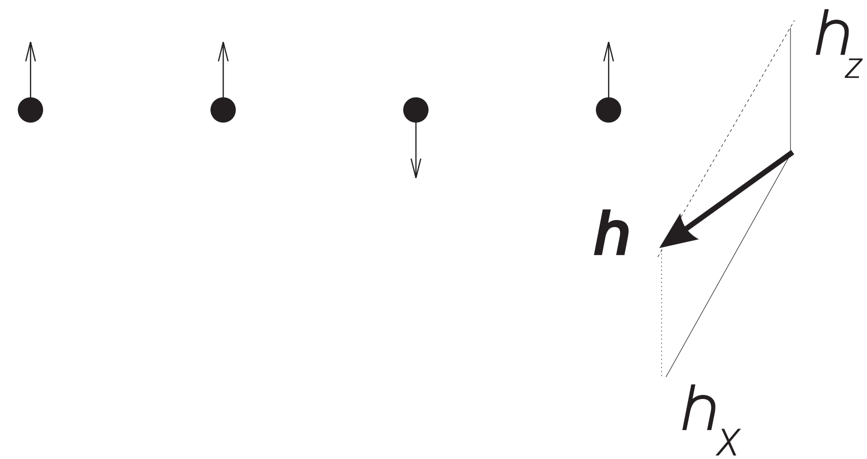
Quantum spin-1/2 Ising chain

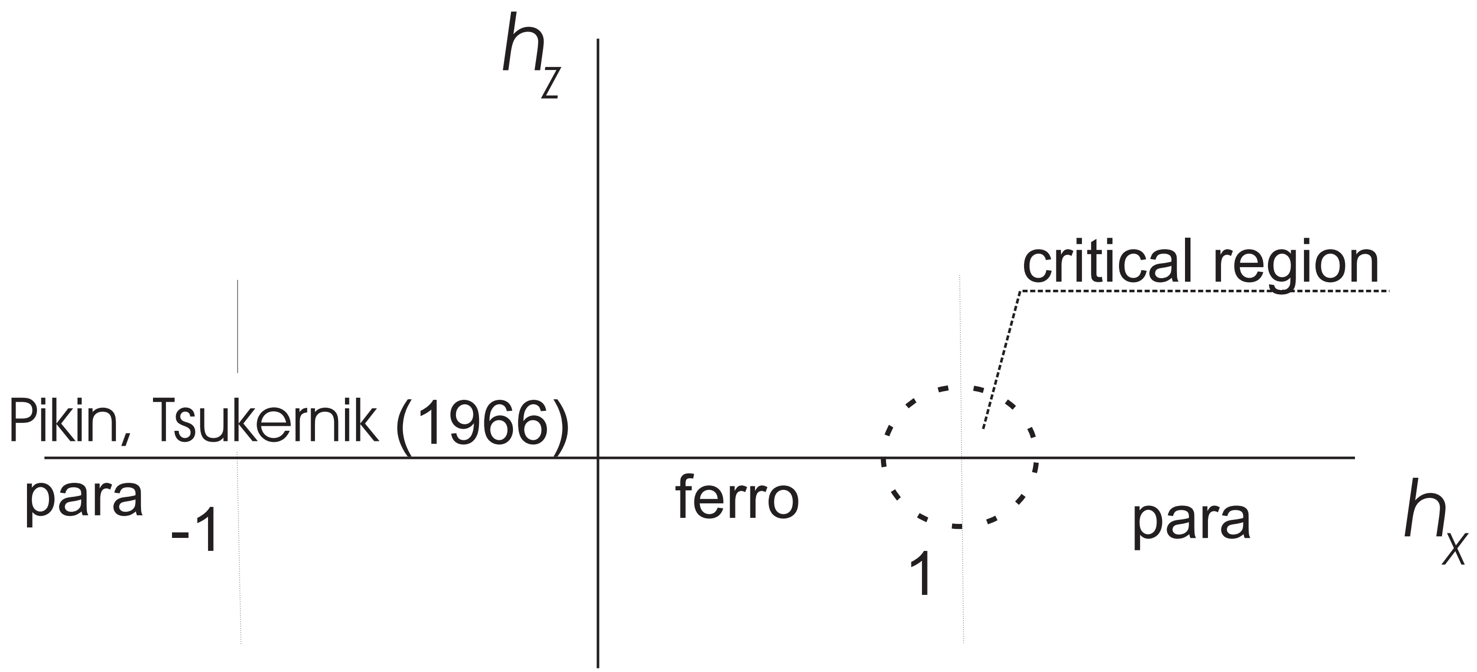


$$\mathcal{H}_{ch} = -\sum_{n=1}^{\mathcal{N}} (\sigma_n^z \, \sigma_{n+1}^z + h_x \sigma_n^x + h_z \sigma_n^z).$$

Here σ_n^x, σ_n^z are the Pauli matrices relating to the n-th site of the chain:

$$\sigma_n^x = \begin{pmatrix} 0 \, 1 \\ 1 \, 0 \end{pmatrix}, \quad \sigma_n^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Quantum Ising spin chain phase diagram at $T = 0$



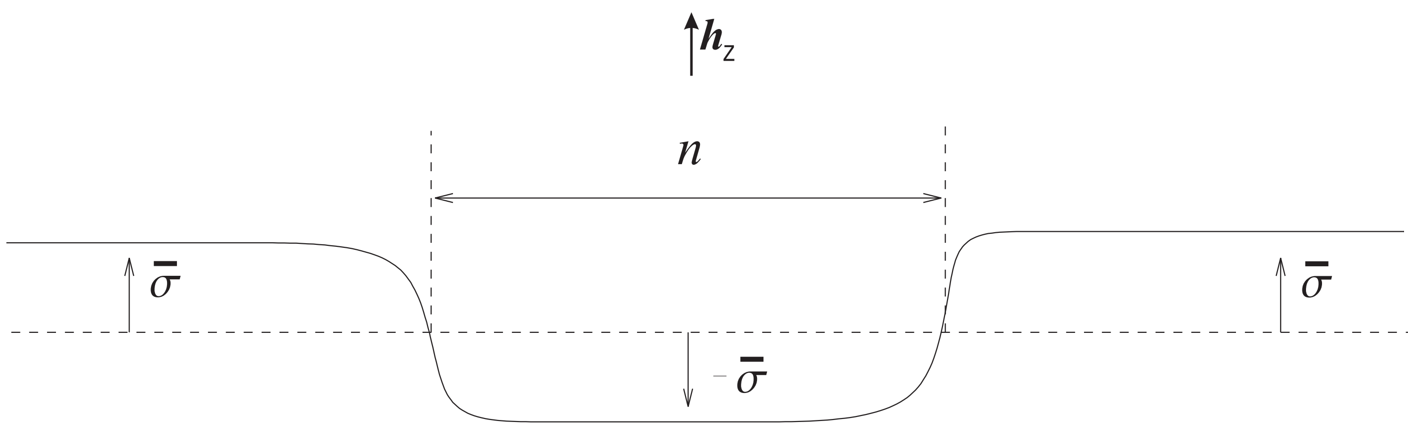
In the critical region quantum Ising spin-1/2 chain is equivalent to the Ising field theory.

Confinement of fermions in the small-h_z ferromagnetic regime

$$\mathcal{H}_{ch} = -\sum_{n=1}^{\mathcal{N}} (\sigma_n^z \sigma_{n+1}^z + h_x \sigma_n^x + h_z \sigma_n^z)$$

1. $h_z = 0$, $h_x < 1$. Two ferromagnetic ground states $|\Phi_\uparrow(0)\rangle$ and $|\Phi_\downarrow(0)\rangle$ with spontaneous magnetizations $\langle \sigma_n^z \rangle = +\bar{\sigma}$ and $\langle \sigma_n^z \rangle = -\bar{\sigma}$ have the same energy $E_\uparrow(0) = E_\downarrow(0)$. Elementary excitations (free fermions) are the domain walls, interpolating between the two degenerate vacua.

2. $0 < h_z \ll 1$, $h_x < 1$. Degeneration is removed: $|\Phi_\uparrow(h_z)\rangle$ is the ground state, $|\Phi_\downarrow(h_z)\rangle$ is the metastable state; $E_\uparrow(h_z) - E_\downarrow(h_z) \approx -2\bar{\sigma} h_z \mathcal{N}$. Two domain walls **attract** one another with the energy $2\bar{\sigma} h_z n$. An isolated domain wall gains infinite energy. Elementary excitations now are coupled pairs of fermions.



McCoy and Wu picture (1978):

The relative motion of two kinks is described by the Schrödinger equation

$$-\frac{1}{m}\frac{d^2}{dx^2}\,\psi_n(x) + \lambda|x|\psi_n(x) = \delta E_n\,\psi(x),$$
$$\psi(x) = -\psi(-x).$$

Energy levels of the kink bound-states:

$$\delta E_n = z_n \lambda^{2/3} m^{-1/3}, \quad n = 1, 2\ldots,$$

where $-z_n$ are zeroes of Airy function, $\mathrm{Ai}(-z_n) = 0$.

