

# Extreme statistics of vicious walkers : from random matrices to Yang-Mills theory on the sphere

G. Schehr

Laboratoire de Physique Théorique  
CNRS-Université Paris Sud-XI, Orsay

- G.S., S.N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- P.J. Forrester, S.N. Majumdar, G.S., arXiv:1009.2362, to appear in Nuclear Physics B

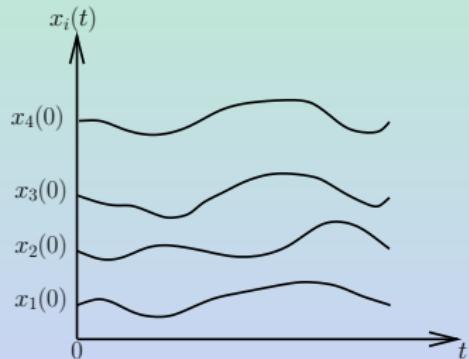
# Non-intersecting Brownian motions in 1d

- $N$  Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t), \langle \zeta_i(t)\zeta_j(t') \rangle = \delta_{i,j}\delta(t-t')$$
$$x_1(0) < x_2(0) < \dots < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \dots < x_N(t), \\ \forall t \geq 0$$



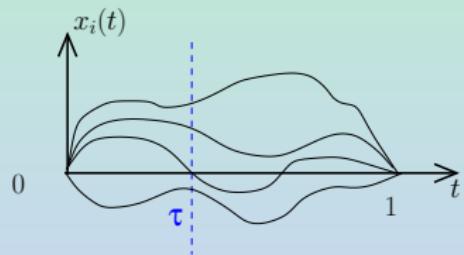
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watermelons

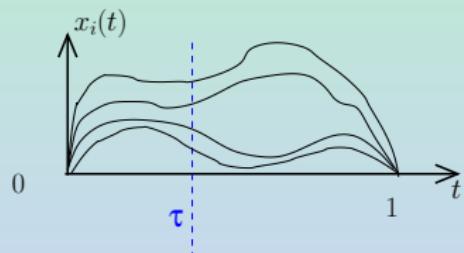
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watermelons "with a wall"

# Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS

VOLUME 48, NUMBER 5

1 MARCH 1968

## Soluble Model for Fibrous Structures with Steric Constraints

P.-G. DE GENNES

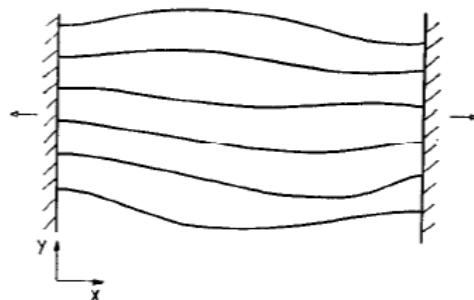
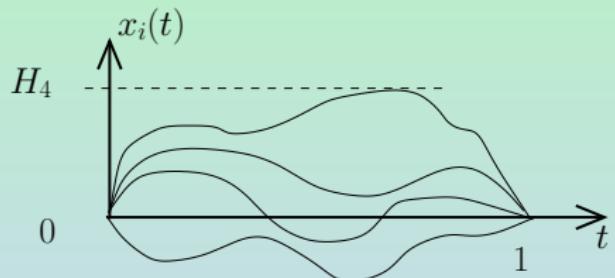


FIG. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

# Vicious walkers in physics

- P.G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- M.E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- C. Krattenthaler, A.J. Guttmann, X.G. Viennot *Vicious walkers, friendly walkers and Young tableaux* (1995)
- K. Johansson, *Directed polymer, random tiling* (2003)
- H. Spohn, M. Praehofer, P. Ferrari et al. *Stochastic growth models* (2006)
- ...

