An Exceptional Algebraic Description of the AdS/CFT Yangian Symmetry

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Collaboration with
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Motivation

The understanding of the all order correspondence of the gauge theory and string theory (AdS/CFT) correspondence.

AdS/CFT correspondence

Type IIB superstring theory on $AdS_5 \times S^5$ $\longleftrightarrow$ 4dim $\mathcal{N} = 4$ $SU(N)$SYM theory

Parameter correspondence

String theory		SYM theory
$\lambda \gg 1$	$\overset{\text{difficulty}}{\longleftrightarrow}$	$\lambda = g_{\mathrm{YM}}^2 N \ll 1$

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$\lambda \gg 1$	$\overset{\text{difficulty}}{\longleftrightarrow}$	$\lambda = g_{\mathrm{YM}}^2 N \ll 1$

Integrability might overcome this difficulty $\Longrightarrow$ It is possible to construct the all-order correspondence in the asymptotic region.

Global symmetry

String theory	$psu(2,2|4)$	SYM theory
	$\cup$	
AdS_5	$so(2,4)$	4dim conf. group
$\times$	$\times$	
S_5	$so(6)$	R-symmetry

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AdS_5	$\mathrm{so}(2,4)$	4dim conf. group
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S_5	$\mathrm{so}(6)$	R-symmetry

One of the concrete realization of the all-order correspondence is $\mathrm{su}(2|2)$ spin-chain model.

[Beisert]

su(2|2) Spin-chain (review)

How does su(2|2) appear **?**

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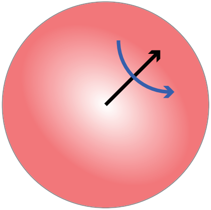
Vacua $|0\rangle := \mathrm{Tr}(\mathcal{Z}\mathcal{Z}\cdots\mathcal{Z})$ is fixed.

$$\mathrm{psu}(2,2|4) \xrightarrow{\text{broken}} [\mathrm{psu}(2|2)]^2 \ltimes \mathbb{R}$$



Why is the ‘Spin-chain’ needed?

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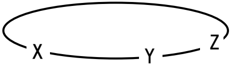
$\widehat{D} \curvearrowright \mathrm{Tr}(\cdots \mathcal{X} \cdots \mathcal{Y} \cdots \mathcal{Z} \cdots) \quad \longleftrightarrow \quad \widehat{H} \curvearrowright$ 

Excitations on the spin-chain
$\simeq$ Fundamental representation $\mathbf{2}|\mathbf{2}$ $|\chi\rangle$ of su(2|2)

[Minahan,Zarembo]

Characteristic feature of the model

- The one-particle dispersion relation (energy) is determined by the representation of the algebra.

$$C = \frac{1}{2}\sqrt{1 + 16g^2\alpha\beta\sin^2(\tfrac{p}{2})}$$

- The representation of the two-particle R-matrix (scattering matrix) is determined by the symmetry of the algebra.

$$[\Delta J^A, R_{12}]|\chi_1\chi_2\rangle = 0 \quad {}^{\forall}J^A \in \mathrm{psu}(2|2) \ltimes \mathbb{R}^3$$

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$$[\Delta J^A, R_{12}]|\chi_1\chi_2\rangle = 0 \quad {}^\forall J^A \in \mathrm{psu}(2|2) \ltimes \mathbb{R}^3$$

The reason is "Off-Shell" formalism.

$\mathrm{Tr}(\cdots\mathcal{X}\cdots\mathcal{Y}\cdots\mathcal{Z}\cdots) \longrightarrow [\ \cdots\mathcal{X}\cdots\mathcal{Y}\cdots\mathcal{Z}\cdots\ ]$ infinite length

The central extension of the algebra: $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$

$$\left(\begin{array}{c|c} L^\alpha{}_\beta & Q^\alpha{}_b \\ \hline S^a{}_\beta & R^a{}_b \end{array}\right) \ltimes C, P, K$$