Experiment: R. Coldea *et al.*, Science **327**, 177 (2010).

Theory: Delfino, Mussardo, Fonseca and Zamolodchikov, Tsvelik, R, ...

Bethe-Salpeter equation approach was introduced by Fonseca and Zamolodchikov (2001) for the Ising field theory.

Ising Field Theory

Ising field theory (IFT) gives the scaling limit of the Ising spin chain model, and of the two-dimensional Ising model as well.

Euclidean action:

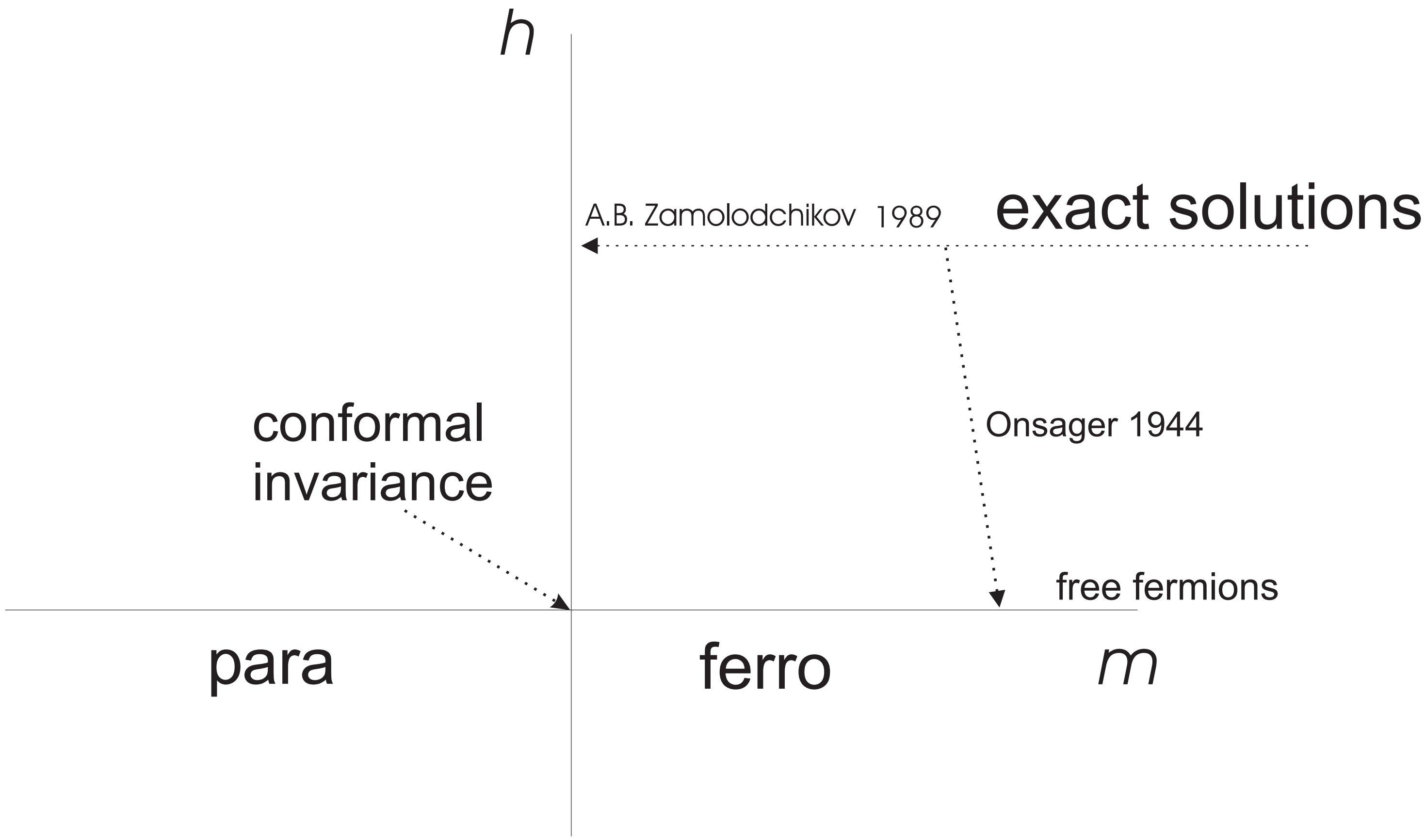
$$\mathcal{A}_{IFT} = \mathcal{A}_{CFT} + \frac{m}{2\pi}\int \varepsilon(x)\, d^2x - h\int \sigma(x)\, d^2x.$$

Relevant fields: Energy density $\varepsilon(x)$ and spin density $\sigma(x)$.

Scaling dimensions: $X_\epsilon = 1$ and $X_\sigma = 1/8$.

$m \sim 1 - h_x$ and $h \sim h_z$ at $h_x \to 1$, $h_z \to 0$.

Ising field theory phase diagram



Physics of Ising field theory is determined by the single scaling parameter η:

$$\eta = \frac{m}{h^{8/15}}$$

.

Quantum Hamiltonian of the Ising field theory

$$\mathcal{H} = \mathcal{H}_{FF} + V.$$

Free fermionic Hamiltonian:

$$\mathcal{H}_{FF} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\, a_p^\dagger a_p,$$

with the dispersion law $\omega(p) = \sqrt{p^2 + m^2}$, and fermionic operators a_p, $a_p^\dagger$:

$$\{a_p, a_{p'}^\dagger\} = 2\pi\delta(p - p'), \quad \{a_p, a_{p'}\} = \{a_p^\dagger, a_{p'}^\dagger\} = 0.$$

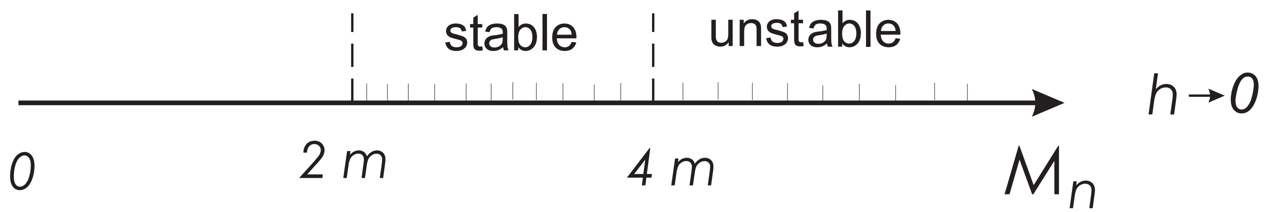
Interaction:

$$V = -h \int_{-\infty}^{\infty} d\mathrm{x}\, \sigma(\mathrm{x}).$$

Formfactors $\langle p_1 ... p_N | \sigma(x) | p_1' ... p_{N'}' \rangle \equiv \langle 0 | a_{p_1} ... a_{p_N} \sigma(x) a_{p_1'}^\dagger ... a_{p_{N'}'}^\dagger | 0 \rangle$ of the order spin operator $\sigma(x)$ are known due to Berg, Karowski, and Weisz (1979).

Meson mass spectrum

At weak magnetic field, the messon masses $M_1, M_2, \ldots, M_n, \ldots$ should fill dense the region above the two kink masses $2m$.



Mesons with masses $M_n < 2M_1$ are stable.

Mesons with masses $M_n > 2M_1$ can decay in two (or more) light mesons.

Perturbation theory for the ferromagnetic Ising field theory at small $h > 0$.

$$\mathcal{H} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\, a_p^\dagger a_p - h \int dx\, \sigma(x),$$

with free fermionic spectrum $\omega(p) = (p^2 + m^2)^{1/2}$.

The meson energy spectrum $E_n(P) = (M_n^2 + P^2)^{1/2}$ is determined by the eigenvalue problem:

$$\mathcal{H} \;|\, \Phi_n(P)\rangle = [E_n(P) + E_{\text{vac}}]\,|\, \Phi_n(P)\rangle, \quad \hat{P}\,|\, \Phi_n(P)\rangle = P\,|\, \Phi_n(P)\rangle,$$