# Extreme statistics of vicious walkers



Maximal height of watermelons

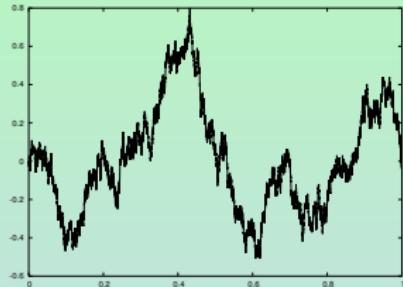
$$x_1(t) < x_2(t) < \dots < x_N(t)$$

$$H_N = \max_{\tau} [x_N(\tau), 0 \leq \tau \leq 1]$$

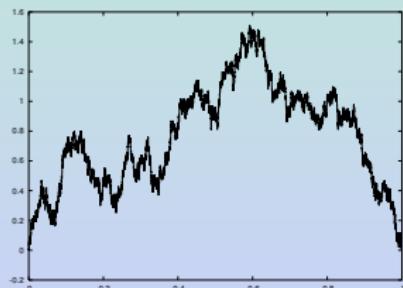
$$\langle H_N \rangle = ?$$

# Extreme statistics of Brownian motion

- Brownian bridge



- Brownian excursion



$$H_1 = \max_{\tau} [x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{8}}$$

$$H_1 = \max_{\tau} [x(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_1 \rangle = \sqrt{\frac{\pi}{2}}$$

de Bruijn, Knuth, Rice '72

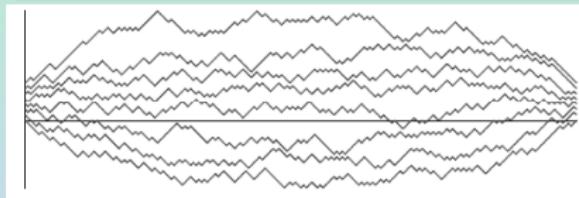
## Watermelon uniform random generation with applications

Nicolas Bonichon\*, Mohamed Mosbah

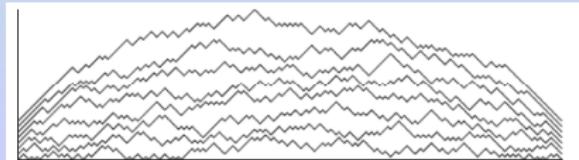
LaBRI-Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence, France

Theoretical Computer Science 307 (2003) 241–256

$$H_N = \max_{\tau} [x_N(\tau), 0 \leq \tau \leq 1]$$



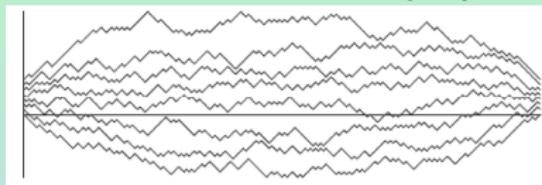
$$\langle H_N \rangle_{\text{num}} \sim \sqrt{0.82 N}$$



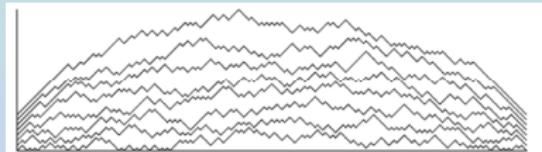
$$\langle H_N \rangle_{\text{num}} \sim \sqrt{1.67 N}$$

## 1 Connection between watermelons and random matrices

⇒ exact asymptotic results for  $\langle H_N \rangle$  for  $N \gg 1$



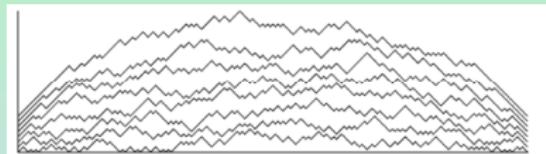
$$\langle H_N \rangle \sim \sqrt{N}$$



$$\langle H_N \rangle \sim \sqrt{2N}$$

2 Connect<sup>o</sup> btw. watermelons with a wall and Yang-Mills theory on the sphere

⇒ exact results for the full distribution of  $H_N$ ,  $N \rightarrow \infty$



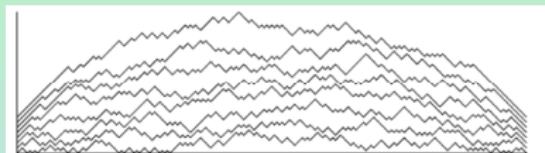
Cumulative distribution

$$F_N(M) = \text{Proba}[H_p \leq M]$$

$$F_N(M) \rightarrow \mathcal{F}_1 \left( 2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right| \right), \quad N \rightarrow \infty$$