Lie superalgebra $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$

$$\left(\begin{array}{c|c} L^{\alpha}{}_{\beta} & Q^{\alpha}{}_{b} \\ \hline S^{a}{}_{\beta} & R^{a}{}_{b} \end{array}\right) \ltimes C, P, K$$

where $\sum_{a=1,2} R^a{}_a = \sum_{\alpha=1,2} L^{\alpha}{}_{\alpha} = 0$.
Commutation relations

$$\mathrm{su}(2)_{R,L}\ [R^a{}_b, R^c{}_d] = \delta^c_b R^a{}_d - \delta^a_d R^c{}_b, \quad [L^{\alpha}{}_{\beta}, L^{\gamma}{}_{\delta}] = \delta^{\gamma}_{\beta} L^{\alpha}{}_{\delta} - \delta^{\alpha}_{\delta} L^{\gamma}{}_{\beta}$$

$$[R^a{}_b, J^c] = \delta^c_b J^a - \tfrac{1}{2}\delta^a_b J^c, \quad [R^a{}_b, J_c] = -\delta^a_c J_b + \tfrac{1}{2}\delta^a_b J_c$$

$$[L^{\alpha}{}_{\beta}, J^{\gamma}] = \delta^{\gamma}_{\beta} J^{\alpha} - \tfrac{1}{2}\delta^{\alpha}_{\beta} J^{c}, \quad [L^{\alpha}{}_{\beta}, J_{\gamma}] = -\delta^{\alpha}_{\gamma} J_{\beta} + \tfrac{1}{2}\delta^{\alpha}_{\beta} J_{\gamma}$$

$$\text{Fermions}\ [Q^{\alpha}{}_a, Q^{\beta}{}_b] = \epsilon^{\alpha\beta}\epsilon_{ab} P, \quad [S^a{}_{\alpha}, S^b{}_{\beta}] = \epsilon^{ab}\epsilon_{\alpha\beta} K$$

$$[Q^{\alpha}{}_a, S^b{}_{\beta}] = \delta^{\alpha}_{\beta} R^a{}_b + \delta^a_b L^{\alpha}{}_{\beta} + \delta^a_b \delta^{\alpha}_{\beta} C$$

[Difficulty] Cartan-Killing form g_{AB} is degenerate.

The exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$

generators $\{\underbrace{R^a{}_b, L^\alpha{}_\beta, C^A{}_B}_{9} \,|\, \underbrace{Q^\alpha{}_a, S^a{}_\alpha}_{8}\}$

$$\begin{array}{ll} \mathrm{su}(2)\times\mathrm{su}(2)\times\mathrm{su}(2) & \subset \mathrm{d}(2,1;\varepsilon) \\ R^a{}_b \quad L^\alpha{}_\beta \quad C^A{}_B & \downarrow \quad \varepsilon \to 0 \\ \mathrm{su}(2)\times\mathrm{su}(2)\times\mathbb{R}^3 & \subset \mathrm{psu}(2|2)\ltimes\mathbb{R}^3 \\ \{C,P,K\} & \text{center} \end{array}$$

The commutation relations of $\{C,P,K\}$ in $\mathrm{d}(2,1;\varepsilon)$.

$$[C,P]=\varepsilon P \quad [C,K]=-\varepsilon K \quad [P,K]=-2\varepsilon C$$
$$[C,Q]=\tfrac{\varepsilon}{2}Q \quad [C,S]=-\tfrac{\varepsilon}{2}S$$

Killing form is not degenerate → There exists the well-defined Casimir operator

$$T^{\mathrm{d}} = RR-(1-\varepsilon)LL+QS-SQ-\tfrac{1}{\varepsilon}(-PK+2CC-KP)$$

The exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$

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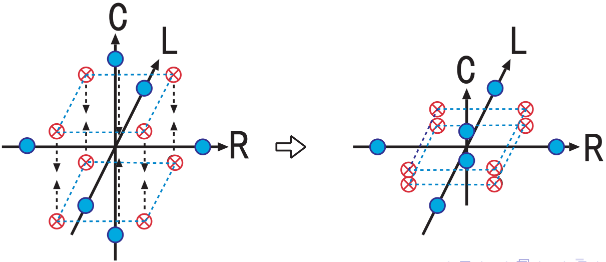
$$\mathrm{su}(2)\times\mathrm{su}(2)\times\mathrm{su}(2) \qquad \subset \mathrm{d}(2,1;\varepsilon)$$

$$R^a{}_b \quad L^\alpha{}_\beta \quad C^A{}_B \qquad \downarrow \quad \varepsilon \to 0$$

$$\mathrm{su}(2)\times\mathrm{su}(2)\times\mathbb{R}^3 \qquad \subset \mathrm{psu}(2|2)\ltimes\mathbb{R}^3$$

$$\{C, P, K\} \qquad \text{center}$$



PLAN

classical r-matrix

S-matrix (R-matrix) : $S_{12} = P_{12} \circ R_{12}$

$$S_{12} : V_1 \otimes V_2 \longrightarrow V_2 \otimes V_1$$

is determined $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$ symmetry on fundamental rep. $\mathbf{2}|\mathbf{2}$ (R_{12}:16×16 matrix).
Is it possible to express R_{12} in terms of generators (the universal R-matrix) ?