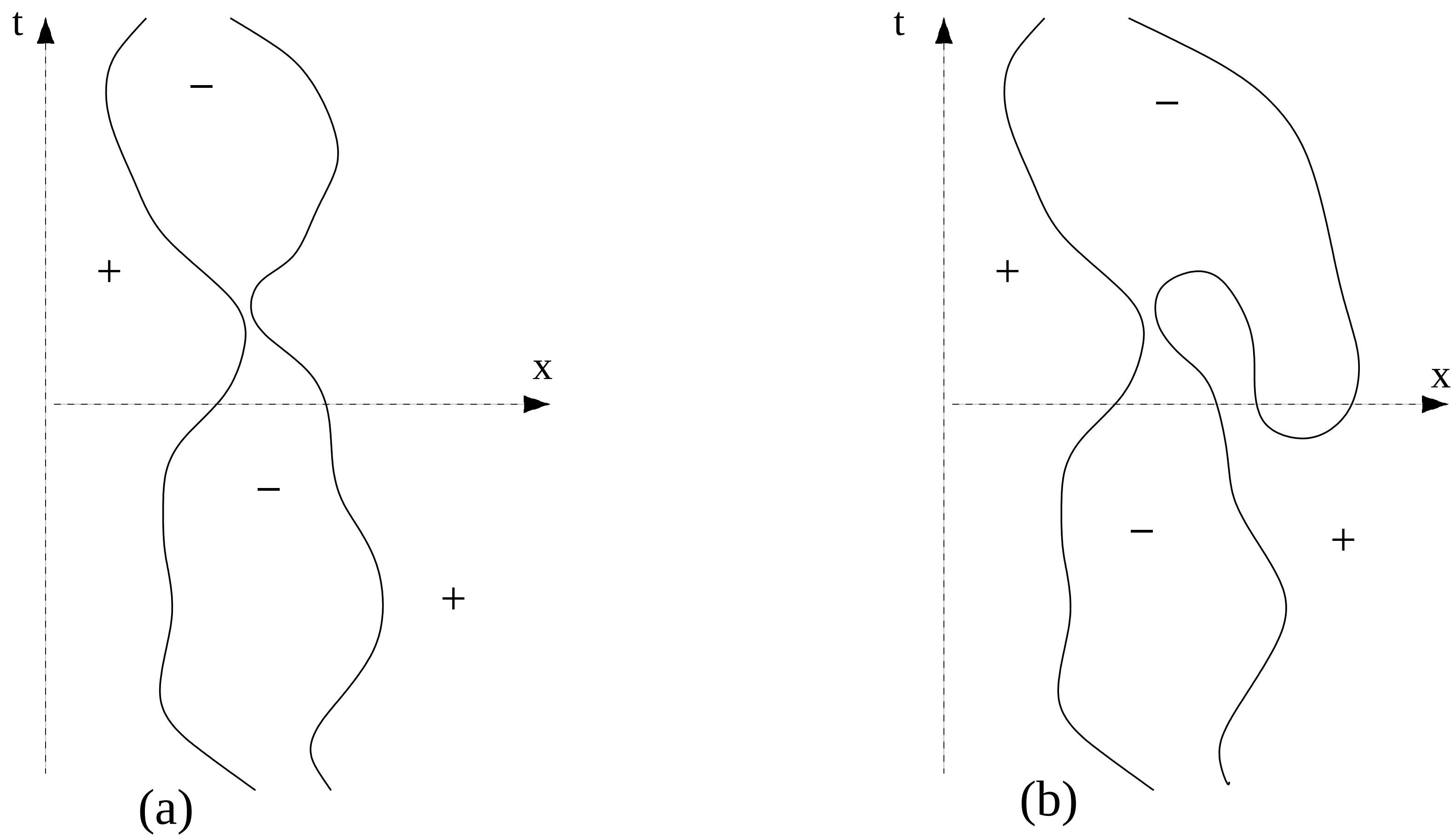
where $\hat{P}$ is the momentum operator, E_{vac} is the ground state energy, and M_n is the meson mass.

Two-quark approximation:

$$\mathcal{P}_2 \mathcal{H} \mathcal{P}_2 \,|\, \widetilde{\Phi}_n(P)\rangle = [\widetilde{E}_n(P) + \widetilde{E}_{\text{vac}}]\,|\, \widetilde{\Phi}_n(P)\rangle, \quad \hat{P}\,|\, \widetilde{\Phi}_n(P)\rangle = P\,|\, \widetilde{\Phi}_n(P)\rangle,$$

where $\mathcal{P}_2$ is the orthogonal projector onto the two-quark subspace $\mathcal{F}_2$ of the Fock space $\mathcal{F}$, $\quad |\, \widetilde{\Phi}_n(P)\rangle \in \mathcal{F}_2$.

In the two-quark approximation $\widetilde{E}_n(P) \neq (\widetilde{M}_n^2 + P^2)^{1/2}$.



Possible world lines of quarks in a meson. (a) Both quarks propagate forwards in time. (b) Creation an annihilation of virtual pairs lead to the presence of more then two quarks in the intermediate state. (from FZ 2006)

Bethe-Salpeter equation (Fonseca, Zamolodchikov 2001, 2006)

In the momentum representation:

$$\left[\omega(P/2+p)+\omega(P/2-p)-\Delta\widetilde{E}(P)\right]\Psi_P(p) = f_0 \int_{-\infty}^{\infty} G_P(p|k)\,\Psi_P(k)\,\frac{dk}{2\pi},$$

where $f_0 = 2h\bar{\sigma}$ is the 'bare string tension',

$$\langle P'/2-p, P'/2+p|\widetilde{\Phi}(P)\rangle = 2\pi\delta(P'-P)\,\Psi_P(p), \quad \Psi_P(-p) = -\Psi_P(p),$$

$$G_P(p|k) = \frac{1}{(p-k)^2} - \frac{1}{(p+k)^2} + G_P^{(reg)}(p|k),$$

and $G_P^{(reg)}(p|k)$ is regular at real p and k.

In the coordinate representation:

$$\left[\omega(P/2+i\partial_x)+\omega(P/2-i\partial_x)+f_0\,|x|-\Delta\widetilde{E}(P)\right]\Psi_P(x)$$

$$= f_0 \int_{-\infty}^{\infty} dx'\,U_P^{(loc)}(x|x')\,\Psi_P(x').$$

Perturbative solutions of the Bethe-Salpeter equation

$$\left[\omega(P/2 + i\partial_x) + \omega(P/2 - i\partial_x) + f_0\,|x| - \Delta\widetilde{E}(P)\right]\Psi_P(x)$$

$$= f_0 \int_{-\infty}^{\infty} dx'\, U_P^{(loc)}(x|x')\,\Psi_P(x').$$

at $f_0 \to 0$.

Meson momentum P is a free parameter. For technical reasons, it is convenient to put $P \to \infty$. Then the meson mass in the two-quark approximation $\widetilde{M}$ is defined by

$$\Delta\widetilde{E}(P) = |P| + \frac{\widetilde{M}^2}{2|P|} + O(|P|^{-3}).$$

$$\widetilde{M}_1 < \widetilde{M}_2 < \ldots < \widetilde{M}_n < \widetilde{M}_{n+1} < \ldots$$

Low energy expansion in $\lambda = f_0/m^2 = 2h\bar{\sigma}/m^2 \to 0$, $n \ll \lambda^{-1}$.

McCoy and Wu (1978), FZ(2001, 2006)

$$\frac{\widetilde{M}_n^2}{4m^2} - 1 = z_n\, t^2 + \frac{z_n^2}{5}\, t^4 - \left(\frac{3\, z_n^3}{175} + \frac{57}{280}\right) t^6 + \left(\frac{23\, z_n^4}{7875} + \frac{1543\, z_n}{12600}\right) t^8 + \frac{13}{1120\,\pi}\, t^9 + \left(-\frac{1894\, z_n^5}{3031875} - \frac{23983\, z_n^2}{242550}\right) t^{10} + \frac{3313\, z_n}{10080\,\pi}\, t^{11} + \ldots,$$

where $t = \lambda^{1/3}$, and $(-z_n)$ is the zero of the Airy function, $\mathrm{Ai}(-z_n) = 0$.

Semiclassical (WKB) expansion in λ: $n \gg 1$

R (2005, 2009), FZ (2006).

$$\frac{\widetilde{M}_n^2}{4m^2} = \cosh^2 \theta_n,$$

where θ_n solves equation

$$\sinh 2\theta_n - 2\theta_n = 2\pi\lambda(n - 1/4) + 2\lambda^2 S_1(\theta_n) + 2\lambda^3 S_2(\theta_n) + O(\lambda^4),$$

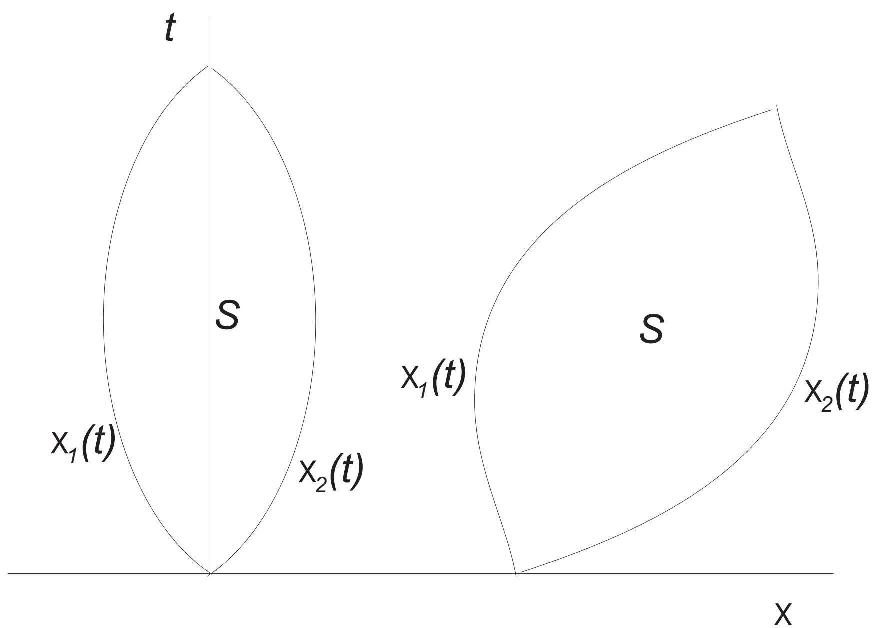
Bohr-Sommerfeld quantization rule

Consider two interacting particles with coordinates $\mathrm{x}_1(t)$, $\mathrm{x}_2(t) \in \mathbb{R}$ described by the classical Lorentz invariant action

$$\mathcal{A}_{\mathrm{cl}} = -m \int_0^{t_m} dt[(1-\dot{\mathrm{x}}_1^2)^{1/2} + (1-\dot{\mathrm{x}}_2^2)^{1/2}] - 2h\bar{\sigma}S,$$

$$\text{with } S = \int_0^{t_m} dt\,[\mathrm{x}_2(t) - \mathrm{x}_1(t)],$$

under the following constrains : $\mathrm{x}_1(0) = \mathrm{x}_2(0)$, $\mathrm{x}_1(t_m) = \mathrm{x}_2(t_m)$, and $\mathrm{x}_1(t) < \mathrm{x}_2(t)$ for $0 < t < t_m$. Typical world paths of the particles look like:



