Q $\longleftrightarrow$ CL $\longleftrightarrow$ SuSy

Quantum mechanical and classical systems' connection with a supersymmetry

E. V. Mozgunov [2,1] and M. V. Feigel'man[1,2]

[1] *L. D. Landau Institute for Theoretical Physics, Kosygin str.2, Moscow 119334, Russia*
[2] *Moscow Institute of Physics and Technology, Moscow 141700, Russia*

1 matrices

This basic knowledge is reviewed, for example, in [Phys. Rev. B 81, 184303 (2010) Quantum mechanical and information theoretic view on classical glass transitions, Claudio Castelnovo, Claudio Chamon, and David Sherrington]

Consider a system whose configurations are labeled by i, and where the probability $P_i(t)$ of the system being in configuration i at time t is governed by the Markov equation

$$\dot{P} = WP \tag{1}$$

where the NxN transition matrix W satisfies probability conservation $\sum_i W_{ij} = 0$ and detailed balance (see below), note that if $\mathrm{diag}(W) < 0$, $\mathrm{offdiag}(W) > 0$, equation (1) is equivalent to random walk on N sites, but it is not necessary.

We want zeroth mode to be $P_i^{(0)} = Z^{-1}e^{-\beta E_i}$ and demand

$$j_{ik} = j_{ki} \quad W_{ik}P_k^{(0)} = W_{ki}P_i^{(0)} \quad W_{ij}e^{-E_j/T} = W_{ji}e^{-E_i/T} \quad \text{detailed balance} \tag{2}$$

We can express the ensemble average of any given observable A as $\bar{A} = \lim_{t\to\infty} A_i P_i(t)$ for any $P(t=0)$. Only the slowest mode survives (its unique existence is ensured by Perron-Frobenius theorem for any irreducible finite system size) and one arrives at the thermodynamic equilibrium expression:

$$\bar{A} = A_i P_i^{(0)} = Z^{-1}A_i e^{-\beta E_i} \tag{3}$$

We investigate the two-time correlation function: system is in the equilibrium $P^{(0)}$

$$\left(A = A_i \ \text{ with probability } \ P_i^{(0)}\right) \ \times \ \left[\text{if } \ P_k = \delta_{ki} \ \text{ at } \ t = 0 \ \text{ then } \ P_m(t) = (e^{Wt})_{mi}\right] \ \times \ (B = B_m) \tag{4}$$

This middle term is the very definition of conditional probability that the system be in configuration m at time t, given that it was in configuration i at time $t = 0$. Now we write it

$$C(\tau) = \sum_{i,j} P_i^{(0)} A_i P_{i\to j}(\tau) B_j = P_i^{(0)} A_i (e^{W t})_{mi} B_m \tag{5}$$

using vector-row I containing 1s

$$C(\tau) = IBe^{W\tau}AP^{(0)} \tag{6}$$

where diagonal matrices $A_{ij} = \delta_{ij}A_i$, $B_{ij} = \delta_{ij}B_i$.

It is possible to construct a real *symmetric* matrix H that has the same eigenvalues as W by means of a similarity transformation

$$H = -S^{-1}WS, \quad S_{ij} = \delta_{ij}e^{-E_i/2T} \tag{7}$$

(it's symmetric due to detailed balance (2)) Also we will need a vector $s^T = IS$ with the elements of S

$$C(\tau) = IBe^{W\tau}AP = s^T S^{-1}BSe^{-H\tau}S^{-1}AP = s^T Be^{-H\tau}AS^{-1}P^{(0)} \tag{8}$$

We take advantage of the fact that $S^{-1}P^{(0)} = s$, and so $Hs = 0$. We can introduce new notation

$$C(\tau) = s^T Be^{-H\tau}As = \langle GS|Be^{-H\tau}A|GS\rangle \tag{9}$$

Now we can form a statement: correlation function in classical stochastic process with transition matrix W are equal to imaginary time correlation function of diagonal operators in quantum system with Hamiltonian H.

To get real-time correlation function, one needs to perform analytic continuation from $\tau = [0, +\infty]$ on $\tau = [-i\infty, +i\infty]$. If it is possible to know $C(\tau)$ analytically , then just $C_q(t) = \langle GS|Be^{-iHt}A|GS\rangle = C(\tau = it)$

Question: what's the requirements from the quantum side?