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Is it possible to express R_{12} in terms of generators (the universal R-matrix) ?
The first step is an analysis of the classical r-matrix,

$$R_{12} = 1 + \hbar r_{12} + \mathcal{O}(\hbar^2),$$

where $\hbar = 1/\lambda$.

classical r-matrix from d$(2,1,\varepsilon)$

Standard form(rational type) : $R_{12} = 1 + \hbar r_{12} + \mathcal{O}(\hbar^2)$

$$r_{12} = \frac{T_{12}^{\mathfrak{g}}}{u_1 - u_2} \quad \text{with } T_{12}^{\mathfrak{g}} = J^A \otimes J^B g_{BA}$$

But it takes the non-canonical form,

$$r_{12} = \frac{T_{12}^{psu} - (u_2/u_1)C \otimes I - (u_1/u_2)I \otimes C}{u_1 - u_2}$$

with the new classical generator I. [Beisert,Spill]

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with the new classical generator I. [Beisert,Spill]

The non-canonical r-matrix can be reproduced from the canonical r-matrix of the exceptional Lie superalgebra. [TM,Moriyama]

$$r_{12}|\chi_1\chi_2\rangle = \frac{T_{12}^{d}}{u_1 - u_2}|\chi_1\chi_2\rangle\Big|_{\varepsilon\to 0}$$

(Note) There are singular terms $\mathcal{O}(1/\varepsilon)$

$$r_{12}|\chi_1\chi_2\rangle = \mathcal{O}(1/\varepsilon) + \mathcal{O}(1) + \mathcal{O}(\varepsilon) + \cdots$$

Evaluation of the $T^d_{12}|\chi_1\chi_2\rangle$

$$T^d_{12}|\chi_1\chi_2\rangle = T^{\mathrm{psu}}_{12}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)|\chi_1\chi_2\rangle$$
$$T^{\mathrm{psu}}_{12} := R\otimes R + L\otimes L + Q\otimes S - S\otimes Q$$

1. Determine the representation of the supercharges Q, S.

$$Q|\phi_n\rangle = a_n|\psi_{n+\frac{1}{2}}\rangle\ , \quad Q|\psi_{n+\frac{1}{2}}\rangle = b_{n+1}|\phi_{n+1}\rangle$$
$$S|\phi_n\rangle = c_n|\psi_{n-\frac{1}{2}}\rangle\ , \quad S|\psi_{n-\frac{1}{2}}\rangle = d_{n-1}|\phi_{n-1}\rangle$$

$$(\because) \quad [Q,Q]=P,\ \ [S,S]=K,\ \ [Q,S]=R+L+C$$

Evaluation of the $T^{d}_{12}|\chi_1\chi_2\rangle$

$$T^{d}_{12}|\chi_1\chi_2\rangle = T^{\mathrm{psu}}_{12}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)|\chi_1\chi_2\rangle$$

1. Determine the representation of the supercharges Q, S.
2. Find the representation of the “central” charges P, C, K.

$$P|\phi_n\rangle = a_n b_{n+1}|\phi_{n+1}\rangle \qquad P|\psi_{n+\frac{1}{2}}\rangle = a_{n+1} b_{n+1}|\psi_{n+\frac{3}{2}}\rangle$$

$$C|\phi_n\rangle = \frac{1}{2}(a_n d_n + b_n c_n)|\phi_n\rangle \qquad C|\psi_{n+\frac{1}{2}}\rangle = \frac{1}{2}(a_n d_n + b_{n+1} c_{n+1})|\psi_{n+\frac{1}{2}}\rangle$$

$$K|\phi_n\rangle = c_n d_{n-1}|\phi_{n-1}\rangle \qquad K|\psi_{n+\frac{1}{2}}\rangle = c_n d_n|\psi_{n-\frac{1}{2}}\rangle$$

c.f. $[C, P] = \varepsilon P, \quad [C, K] = -\varepsilon K, \quad [P, K] = 2\varepsilon C$

Evaluation of the $T^d_{12}|\chi_1\chi_2\rangle$

$$T^d_{12}|\chi_1\chi_2\rangle = T^{\rm psu}_{12}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)|\chi_1\chi_2\rangle$$