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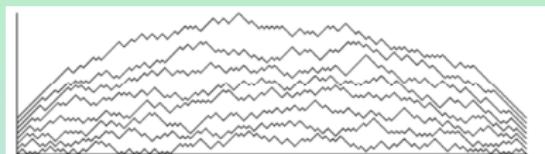
$$F_N(M) \rightarrow \mathcal{F}_1 \left( 2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right| \right), \quad N \rightarrow \infty$$

$$\mathcal{F}_1(t) = \exp \left( -\frac{1}{2} \int_t^{\infty} ((s-t) q^2(s) - q(s)) \, ds \right)$$

$$q''(t) = 2q^3(t) + tq(t), \quad q(t) \sim \text{Ai}(t), \quad t \rightarrow \infty$$

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$\mathcal{F}_1$  is the Tracy-Widom distribution of the largest eigenvalue of GOE random matrices

# Outline

- 1 Vicious walkers and random matrices
- 2 Path integral approach to vicious walkers problem
  - Transition probability
  - A regularization procedure for watermelons
  - Joint probability distribution for watermelons
- 3 Distribution of the maximal height
  - Watermelons without wall
  - Watermelons with a wall
- 4 Connection with Yang-Mills theory on the sphere
  - Partition function of 2d Yang-Mills theory
  - Large  $N$  limit and Tracy-Widom distribution
- 5 Conclusion

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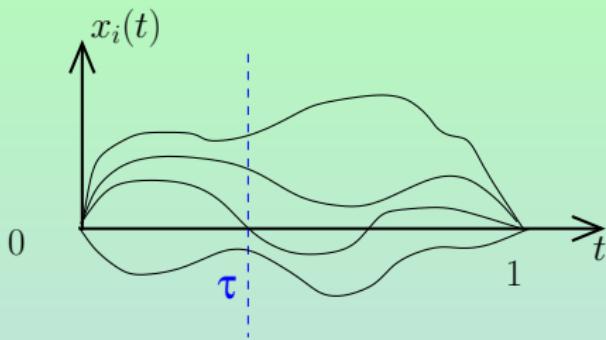
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# Non intersecting Brownian motions and RMT



- Joint probability of  $x_1(\tau), x_2(\tau), \dots, x_p(\tau)$  at fixed time  $\tau$

$$P_{\text{joint}}(x_1, x_2, \dots, x_N, \tau) \propto \sigma(\tau)^{-N^2} \prod_{i < j=1}^N (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}$$

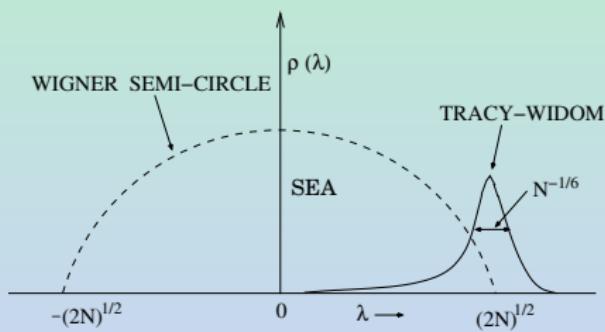
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions  $\frac{x_i}{\sigma(\tau)}$  are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE,  $\beta = 2$ )

# Asymptotic behavior of $\langle H_N \rangle$

- The rescaled positions  $\frac{x_i}{\sigma(\tau)}$  are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE,  $\beta = 2$ )
- Mean density  $\rho(\lambda)$  of eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_N$  for GUE

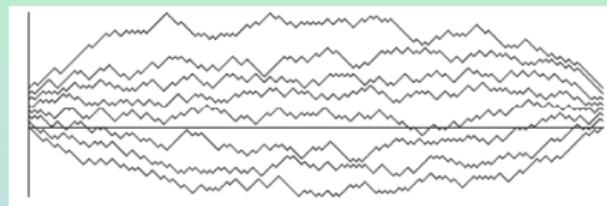
$$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^N \langle \delta(\lambda - \lambda_\alpha) \rangle$$