The Bohr-Sommerfeld quantization condition can be written in a relativistic invariant form:

$$2h\,\bar{\sigma}\,S = 2\pi\Big(n - \frac{1}{4}\Big).$$

It leads to the mass spectrum

$$\frac{M_n^2}{4m^2} = \cosh^2\theta_n,$$

where

$$\sinh 2\theta_n - 2\theta_n = 2\pi\lambda(n - 1/4).$$

This reproduces to the leading order the semiclassical spectrum following from the Bethe-Salpeter equation.

Multi-quark corrections

Exact eigenvalue problem:

$$\mathcal{H} \ | \Phi_n(P)\rangle = [E_n(P) + E_{\rm vac}] \ | \Phi_n(P)\rangle, \quad \hat{P} \ | \Phi_n(P)\rangle = P \ | \Phi_n(P)\rangle,$$

The exact meson eigenvector $|\Phi_n(P)\rangle$ contains four-quark, six-quark, ... contributions, which were ignored in the two-quark approximation.

Renormalized Bethe-Salpeter equation (FZ 2006):

$$[\varepsilon(P/2 + p) + \varepsilon(P/2 - p) - \Delta E(P)] \, \Psi_P(p) = f \int_{-\infty}^{\infty} \mathbb{G}_P(p|k) \, \Psi_P(k) \, \frac{dk}{2\pi},$$

$\Delta E(P) = (P^2 + M^2)^{1/2}$,

$\varepsilon(p)$ is the renormalized quark dispersion law, $\varepsilon(p) \neq (p^2 + m_q^2)^{1/2}$,

f is the the renormalized string tension,

$\mathbb{G}_P(p|k)$ is the renormalized kernel:

$$\mathbb{G}_P(p|k) = \frac{1}{(p-k)^2} - \frac{1}{(p+k)^2} + \mathbb{G}_P^{(reg)}(p|k),$$

where $\mathbb{G}_P^{(reg)}(p|k) = G_P^{(reg)}(p|k) + \Delta\mathbb{G}_P^{(reg)}(p|k)$.

Renormalization of the quark dispersion law:

$$\varepsilon(p) = (p^2 + m^2)^{1/2} + \delta\varepsilon(p) = (p^2 + m_q^2)^{1/2} + \Delta\varepsilon(p).$$

The second order term determines the leading correction to the quark energy $\delta_2\varepsilon(p) \sim h^2$:

$$\delta\epsilon(p) = \delta_2\epsilon(p) + \delta_3\epsilon(p) + \dots .$$

Formfactor expansion for $\delta_2\varepsilon(p)$:

$$\delta_2\varepsilon(p) = \delta_{2,3}\varepsilon(p) + \delta_{2,5}\varepsilon(p) + \dots ,$$

$$\delta_{2,n}\varepsilon(p) = -\frac{h^2}{n!}\int_{-\infty}^{\infty} \frac{dq_1 \dots dq_n}{(2\pi)^{n-1}} \frac{\delta(q_1 + \dots + q_n - p)}{\omega(q_1) + \dots + \omega(q_n) - \omega(p)}$$

$$\cdot \lim_{k\to p} \langle p|\sigma(0)|q_1, \dots, q_n\rangle\langle q_n, \dots, q_1|\sigma(0)|k\rangle,$$

Integral representation for $\delta_2\varepsilon(p)$ (FZ 2003):

$$\delta_2\varepsilon(p) = -h^2 \int_{-\infty}^{\infty} d\mathrm{x} \int_0^{\infty} d\mathrm{y} \lim_{k\to p} \langle p|\sigma(\mathrm{x}, \mathrm{y})(1 - \mathcal{P}_1)\sigma(0, 0)|k\rangle.$$

where $\sigma(\mathrm{x}, \mathrm{y}) = \exp(-i\mathrm{x}\hat{P} + \mathrm{y}\mathcal{H}_0)\sigma(0)\exp(i\mathrm{x}\hat{P} - \mathrm{y}\mathcal{H}_0)$.

Renormalized quark mass:

$$m_q^2 = m^2(1 + a_2\,\lambda^2 + a_4\,\lambda^4 + \ldots),$$
$$a_2 = a_{2,3} + a_{2,5} + \ldots,$$

Three-fermion contribution to the formfactor expansion of $\delta_2\varepsilon(p)$:

$$\delta_{2,3}\varepsilon(p) = -\frac{h^2}{3!}\int_{-\infty}^{\infty}\frac{dq_1 dq_2 dq_3}{(2\pi)^2}\frac{\delta(q_1+q_2+q_3-p)}{\omega(q_1)+\omega(q_2)+\omega(q_3)-\omega(p)}$$
$$\cdot \lim_{k\to p}\langle p|\sigma(0)|q_1,q_2,q_3\rangle\langle q_3,q_2,q_1|\sigma(0)|k\rangle.$$

Numerical values:

$$a_{2,3} \approx 0.07 \quad \text{(FZ 2001)},$$
$$a_{2,3} = \frac{1}{16} + \frac{1}{12\pi^2} = 0.07094\ldots \quad \text{(exact value R 2009)},$$
$$a_2 = 0.071010809 \quad \text{(FZ 2003)}.$$

$$\Delta\varepsilon(p) = -\frac{\lambda^2}{8}\frac{p^2 m^4}{(p^2+m^2)^{5/2}} + O(\lambda^3), \quad \text{(FZ 2006)}$$

Renormalized string tension (FZ, 2006):

$$f = f_0 \left(1 + c_2\,\lambda^2 + c_4\,\lambda^4 + \cdots\right),$$

$$E_{vac} = L\,m^2 \left(-\frac{1}{2}\,\lambda + \tilde{g}_2\,\lambda^2 + \tilde{g}_3\,\lambda^3 + \tilde{g}_4\,\lambda^4 + \dots\right),$$

$$c_{2k} = -2\,\tilde{g}_{2k+1}, \quad c_2 = -0.003889\dots$$

Renormalization of short-range interaction between quarks $\Delta\mathbb{G}_P^{(reg)}(p|k)$, (R, 2009).

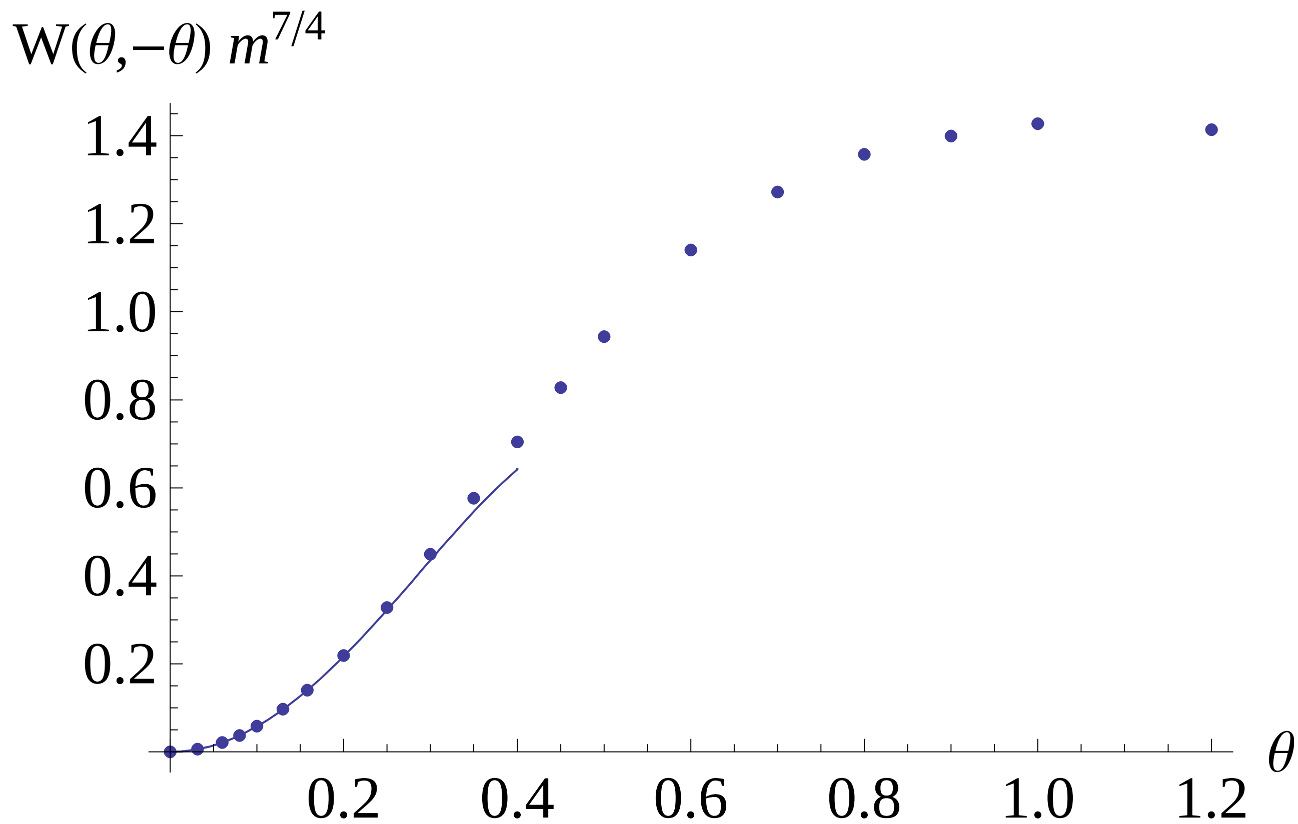
The corresponding third-order correction to the meson mass $M_n = 2m\cosh\theta$ is determined by its diagonal part $\Delta\mathbb{G}_P^{(reg)}(p|p)$.