1. Determine the representation of the supercharges Q, S.
2. Find the representation of the "central" charges P, C, K.
3. Shift the indices $|\psi_{n+\frac{1}{2}}\rangle \to |\psi_n\rangle$

$$P|\phi_n\rangle = a_n b_{n+1}|\phi_{n+1}\rangle \qquad P|\psi_n\rangle = a_{n+\frac{1}{2}} b_{n+\frac{1}{2}}|\psi_{n+1}\rangle$$

$$C|\phi_n\rangle = \frac{1}{2}(a_n d_n + b_n c_n)|\phi_n\rangle \qquad C|\psi_n\rangle = \frac{1}{2}(a_n d_n + b_n c_n)|\psi_n\rangle$$

$$K|\phi_n\rangle = c_n d_{n-1}|\phi_{n-1}\rangle \qquad K|\psi_n\rangle = c_{n-\frac{1}{2}} d_{n-\frac{1}{2}}|\psi_{n-1}\rangle$$

Evaluation of the $T^d_{12}|\chi_1\chi_2\rangle$

$$T^d_{12}|\chi_1\chi_2\rangle = T^{\text{psu}}_{12}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)|\chi_1\chi_2\rangle$$

1. Determine the representation of the supercharges Q, S.
2. Find the representation of the "central" charges P, C, K.
3. Shift the indices $|\psi_{n+\frac{1}{2}}\rangle \longrightarrow |\psi_n\rangle$
4. Solve the diagonalization problem

$$\frac{1}{\varepsilon}T^C_{12} := \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)$$

$$\frac{1}{\varepsilon}T^C_{12}|\chi_n\chi'_m\rangle = r_n|\chi_{n+1}\chi'_{m-1}\rangle + s_n|\chi_n\chi'_m\rangle + t_n|\chi_{n-1}\chi'_{m+1}\rangle$$

Eigenvector $|\chi\chi'\rangle = \sum_{l=-\infty}^{\infty}(1+\varepsilon f_{n-l})|\chi_{n-l}\chi'_{m+l}\rangle$

Eigenvalue $\frac{1}{\varepsilon}T^C_{12}|\chi\chi'\rangle = (r_n + s_n + t_n)|\chi\chi'\rangle$

Evaluation of the $T^d_{12}|\chi_1\chi_2\rangle$

$$T^d_{12}|\chi_1\chi_2\rangle = T^{\mathrm{psu}}_{12}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P \otimes K - 2C \otimes C + K \otimes P)|\chi_1\chi_2\rangle$$

1. Determine the representation of the supercharges Q, S.
2. Find the representation of the "central" charges P, C, K.
3. Shift the indices $|\psi_{n+\frac{1}{2}}\rangle \rightarrow |\psi_n\rangle$
4. Solve the diagonalization problem
5. Normalization

The singular term $\mathcal{O}(1/\varepsilon)$ is nothing but a normalization factor.

$$T^C_{12}|\phi_1\phi_2\rangle = [\frac{1}{\varepsilon}T - 2(\frac{u_1}{u_2}I \otimes C + \frac{u_2}{u_1}C \otimes I)]|\phi_1\phi_2\rangle$$
$$T^C_{12}|\psi_1\psi_2\rangle = [\frac{1}{\varepsilon}T + 0]|\psi_1\psi_2\rangle$$
$$T^C_{12}|\phi_1\psi_2\rangle = [\frac{1}{\varepsilon}T - 2\frac{u_1}{u_2}I \otimes C]|\phi_1\psi_2\rangle$$
$$T^C_{12}|\psi_1\phi_2\rangle = [\frac{1}{\varepsilon}T - 2\frac{u_2}{u_1}C \otimes I]|\psi_1\phi_2\rangle$$

Evaluation of the $T_{12}^{d}|\chi_1\chi_2\rangle$

$$T_{12}^{d}|\chi_1\chi_2\rangle = T_{12}^{\mathrm{psu}}|\chi_1\chi_2\rangle + \frac{1}{\varepsilon}(P\otimes K - 2C\otimes C + K\otimes P)|\chi_1\chi_2\rangle$$

1. Determine the representation of the supercharges Q, S.
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3. Shift the indices $|\psi_{n+\frac{1}{2}}\rangle \to |\psi_n\rangle$
4. Solve the diagonalization problem
5. Normalization

$$T_{12}^{C}|\phi_1\phi_2\rangle = -\left(\frac{u_1}{u_2} I\otimes C + \frac{u_2}{u_1} C\otimes I\right)|\phi_1\phi_2\rangle$$
$$T_{12}^{C}|\psi_1\psi_2\rangle = -\left(\frac{u_1}{u_2} I\otimes C + \frac{u_2}{u_1} C\otimes I\right)|\psi_1\psi_2\rangle$$
$$T_{12}^{C}|\phi_1\psi_2\rangle = -\left(\frac{u_1}{u_2} I\otimes C + \frac{u_2}{u_1} C\otimes I\right)|\phi_1\psi_2\rangle$$
$$T_{12}^{C}|\psi_1\phi_2\rangle = -\left(\frac{u_1}{u_2} I\otimes C + \frac{u_2}{u_1} C\otimes I\right)|\psi_1\phi_2\rangle$$