# Asymptotic behavior of $\langle H_N \rangle$

- Consequences for watermelons without wall for large  $N$

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}} \chi_2$$



$$\text{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi),$$

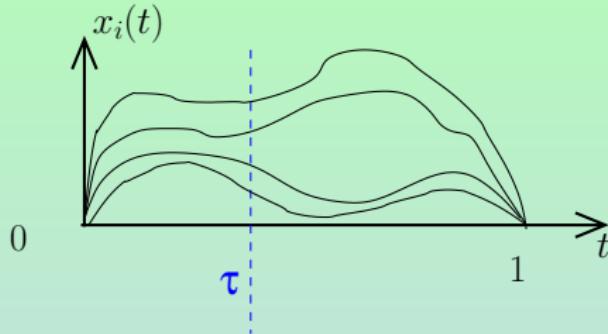
Tracy-Widom distribution for  $\beta = 2$

- The maximal height is reached for  $\tau = 1/2$

$$H_N = \max_{\tau} [x_p(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{N} \quad \text{vs.} \quad \langle H_N \rangle_{\text{num}} \sim \sqrt{0.82 N}$$

# Joint probability for watermelons with a wall



- At fixed time  $\tau$

$$P_{\text{joint}}(\mathbf{x}, \tau) \propto \sigma(\tau)^{-N(2N+1)} \prod_{i=1}^N x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

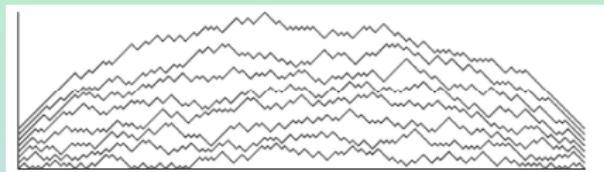
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- $y_i = \frac{x_i^2}{2\sigma^2(\tau)}$  are distributed like the eigenvalues of Wishart matrices, with  $\beta = 2$  and  $M - N = \frac{1}{2}$ ,

# Asymptotic behavior of $\langle H_N \rangle$

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$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim 2\sqrt{N} + \frac{N^{-1/6}}{2^{2/3}} \chi_2$$



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- The maximal height is reached for  $\tau = 1/2$

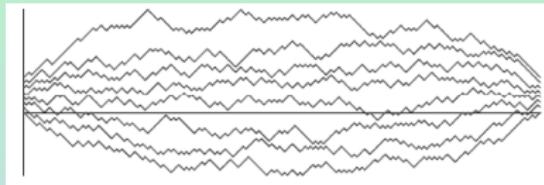
$$H_N = \max_{\tau} [x_N(\tau), 0 \leq \tau \leq 1]$$

$$\langle H_N \rangle = \langle x_N(\tau = \frac{1}{2}) \rangle \sim \sqrt{2N} \quad vs. \quad \langle H_N \rangle_{\text{num}} \sim \sqrt{1.67 N}$$

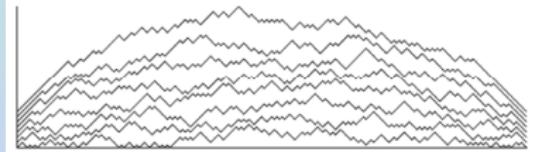
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- Connection between **watermelons** and **random matrices**

⇒ exact asymptotic results for  $\langle H_N \rangle$  for  $N \gg 1$



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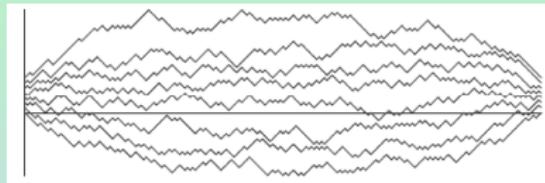


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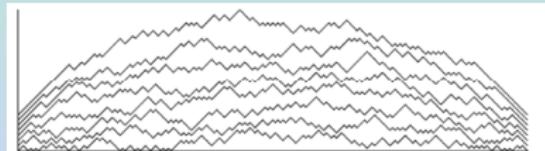
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What about the fluctuations of  $H_N$  ?