$$\frac{\delta_3 M_n^2}{m^2} = -\frac{\lambda^3}{4\sinh^2\theta}\,\frac{m^2\,W_{irr}(\beta+\theta,\beta-\theta)}{\bar{\sigma}^2},$$

$$W_{irr}(\beta_1,\beta_2) = \int_{\infty}^{\infty} d\mathrm{x} \int_0^{\infty} d\mathrm{y} \lim_{\substack{\beta_1' \to \beta_1 \\ \beta_2' \to \beta_2}} \langle \beta_2, \beta_1 | \sigma(\mathrm{x},\mathrm{y})\sigma(0,0) | \beta_1', \beta_2' \rangle_{irr},$$

$|\beta\rangle = (m\cosh\beta)^{1/2}|p(\beta)\rangle, \quad p(\beta) = m\sinh\beta,$

$W_{irr}(\beta_1+\beta,\beta_2+\beta) = W_{irr}(\beta_1,\beta_2).$



Small-θ fit:

$$W_{irr}(\theta, -\theta)\, m^{7/4} = 4\, \bar{s}^2 (B_2\, \theta^2 + B_4\, \theta^4) + O(\theta^6),$$

where $B_2 = 0.8, \quad B_4 = -1.6, \quad \bar{s} = \frac{\bar{\sigma}}{m^{1/8}} = 1.35783834\ldots$

Multi-quark correction to the meson mass in the low-energy region to λ^3-order compared with numerical TFFSA results (FZ, 2006)

$$\frac{M_n(\eta) - \widetilde{M}_n(m, f_0)}{h^{8/15}} = \eta \left[a_2\, t^6 + \frac{z_n}{6} (4c_2 - a_2)\, t^8 - \frac{B_2}{4}\, t^9 + O(t^{10}) \right],$$

$t = \lambda^{1/3}$, $a_2 = 0.0710809...$, $c_2 = -0.003889$, $B_2 = 0.8$, $\eta = m/h^{8/15}$.

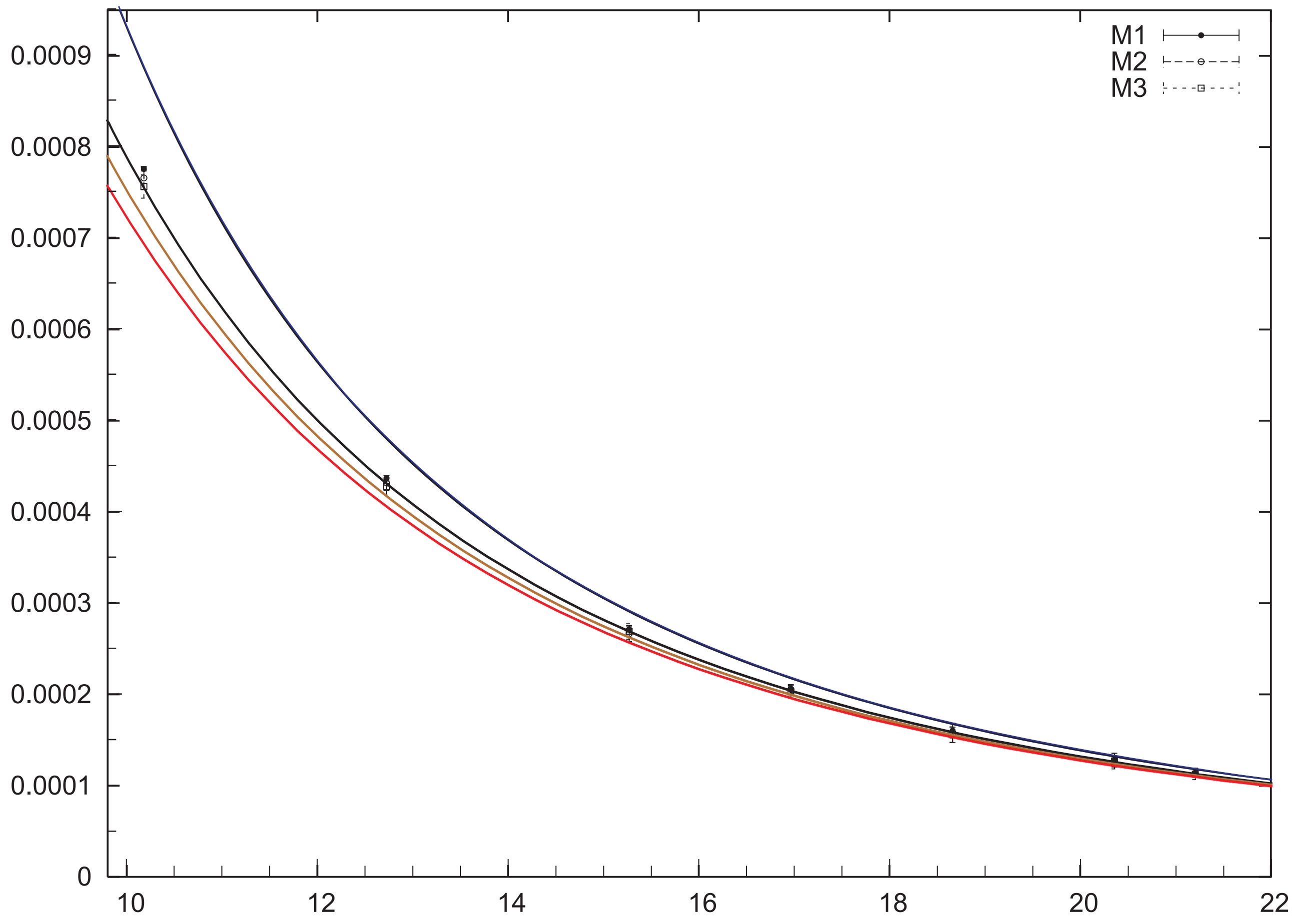
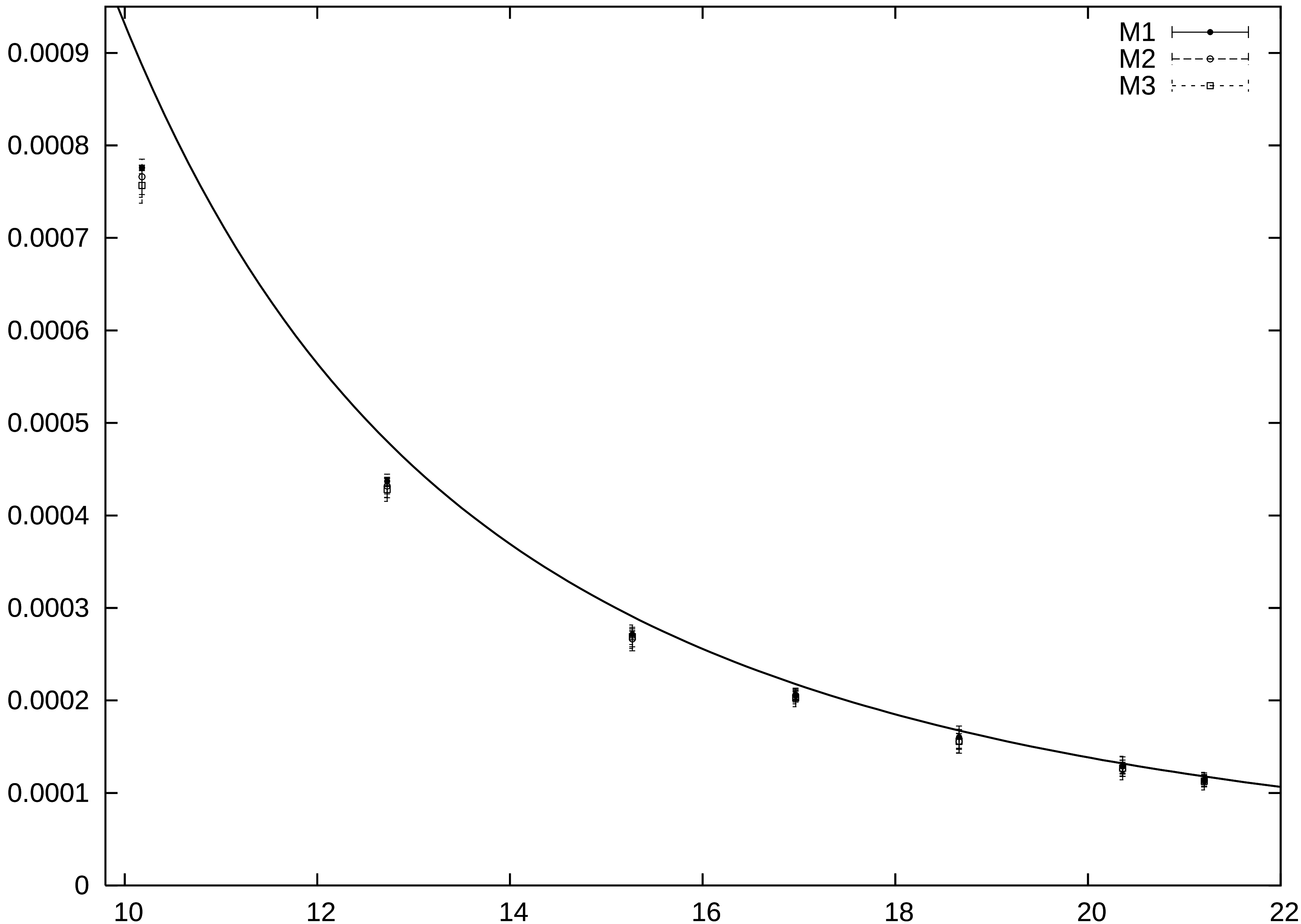
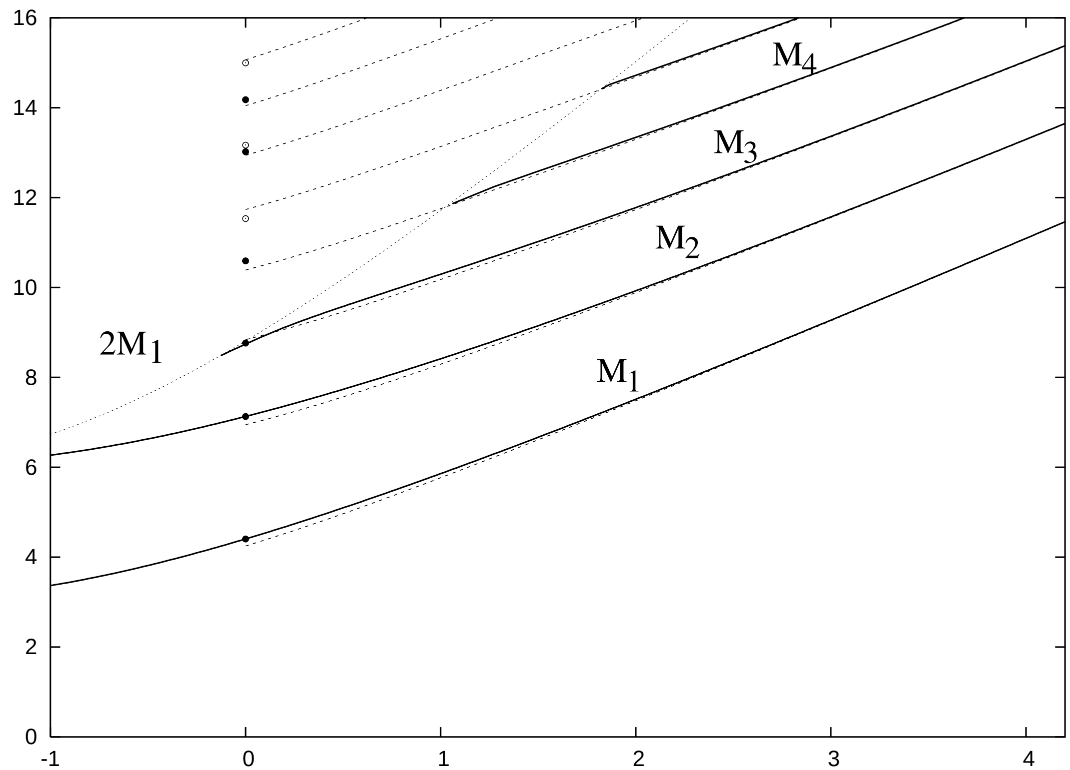