Answer: very strict. No magnetic field ($H = H^T$), must know the ground state, and it should have all amplitudes nonzero (existence of S).

2 Fokker-Planck

$$\text{Now lets describe continual} \quad \dot{\varphi} = -\frac{\partial V}{\partial \varphi} + \xi(t) \tag{10}$$

$$\text{stochastic evolution} \quad \langle \xi(t)\xi(t') \rangle = 2T\delta(t-t') \tag{11}$$

2 ways from here: considering ensemble of particles and differentials or discretizing space. Anyway we get

$$\text{Fokker-Planck eqn.} \quad \dot{P}(\varphi,t) = \frac{\partial}{\partial \varphi}(T\frac{\partial}{\partial \varphi} + \frac{\partial V}{\partial \varphi})P(\varphi,t) \tag{12}$$

to see that it is of form (1), apply any scheme $\varphi = nh$

$$(12) \;\rightarrow\; \dot{P}_n(t) = W_{nm}Pm \tag{13}$$

$\partial^2/\partial\varphi^2 \rightarrow \frac{1}{h^2}(\;\; 1 \;\; -2 \;\; 1 \;\;)$etc. Get three-diagonal matrix. Check $\sum_i W_{ij} = 0$ and detailed balance (2) for $E = V$. To get a hamiltonian like in (7) we need $S_i = e^{-E_i/2T} \rightarrow S(\varphi) = e^{-V/2T}, \quad P = S\Psi$

$$\dot{P} = \frac{\partial}{\partial \varphi}(T\frac{\partial}{\partial \varphi} + \frac{\partial V}{\partial \varphi})P \quad \rightarrow \quad -\dot{\Psi} = -T(\frac{\partial}{\partial \varphi} - \frac{1}{2T}\frac{\partial V}{\partial \varphi})(\frac{\partial}{\partial \varphi} + \frac{1}{2T}\frac{\partial V}{\partial \varphi})\Psi \tag{14}$$

conventional Shroedinger eqn. form after $(\tau \rightarrow it)$

$$iT\dot{\Psi} = -T^2\frac{\partial^2 \Psi}{\partial \varphi^2} + U(\varphi)\Psi \tag{15}$$

T – Planck constant! Potential

$$U(\varphi) = \frac{1}{4}\left(V'\right)^2 - \frac{T}{2}V'' \tag{16}$$

But note the form that appeared in the middle:

$$iT\dot{\Psi} = (iT\frac{\partial}{\partial \varphi} - \frac{i}{2}\frac{\partial V}{\partial \varphi})(i\frac{\partial}{\partial \varphi} + \frac{i}{2T}\frac{\partial V}{\partial \varphi})\Psi = q^\dagger q\Psi \tag{17}$$

Where $q = iT\frac{\partial}{\partial \varphi} + \frac{i}{2T}\frac{\partial V}{\partial \varphi}, \quad q^\dagger = iT\frac{\partial}{\partial \varphi} - \frac{i}{2T}\frac{\partial V}{\partial \varphi}$. This operator annihilates GS, but that's not enough for supersymmetry! Supercharge Q must $Q^2 = 0$, and Hamiltonian must be $H = \{Q, Q^\dagger\}$. Things become even worse when we consider N particles (labeled j) in d dimensions (labeled α):

$$H = \sum_{j,\alpha} q^\dagger_{j,\alpha} q_{j,\alpha} \tag{18}$$

Trick: introduce fermionic operator $a_{j,\alpha}$ for every degree of freedom. Now we can define supercharge:

$$Q = \sum_{j,\alpha} q_{j,\alpha} a^\dagger_{j,\alpha}, \quad Q^\dagger = \sum_{j,\alpha} q^\dagger_{j,\alpha} a_{j,\alpha} \tag{19}$$

Supersymmetric Hamiltonian

$$H = \{Q, Q^\dagger\} = \sum_{j,\alpha} q^\dagger_{j,\alpha} q_{j,\alpha}(a^\dagger_{j,\alpha} a_{j,\alpha} + 1) + q_{j,\alpha} q^\dagger_{j,\alpha} a^\dagger_{j,\alpha} a_{j,\alpha} + \sum_{j\alpha \neq j'\alpha'} (q..) a^\dagger_{j,\alpha} a_{j',\alpha'} \tag{20}$$

Now lets project our model onto zero-fermion sector and we get

$$\langle 0_F | H | 0_F \rangle = \sum_{j,\alpha} q^\dagger_{j,\alpha} q_{j,\alpha} \tag{21}$$

3 models

But was it possible to do without this doubling trick?