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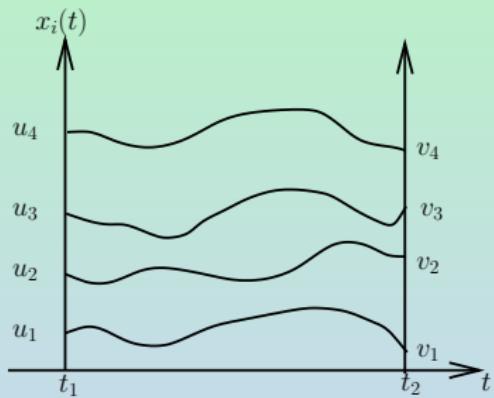
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# Transition probability



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- $N$  non constrained Brownian motions : path integral

$$\mathcal{P}_N(\mathbf{v}, t_2 | \mathbf{u}, t_1) \propto \int_{\mathbf{x}(t_1)=\mathbf{u}}^{\mathbf{x}(t_2)=\mathbf{v}} \mathcal{D}\mathbf{x}(t) \exp \left[ -\frac{1}{2} \sum_{i=1}^N \int_{t_1}^{t_2} (\dot{x}_i)^2 dt \right]$$
$$\mathcal{P}_N(\mathbf{v}, t_2 | \mathbf{u}, t_1) \propto \langle \mathbf{v} | \exp \left[ -\hat{H}_0(t_2 - t_1) \right] | \mathbf{u} \rangle, \quad \hat{H}_0 = \sum_{i=1}^N -\frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

$$\mathcal{P}_N(\mathbf{v}, t_2 | \mathbf{u}, t_1) = \sum_E e^{-E(t_2 - t_1)} \Psi_E^*(\mathbf{u}) \Psi_E(\mathbf{v})$$
$$\Psi_E(\mathbf{u}) = \langle \mathbf{u} | E \rangle, \quad \hat{H}_0 | E \rangle = E | E \rangle$$

# Transition probability

$$\mathcal{P}_N(v_1, v_2, \dots, v_N, t_2 | u_1, u_2, \dots, u_N, t_1) = ?$$

- $p$  non constrained Brownian motions : path integral

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$$\Psi_E(\mathbf{u}) = \langle \mathbf{u} | E \rangle, \quad \hat{H}_0 | E \rangle = E | E \rangle$$

$$\Psi_E(\mathbf{u}) \equiv \Psi_E(u_1, u_2, \dots, u_N) = \prod_{i=1}^N \psi_{E_i}(u_i), \quad E = \sum_{i=1}^N E_i$$

$\psi_{E_i}(u_i)$   $\equiv$  one particle eigenfunction

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$$\mathcal{P}_N(v_1, v_2, \dots, v_N, t_2 | u_1, u_2, \dots, u_N, t_1) = ?$$

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$$\begin{aligned} \Psi_E(\mathbf{u}) \equiv \Psi_E(u_1, u_2, \dots, u_N) &= \frac{1}{\sqrt{N!}} \det_{1 \leq i, j \leq N} \psi_{E_i}(u_j) \\ &\equiv \text{SLATER DETERMINANT} \end{aligned}$$

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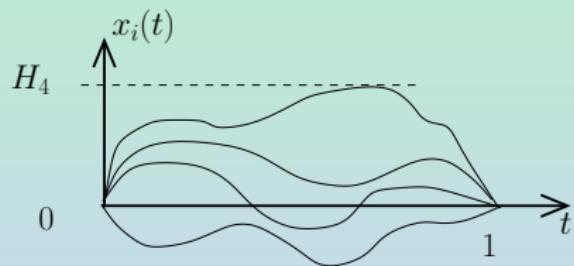
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# Watermelons configurations : regularization procedure