Figure 7 from FZ, hep-th/0612304, 2006

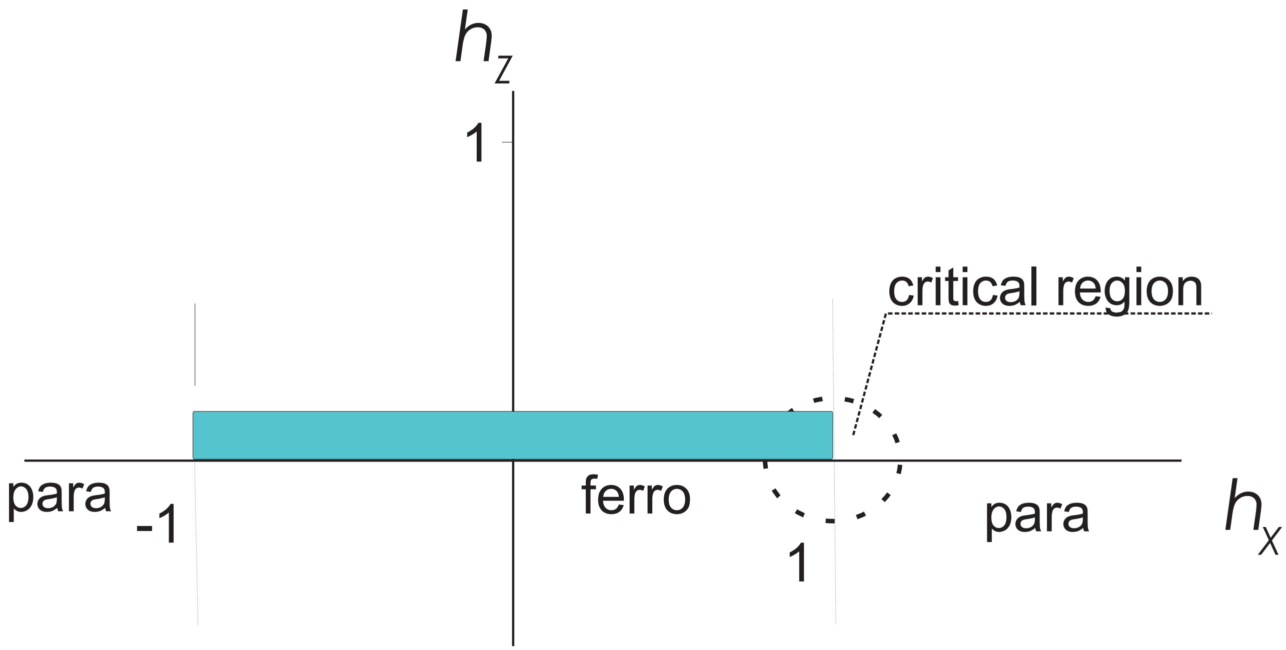


The mass spectrum of meson masses $M_n(\eta)$ of IFT at positive (and some negative) $\eta = \frac{m}{h^{8/15}}$. From FZ 2006.



Solid lines - numerical data. Dashed lines - masses from Bethe-Salpeter equation with renormalized string tension. Bullets $\bullet$ indicate exact masses of eight particles at $\eta = 0$. Circles $\circ$ indicate positions of $M_1 + M_2$,, $M_1 + M_3$, etc. at $\eta = 0$.

$$\mathcal{H}_{ch} = -\sum_{n=1}^{\mathcal{N}} (\sigma_n^z \sigma_{n+1}^z + h_x \sigma_n^x + h_z \sigma_n^z)$$



Ferromagnetic phase, $-1 < h_x < 1$. Weak-field limit: $h_z \ll 1$.

How the lattice discreteness modifies the excitation dispersion law beyond the critical region? R (2008).

Of course, beyond the critical region $E_n(P) \neq \sqrt{P^2 + M_n^2}$.