What we have shown is

the non-canonical r-matrix can be reproduced from the exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$ via the standard definition.

$$r_{12}|\chi_1\chi_2\rangle = \frac{T^{d}_{12}}{u_1-u_2}|\chi_1\chi_2\rangle\Big|_{\varepsilon\to 0}$$
$$= \frac{T^{psu}_{12} - (u_2/u_1)C\otimes I - (u_1/u_2)I\otimes C}{u_1-u_2}|\chi_1\chi_2\rangle$$

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Yangian symmetries of the R-matrix

The Yangian algebra is a symmetry

$$[\Delta\widehat{J}^A, R_{12}]|\chi_1\chi_2\rangle = 0 \ , \quad {}^{\forall}\widehat{J}^A \in Y(\mathrm{psu}(2|2) \ltimes \mathbb{R}^3)$$

Coproducts (acting on 2-body state):

$$\Delta\widehat{J}^A = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \underbrace{\tfrac{\hbar}{2} J^B \otimes J^C f_{CB}{}^A}_{\text{non-local}}$$
$$= \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \tfrac{\hbar}{4}[T_{12}, 1 \otimes J^A - J^A \otimes 1].$$

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Evaluation representation

$$\widehat{J}|\chi\rangle = uJ|\chi\rangle \quad (u : \text{evaluation parameter})$$

[Beisert]

Yangian symmetries from d$(2,1;\varepsilon)$

The su(2) outer automorphism $B^A{}_B$ is needed. su(2) $\ltimes$ psu(2|2)

$$\Delta\widehat{J}^A = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \frac{\hbar}{4}[T_{12}, 1 \otimes J^A - J^A \otimes 1]$$

with $$T_{12}^{\mathrm{out}} = T_{12}^{\mathrm{psu}} + B^A{}_B \otimes C^B{}_A + C^A{}_B \otimes B^B{}_A$$

Yangian symmetries from d$(2,1;\varepsilon)$

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with $\quad T_{12}^{\mathrm{out}} = T_{12}^{\mathrm{psu}} + B^A{}_B \otimes C^B{}_A + C^A{}_B \otimes B^B{}_A$

d$(2,1;\varepsilon)$ has the well-defined Casimir operator.

$$T_{12}^{\mathrm{d}} = T_{12}^{\mathrm{psu}} + \frac{1}{\varepsilon}(C^A{}_B \otimes C^B{}_A + C^A{}_B \otimes C^B{}_A)$$

$$C^A{}_B = \begin{pmatrix} C & P \\ -K & -C \end{pmatrix}$$

The Yangian coproducts of the exceptional algebra recover the Yangian symmetries of the R-matrix in the $\varepsilon \to 0$ limit.

$$[\Delta\widehat{J}^A|_{\varepsilon\to 0}, R_{12}]|\chi_1\chi_2\rangle = 0 \qquad \text{[TM,Moriyama]}$$

(Note) There is no singular term $\mathcal{O}(1/\varepsilon)$. $\to$ $\Delta\widehat{J}^A = \mathcal{O}(1) + \mathcal{O}(\varepsilon)$

Secret symmetry

$$\Delta\widehat{I} := \widehat{I} \otimes 1 + 1 \otimes \widehat{I} + \frac{\hbar}{2}(Q^{\alpha}{}_{a} \otimes S^{a}{}_{\alpha} + S^{a}{}_{\alpha} \otimes Q^{\alpha}{}_{a})$$

$$I \notin \mathrm{su}(2|2) \quad (\because) I|\phi\rangle = +\bullet\,|\phi\rangle, I|\psi\rangle = -\bullet\,|\psi\rangle$$ "Secret"

Symmetry of the R-matrix [MT,Moriyama,Torrielli]

$$[\Delta\widehat{I}, R_{12}]|\chi_1\chi_2\rangle = 0$$ "Symmetry"

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Symmetry of the R-matrix [MT,Moriyama,Torrielli]

$$[\Delta\widehat{I}, R_{12}]|\chi_1\chi_2\rangle = 0$$ "Symmetry"