Yes. Take matrix Q: $Q^2=0$ and you know that $Q\Psi_0=0$. Then you can construct $H=\{Q,Q^\dagger\}$. If Q is real (or pure im) then H is symmetric. To make it classical $S=\mathrm{diag}(\Psi_0)$. Note that you need S^{-1} so all amplitudes in Ψ_0 must be nonzero (we're not in continuum limit yet). So $W=-SHS^{-1}$, and if all the signs satisfy:

$\mathrm{diag}(W)<0$, $\mathrm{offdiag}(W)>0$, you can simulate it via Monte Carlo.

For two known examples: chain and ladder of hardcore fermions, incredibly, this is indeed the case. It also works for Rokhsar-Kivelson point of dimer Hamiltonian.

Question: both the $\sum_{j,\alpha} q^\dagger_{j,\alpha} q_{j,\alpha}$ Hamiltonians and direct $\{Q,Q^\dagger\}$ Hamiltonians has their continuum limit, yet very different. In latter case the number of lattice sites goes to infinity, and what we arrive is said [A supersymmetric model for lattice fermions, L. Huijse] to be (N=2 supersymmetry). In former case the indices space j,α approachs continuum limit. Corresponding field theory was explicitly constructed in [Hidden supersymmetry in stochastic dissipative dynamics, Feigelman, M. V.; Tsvelik, A. M., ZhETP, vol. 83, Oct. 1982, p. 1430-1443. http://www.jetp.ac.ru/cgi-bin/dn/e_056_04_0823.pdf] So are they of the same class of SuSy theories?

My model has $H=\sum_{j,\alpha} q^\dagger_{j,\alpha} q_{j,\alpha}$ in 2D

$$\text{where}\quad i\hat{\mathbf{q}}_j=\nabla_j-\alpha\left(\sum_{k\neq j}\frac{\mathbf{r}_j-\mathbf{r}_k}{|\mathbf{r}_j-\mathbf{r}_k|^2}-\pi n r_j\right) \tag{22}$$

ground state is known exactly

$$\Psi_0(r_1,...r_N)=c\prod_{j>k}|r_j-r_i|^{2\alpha}e^{-\pi\alpha n\sum_i r_i^2} \tag{23}$$

By its form we see that particles are bosons. Expansion of H

$$H=\sum_{j=1}^{N}(\frac{p_j^2}{2m}+V_1(\mathbf{r}_j))+\sum_{j,k=1}^{N}V_2(\mathbf{r}_j,\mathbf{r}_k)+\sum_{j,k,n=1}^{N}V_3(\mathbf{r}_j,\mathbf{r}_k,\mathbf{r}_n) \tag{24}$$

$$H=\frac{1}{2m}\sum_i\left[-\nabla_i^2+2\alpha\sum_{j\neq i}\nabla_i^2\ln|r_i-r_j|+4\alpha^2\sum_{j\neq i,k\neq i}\nabla_i\ln|r_i-r_j|\nabla_i\ln|r_i-r_k|+(2\pi\alpha n)^2r_i^2\right] \tag{25}$$

V_3 (integrated over 3rd particle with density n) $\to -\sum\limits_{j,k=1}^{N}\frac{g^2}{2\pi}\ln|\mathbf{r}_j-\mathbf{r}_k|$ where $g=4\pi\alpha\sqrt{\frac{n}{m}}$