Fermionic representation in the thermodynamic limit

$$\mathcal{H}_{ch} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, \omega(\theta)\, a^{\dagger}(\theta)\, a(\theta) - h_z \sum_{n=-\infty}^{\infty} \sigma_n^z,$$

where θ is the quasi-momentum, fermionic operators $a^{\dagger}(\theta)$, $a(\theta)$ satisfy the canonical anticommutation relations

$$\{a(\theta)\ , a(\theta')\} = \left\{a^{\dagger}(\theta)\ , a^{\dagger}(\theta')\right\} = 0, \quad \left\{a^{\dagger}(\theta)\ , a(\theta')\right\} = 2\pi\delta(\theta-\theta'),$$

and the free-fermion dispersion law is given by

$$\omega(\theta) = 2\left[(1-h_x)^2 + 4h_x \sin^2\frac{\theta}{2}\right]^{1/2}.$$

Operators σ_n^z can be completely characterized by the form-factors

$$\langle\theta_1,\ldots,\theta_K|\sigma_n^z|\theta_1'\ldots,\theta_N'\rangle = \langle 0|a(\theta_1),\ldots,a(\theta_K)\,\sigma_n^z\, a^{\dagger}(\theta_1'),\ldots,a^{\dagger}(\theta_N')|0\rangle,$$

which are explicitly known.

Eigenvalue problem

The kink bound-state energy spectrum can be formally defined as the solution of the eigenvalue problem

$$(\mathcal{H}_{ch} - E_{\rm vac}) \mid \Phi_n(\Theta)\rangle = E_n(\Theta) \mid \Phi_n(\Theta)\rangle,$$
$$\hat{T}_1|\Phi_n(\Theta)\rangle = \exp(i\Theta)\,|\Phi_n(\Theta)\rangle$$

where $E_{\rm vac}$ is the ground state energy, and $|\Phi_n(\Theta)\rangle$ is the kink bound-state eigenvector with quasimomentum Θ, $\hat{T}_1$ is the lattice-site translation operator. Two-fermion approximation

$$\mathcal{P}_2\mathcal{H}_{ch}\mathcal{P}_2 \mid \Phi_n(\Theta)\rangle = E_n(\Theta) \ \mid \Phi_n(\Theta)\rangle,$$

where $\mathcal{P}_2$ is the orthogonal projector onto the two-fermion subspace $\mathcal{F}_2$ of the Fock space $\mathcal{F}$.

Bethe-Salpeter equation

In the momentum representation this equation takes the form:

$$[\epsilon(z;\Theta) - E_n(\Theta)]\,\phi_n(z) = -\chi\, z \oint_{S_1} \frac{dz'}{\pi i}\,\phi_n(z')\left[\frac{1}{(z'-z)^2} + \Delta G(z,z';\Theta)\right] \quad (1)$$

$$\epsilon(e^{i\theta},\Theta) = \omega(\theta+\Theta/2) + \omega(\theta-\Theta/2),$$

where $\chi = 2h_z\bar{\sigma}$, $z = \exp(i\theta)$, $z' = \exp(i\theta')$, S_1 is the unit circle, $z, z' \in S_1$,

$$\epsilon(z;\Theta) = 2\sqrt{h_x}\left\{\left[h_x + \frac{1}{h_x} - zv - \frac{1}{zv}\right]^{1/2} + \left[h_x + \frac{1}{h_x} - \frac{z}{v} - \frac{v}{z}\right]^{1/2}\right\},$$

and $v = \exp(i\Theta/2)$.

Equation (1) is analogous to the Bethe-Salpeter equation in the Ising field theory.

Perturbative solution of (1) was obtained in the leading order in χ. In the leading order in χ function $\Delta G(z,z';\Theta)$, which is regular at $z = z'$, can be omitted in (1).

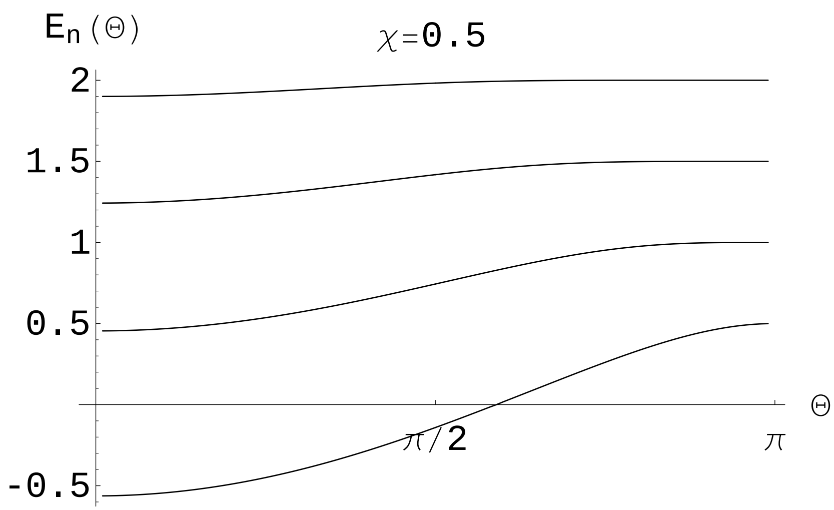
Toy models

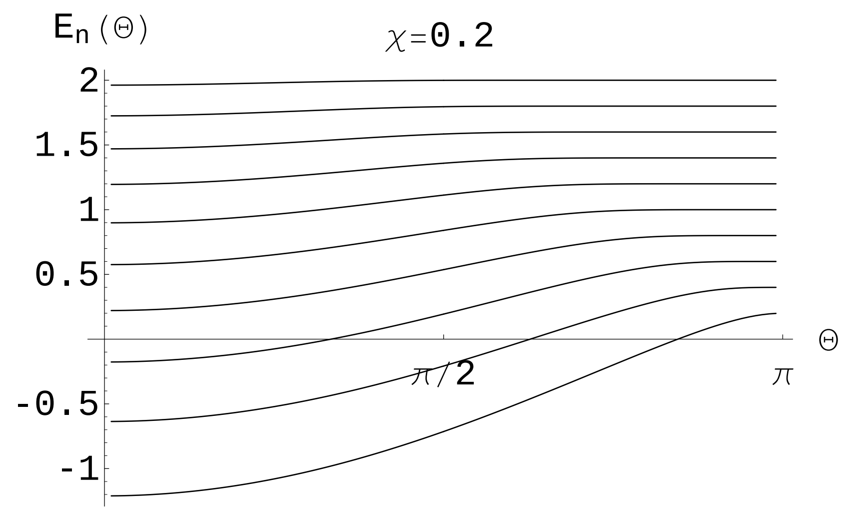
$$[\epsilon(z;\Theta) - E_n(\Theta)]\,\phi_n(z) = -\chi\, z \oint_{S_1} \frac{dz'}{\pi i}\,\frac{\phi_n(z')}{(z'-z)^2},$$

1. First Toy Model.

$$\omega(\theta) = -\cos\theta \implies \epsilon(z;\Theta) = -\cos(\Theta/2)\,(z + z^{-1}).$$

$$E_n(\Theta) = -\nu_n\,\chi, \quad \text{where } J_{\nu_n}\left[\frac{2\cos(\Theta/2)}{\chi}\right] = 0, \;\; n = 1, 2, \ldots.$$





2. Second Toy Model.

$$\omega(\theta) = -\cos\theta - \gamma\cos(2\,\theta) \implies$$

$$\epsilon(z;\Theta) = -\cos(\Theta/2)\,(z + z^{-1}) - \gamma\cos(\Theta)\,(z^2 + z^{-2}).$$

Neutron scattering experiments on $1d$-quantum ferromagnet $CoNb_2O_6$:

Coldea *et al.*, Quantum criticality in an Ising chain: Experimental evidence for emergent E8 symmetry. *Science*, **327**, 177 (2010).

Effective Hamiltonian introduced by Coldea *et al.* :

$$H|j,l\rangle = J|j,l\rangle - \alpha[|j,l+1\rangle + (|j,l-1\rangle + |j+1,l-1\rangle)(1-\delta_{l,1}) + |j-1,l+1\rangle] + h_z l|j,l\rangle - \beta\delta_{l,1}(|j-1,1\rangle + |j+1,1\rangle) + \beta'|j,1\rangle\delta_{l,1},$$

where $j = 0, \pm 1, \pm 2, \ldots$, and $l = 1, 2, \ldots$,

$$|j,l\rangle = |\ldots\uparrow\uparrow\uparrow\overset{j}{\downarrow}\downarrow\ldots\downarrow\overset{j+l}{\uparrow}\uparrow\uparrow\ldots\rangle,$$

At $\beta = \beta' = 0$ this model is equivalent to the First Toy model of (R 2008).