$(\because)$

$$[Q,S] \sim R + L + C \to f_{C}{}^{QS} \neq 0$$

$$r_{12} = \frac{J^A \otimes J^B g_{BA}}{u_1 - u_2} = \frac{T^{psu}_{12} - C\otimes I - I\otimes C}{u_1 - u_2} \to g_{CI} \neq 0$$

$$\Rightarrow f_{QS}{}^{I} \neq 0$$

Substitute $f_{QS}{}^{I}$ for the formula

$$\Delta\widehat{J}^A = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \tfrac{\hbar}{2} J^B \otimes J^C f_{CB}{}^{A}$$

PLAN

Yangian symmetries of the R-matrix

The R-matrix enjoys the Yangian symmetries

$$[\Delta\widehat{J}^A, R_{12}]|\chi_1\chi_2\rangle = 0\ , \quad {}^\forall\widehat{J}^A \in Y(\mathrm{psu}(2|2) \ltimes \mathbb{R}^3)$$

with the coproducts (acting on 2-body state)

$$\Delta\widehat{J}^A = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \underbrace{\tfrac{\hbar}{2} J^B \otimes J^C {f_{CB}}^A}_{\text{non-local}}$$

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with the coproducts (acting on 2-body state)

$$\Delta\widehat{J}^A = \widehat{J}^A \otimes 1 + 1 \otimes \widehat{J}^A + \underbrace{\frac{\hbar}{2} J^B \otimes J^C f_{CB}{}^A}_{\text{non-local}}$$

if we assume the Evaluation representation of the Yangian algebra!

$$\widehat{J}|\chi\rangle = uJ|\chi\rangle \quad (u : \text{evaluation parameter})$$

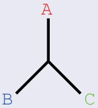
[Beisert]

We would like to show that the Evaluation representation is compatible with the all defining relation of the Yangian algebra.

Notations

Structure constants

$$f_{ABC} \quad \longleftrightarrow$$



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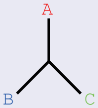
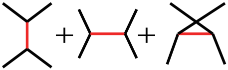


(Ex.) Jacobi identity

$$f^{(AB|D} f_D{}^{|C)E} = 0 \qquad \longleftrightarrow$$



$$= 0$$

Yangian algebra

The generators

Grade-0 J ($\leftarrow$ Lie algebra)

Grade-1 $\widehat{J}$

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Commutation relations

$$[J^A, J^B] = J^C f_C{}^{AB}, \quad [J^A, \widehat{J}^B] = \widehat{J}^C f_C{}^{AB}$$

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Serre relation

$$[\widehat{J}^A, [\widehat{J}^B, J^C]] + \text{cyclic} = -\frac{\hbar^2}{24} f_L{}^{AI} f_M{}^{BJ} f_N{}^{CK} f_{IJK} \{J^L, J^M, J^N\}$$

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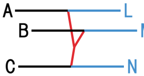

$$\{J^L, J^M, J^N\}$$

The proof of the Serre relation

What we'd like to do is

to show that the evaluation representation $\widehat{J}|\chi\rangle = uJ|\chi\rangle$ is consistent with the all defining relations, especially the Serre relation.

$$([\widehat{J}^A, [\widehat{J}^B, J^C]] + \text{cyclic})|\chi\rangle =$$

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Since $\quad (\text{LHS}) = u^2 \times (\text{Jacobi id})|\chi\rangle = 0$,

we have to show $\quad (\text{RHS}) =$ 

$\{J^L, J^M, J^N\}|\chi\rangle = 0$.

Difficulty(1)

The Killing form of the $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$ is degenerate.
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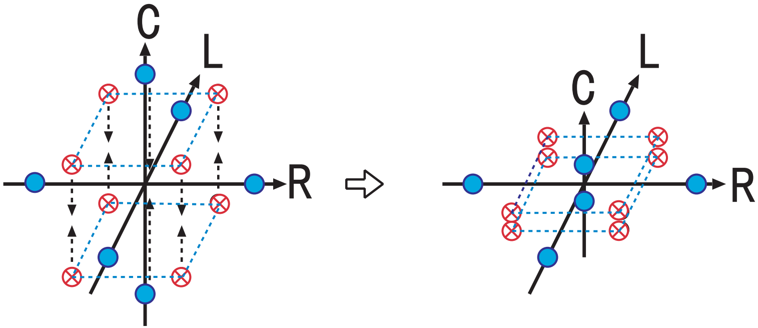
$\longrightarrow$ We cannot raise or lower the indices of $f_C{}^{AB}$.
[Solution] Using the exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$,

- which has the non-degenerate Killing form and
- reproduce $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$ in $\varepsilon \to 0$ limit.