Using procedure described above, we can map this system dynamics onto a set of Langevin eqns:

$$\frac{dr_{j,\alpha}}{dt}=-\frac{\partial V\{\mathbf{r}_i\}}{\partial r_{j,\alpha}}+\xi_{j,\alpha}(t) \tag{26}$$

$$\text{where}\quad \overline{\xi_{j,\alpha}(t)\xi_{k,\beta}(t')}=2T\delta_{jk}\delta_{\alpha\beta}\delta(t-t'). \tag{27}$$

$$V\{\mathbf{r}_i\}=-4\alpha T\sum_{j>k}\ln|\mathbf{r}_j-\mathbf{r}_k|+2\pi\alpha T n\sum_i r_i^2. \tag{28}$$

This is done in

Excitation spectrum of a 2d long-range Bose-liquid with a supersymmetry (to be published)

Mozgunov EV, Feigel'man MV
Landau institute for Th. Physics

4 entanglement

Here I present a method for fast numerical calculation of the quantity $\mathrm{tr}\rho^2$, used for characterizing of entanglement in Renyi entropy (reduced density matrix $\rho = \int da_2 \psi_0(a_1, a_2)\psi_0(b_1, a_2)$, gr. state is real):

$$\mathrm{tr}\rho^2 = \int\int da_1 db_1 \rho(a_1, b_1)\rho(b_1, a_1) =$$

$$= \int da_1 da_2 db_1 db_2 \psi_0(a_1, a_2)\psi_0(b_1, a_2)\psi_0(b_1, b_2)\psi_0(a_1, b_2)$$

Twice evaluation of Hilbert space integral takes $\mathrm{Np}^{2\mathrm{Np}}$ (Np - number of particles, exact ground state wavefunction is known ψ_0(see above)).To get rid of exponential difficulty of the problem, I present rather clumsy but feasible **technique**.

Exploiting the same equivalence Shroedinger eqn. -> Fokker-Planck eqn. -> Langevin dynamics makes it possible to estimate any averages of the form:

$$\int da_1 da_2 \psi_0(a_1, a_2) f(a_1, a_2)\psi_0(a_1, a_2)$$

Where f is arbitrary function of coordinates. I emphasize that if difficulty of calculation $f(a_1, a_2)$ scales as Np^k (below check $k = 2$), then averaging takes $\mathrm{N}_{it}\mathrm{Np}^k$, where N_{it} - number of steps in computer program needed for convergence of the algorithm. Main assumption of Monte-Carlo calculations states that it converges sufficiently faster than $O(\exp(\mathrm{Np}))$, required for direct integration.

Now the formalism. $\mathrm{tr}\rho^2$ contains an integral

$$\int db_1 db_2 \psi_0(a_1, b_2)\psi_0(b_1, b_2)\psi_0(b_1, a_2) =$$

rewrite it as ground state average:

$$= \int db_1 db_2 \psi_0(b_1, b_2)\frac{\psi_0(a_1, b_2)\psi_0(b_1, a_2)}{\psi_0(b_1, b_2)}\psi_0(b_1, b_2) =$$

Fixing a_1, a_2, one can MC-estimate this average:

$$= \langle\frac{\psi_0(a_1, b_2)\psi_0(b_1, a_2)}{\psi_0(b_1, b_2)}\rangle_b = A(a_1, a_2)$$

Let us come back to the main formula $\mathrm{tr}\rho^2$ with this knowledge:

$$\mathrm{tr}\rho^2 = \int da_1 da_2 \psi_0(a_1, a_2)A(a_1, a_2) =$$

Do the same trick once more:

$$= \int da_1 da_2 \psi_0(a_1, a_2)\frac{A(a_1, a_2)}{\psi_0(a_1, a_2)}\psi_0(a_1, a_2) =$$

$$= \langle\frac{A(a_1, a_2)}{\psi_0(a_1, a_2)}\rangle_a$$

Note that estimating A needs to be done for every a_1, a_2. But this enters only as a factor in overall computation time. So, entanglement Renyi entropy is estimated for ~polynomial number of steps.

$$\mathrm{tr}\rho^2 = \langle\frac{\langle\frac{\psi_0(a_1, b_2)\psi_0(b_1, a_2)}{\psi_0(b_1, b_2)}\rangle_b}{\psi_0(a_1, a_2)}\rangle_a$$