Eigenvalue problem $H|\Psi_n(P)\rangle = E_n(P)|\Psi_n(P)\rangle$ reduces to:

$$\left(-\lambda_n + \mu\, l + \frac{a\,\delta_{l,1}}{2}\right)\psi_n(l) - \frac{\psi_n(l+1)+\psi_n(l-1)}{2} = 0, \qquad l = 1, 2, \ldots,$$

with boundary conditions

$$\lim_{l\to+\infty}\psi_n(l) = 0, \quad \psi_n(0) = 0.$$

and

$$\lambda_n = \frac{E_n(P) - J}{4\alpha\cos(P/2)}, \quad a = \frac{\beta' - 2\beta\cos P}{2\alpha\cos(P/2)}, \quad \mu = \frac{h_z}{4\alpha\cos(P/2)}.$$

Eigenvalues: $\lambda_n = -\mu\,\nu_n$, where ν_n solve equation:

$$J_{\nu_n}(1/\mu) + a\,J_{\nu_n+1}(1/\mu) = 0,$$

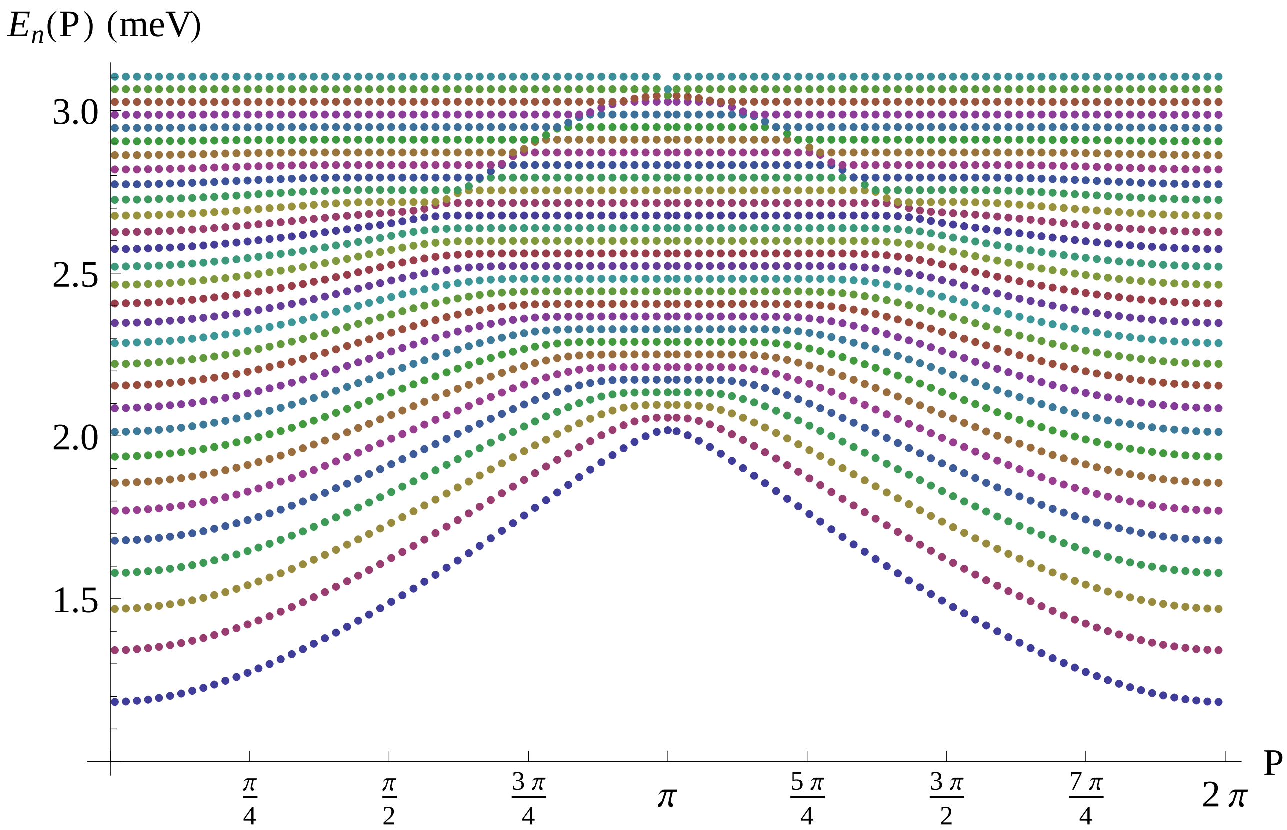
Intensities:

$$I_n(P) \equiv \frac{|\psi_n(l=1,P)|^2}{\sum_{l=1}^{\infty}|\psi_n(l,P)|^2} = 2\mu\left\{\frac{\partial}{\partial\nu}\left[\frac{J_\nu(1/\mu)}{J_{\nu+1}(1/\mu)}\right]\right\}^{-1}\Bigg|_{\nu\to\nu_n},$$

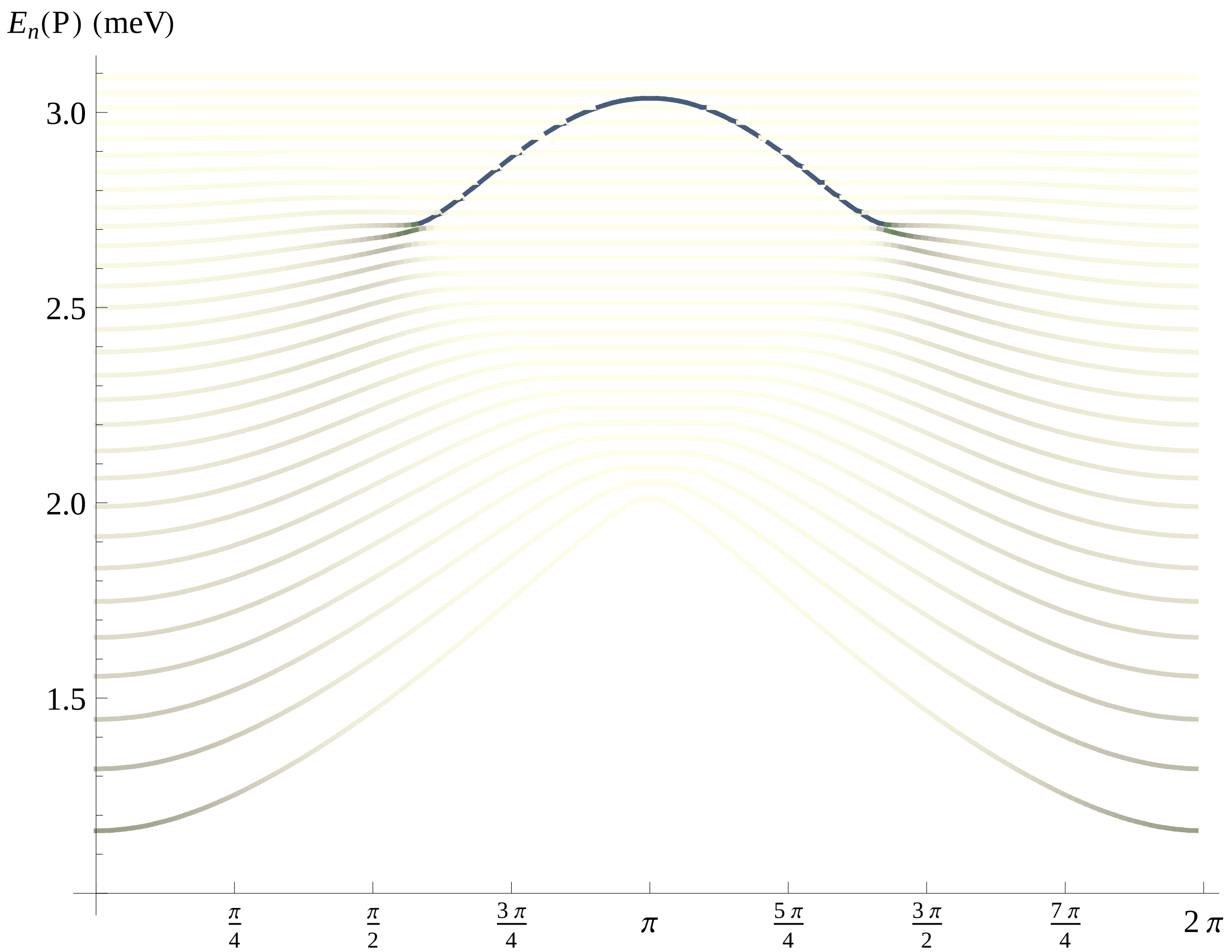
Energy spectra of 30 lowest modes with the Hamiltonian parameter values

$$J = 1.94\,\text{meV}, \quad \alpha = 0.12J, \quad \beta = 0.17\,J, \quad \beta' = 0.21\,J, \quad h_z = 0.02J,$$

chosen by Coldea *et al* to give the best fit of the experimental results.



The same modes with darkness of the curves modulated according to the mode intensities $I_n(P)$.



Articles by P. Fonseca and A.B. Zamolodchikov

1. Ising field theory in a magnetic field: analytic properties of the free energy. J. Stat. Phys. **110**, 527 (2003); hep-th/0112167 (2001).
2. Ward identities and integrable differential equations in the Ising field theory. hep-th/0309228 (2003).
3. Ising spectroscopy I: Mesons at $T < T_c$. hep-th/0612304 (2006).

Conclusion

Bethe-Salpeter equation, which is based on the two-particle approximation, provides an effective technique to describe confinement both in the Ising field theory and in the Ising spin chain. In higher order in magnetic field, multi-particle corrections become important.

It would be interesting to extend this technique to the models exhibiting confinement, which are integrable but not free at $h = 0$.