Difficulty(2)

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[Solution] 3 dim γ-matrix formalism
$d(2,1;\varepsilon) \supset su(2)_R \times su(2)_L \times su(2)_C, \quad su(2) = so(3)$

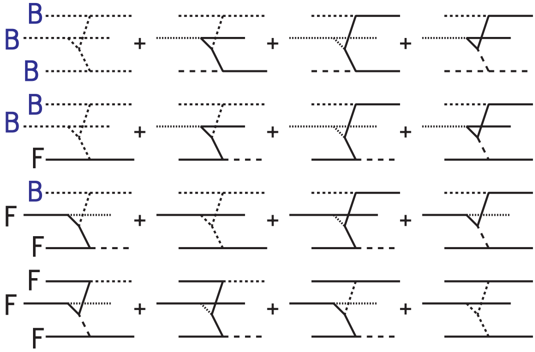
$$(\gamma^{\mathbf{A}})^K{}_L = (\gamma^A_{A'})^K{}_L = -\sqrt{2}\Big(\delta^K_{A'}\delta^A_L - \frac{1}{2}\delta^K_L\delta^A_{A'}\Big)$$

Clifford algebra

$$(\gamma^{\mathbf{A}})^K{}_L(\gamma^{\mathbf{B}})^L{}_M + (\gamma^{\mathbf{B}})^K{}_L(\gamma^{\mathbf{A}})^L{}_M = 2\delta^K_M g^{\mathbf{AB}}$$

$$\text{with } g^{\mathbf{AB}} = \delta^B_{A'}\delta^A_{B'} - \frac{1}{2}\delta^A_{A'}\delta^B_{B'}$$

Computing the all possible combinations (Totally $66 \times 2 = 132$ terms)



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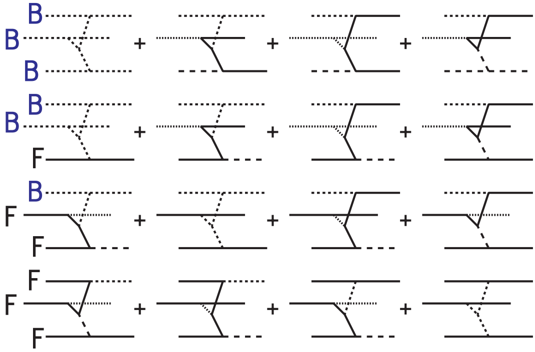
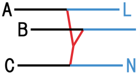


we arrive at the result

$$([\widehat{J}^A, [\widehat{J}^B, J^C]] + \text{cyclic})|\chi\rangle = -\frac{\hbar^2}{24}$$ 

$$\{J^L, J^M, J^N\}|\chi\rangle = 0.$$

[TM, Moriyama]

Summary

The fundamental representation $|\chi\rangle$ of the psu(2|2) $\ltimes \mathbb{R}^3$ (one-magnon state of the spin-chain) can be lifted to the Evaluation representation of the Yangian algebra.
This is also true for the infinite dimensional representation $|\chi_n\rangle$ of the exceptional algebra d$(2,1;\varepsilon)$.

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[Recall] The meaning of the Serre relation

(1) It assures the homomorphism of the coproduct Δ.

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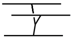
$\rightarrow$ Construction of the grade-2 generators

The higher grade generators in general case

The definition of the grade-2 generators $\widehat{\widehat{J}}$,

$$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}^A f_A{}^{BC} + X^{BC} \quad \text{with} \quad X^{(A|D} f_D{}^{|BC)} =$$  $$J^3$$

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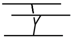
$$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}^A f_A{}^{BC} + X^{BC} \quad \text{with} \quad X^{(A|D} f_D{}^{|BC)} =$$  J^3

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$$X^{BC} f_{CBD} = 0$$

which gives the standard form of the grade-2 generators,

$$\widehat{\widehat{J}}^A = \frac{1}{c_2} [\widehat{J}^B, \widehat{J}^C] f_{CB}{}^A \quad \text{with } f_A{}^{BC} f_{CBD} = c_2 g_{AD}.$$

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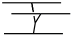
$$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}^A f_A{}^{BC} + X^{BC} \quad \text{with} \quad X^{(A|D} f_D{}^{|BC)} =$$  J^3

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In the case of $d(2,1;\varepsilon)$ $(\xrightarrow{\varepsilon \to 0} psu(2|2) \ltimes \mathbb{R}^3)$, $c_2 = 0$
$\longrightarrow$ The general framework does not work! (The gauge condition is too strong...)

In the case of the exceptional algebra

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Because it is consistent with the Evaluation representation,

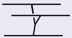
$$[\widehat{J}^B, \widehat{J}^C]|\chi\rangle = \widehat{\widehat{J}}^A f_A{}^{BC}|\chi\rangle + X^{BC}|\chi\rangle.$$

The grade-2 generators

$$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}^A f_A{}^{BC} + X^{BC}$$

Question

What is X^{BC} which satisfies the following two conditions simultaneously?

1. The Serre relation $X^{(A|D} f_D{}^{|BC)} =$  J^3 and
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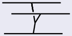
1. The Serre relation $X^{(A|D} f_D{}^{|BC)} =$  J^3 and
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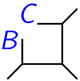
Answer

$X^{BC} =$ 

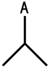
$J^3 - [J^B, J^C]$

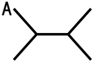
Construction of the coproducts

By the definition of the grade-2 generators

$$[\widehat{J}^B, \widehat{J}^C] = \widehat{\widehat{J}}^A f_A{}^{BC} + X^{BC},$$

we can compute its coproducts as

$$\Delta\widehat{\widehat{J}}^A = \widehat{\widehat{J}}^A \otimes 1 + 1 \otimes \widehat{\widehat{J}}^A + \frac{\hbar}{2}$$ 

$$(\widehat{J} \otimes J + J \otimes \widehat{J}).$$
$$+ \frac{\hbar^2}{24}$$ 

$$(J^2 \otimes J + J \otimes J^2)$$

This is the symmetry of the R-matrix

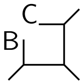
$$[\Delta\widehat{\widehat{J}}^A, R_{12}]|\chi_1\chi_2\rangle = 0$$

even though $c_2 = 0$.

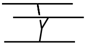
The Serre relation revisited

What is the meaning that $\widetilde{X}^{BC}$ reduces the commutator $[J^B, J^C]$ on the representation?

$\widetilde{X}^{BC} :=$



$J^3 \quad \Rightarrow \quad [\hat{J}^{(A},[\hat{J}^B, J^{C)}]] =$



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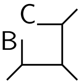
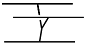
$\widetilde{X}^{BC} :=$  $J^3 \quad \Rightarrow \quad [\hat{J}^{(A}, [\hat{J}^B, J^{C)}]] =$  $J^3 = \widetilde{X}^{(A|D} f_D{}^{|BC)}$

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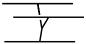 $J^3|\chi\rangle = \widetilde{X}^{(A|D} f_D{}^{|BC)}|\chi\rangle = J^E f_E{}^{(A|D} f_D{}^{|BC)}|\chi\rangle = 0.$

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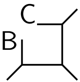
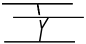
$\widetilde{X}^{BC} :=$  $J^3 \quad \Rightarrow \quad [\hat{J}^{(A}, [\hat{J}^B, J^{C)}]] =$  $J^3 = \widetilde{X}^{(A|D} f_D{}^{|BC)}$

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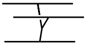 $J^3|\chi\rangle = \widetilde{X}^{(A|D} f_D{}^{|BC)}|\chi\rangle = J^E f_E{}^{(A|D} f_D{}^{|BC)}|\chi\rangle = 0.$

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The relation $\widetilde{X}^{AD}|\chi\rangle = J^E f_E{}^{AD}|\chi\rangle$ is a sufficient condition so that the evaluation representation holds.

Conclusion

We have investigated the Yangian symmetries in the AdS/CFT spin chain model from the viewpoint of the exceptional Lie superalgebra $\mathrm{d}(2,1;\varepsilon)$.

1. We have reproduced the (non-canonical) classical r-matrix from the canonical r-matrix of the exceptional algebra $\mathrm{d}(2,1;\varepsilon)$ in the $\varepsilon \to 0$ limit.
2. We have recovered the Yangian symmetries from the exceptional algebra without using the su(2) outer automorphism.
3. The R-matrix has a secret symmetry $I \notin \mathrm{su}(2|2)$, which is the quantum analogue of the additional operator in the classical r-matrix.
4. The ∞-dim. representation of the exceptional algebra and the fundamental representation of the $\mathrm{psu}(2|2) \ltimes \mathbb{R}^3$ can be lifted to the evaluation representation of the Yangian algebra.

Future

- The construction of the universal R-matrix
- The $\mathrm{d}(2,1,\varepsilon)$ spin-chain model and its application to the AdS_3/CFT_2 correspondence
- The Yangian symmetries as a special limit of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with $q = e^{\hbar}$