- Brownian motion has an infinite density of zero-crossings



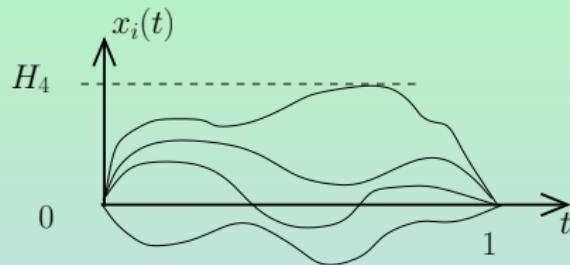
Such configurations are ill-defined for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$

$$\text{AND } x_i(t = 0^+) < x_{i+1}(t = 0^+)$$

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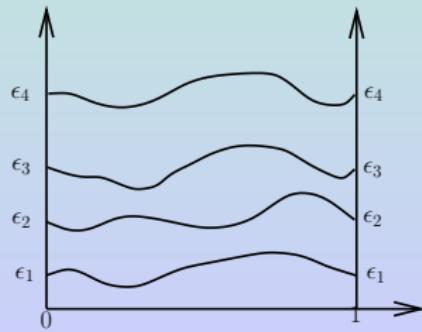


Such configurations are **ill-defined** for Brownian motion :

$$x_i(0) = x_{i+1}(0)$$

AND  $x_i(t = 0^+) < x_{i+1}(t = 0^+)$

- A need for regularization : introduce cut-offs  $\epsilon_i$ 's



Only at the end take the limit  
 $\epsilon_i \rightarrow 0$

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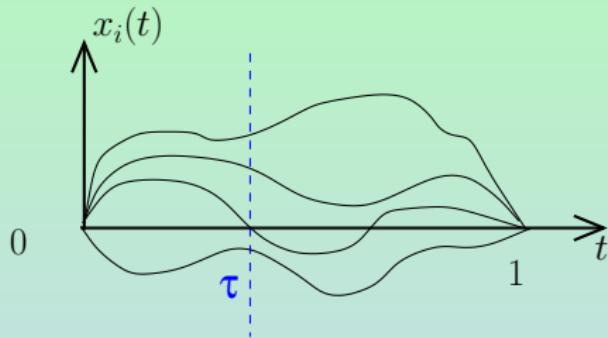
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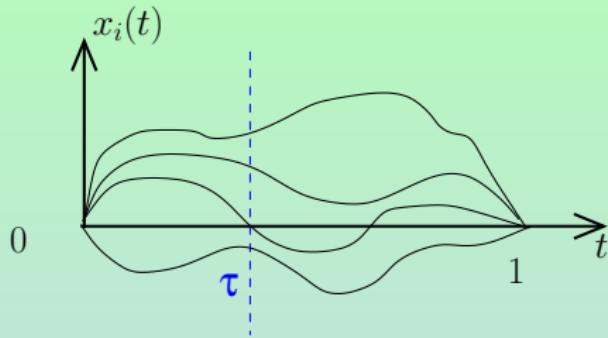
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# Joint probability for watermelons without wall



$$P_{\text{joint}}(\mathbf{x}, \tau) = \lim_{\epsilon_i \rightarrow 0} \frac{\langle \epsilon | e^{-\tau \hat{H}_0} | \mathbf{x} \rangle \langle \mathbf{x} | e^{-(1-\tau) \hat{H}_0} | \epsilon \rangle}{\langle \epsilon | e^{-\hat{H}_0} | \epsilon \rangle}, \quad \hat{H}_0 = - \sum_{i=1}^N \frac{1}{2} \frac{\partial^2}{\partial x_i^2}$$

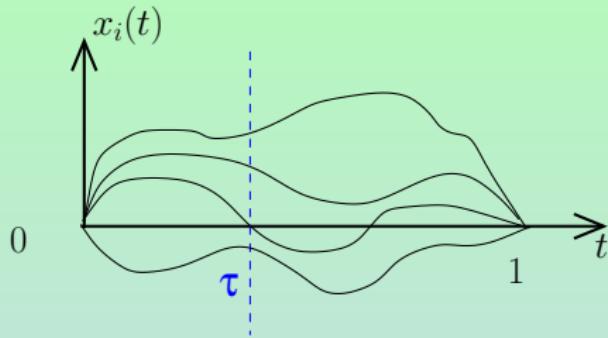
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$$\begin{aligned} \langle \epsilon | e^{-\tau \hat{H}_0} | \mathbf{x} \rangle &= \int_{-\infty}^{\infty} dk_1 \cdots \int_{-\infty}^{\infty} dk_N \left( \det_{1 \leq m, n \leq N} e^{-ik_m \epsilon_n} \right) \left( \det_{1 \leq m, n \leq N} e^{ik_m x_n} \right) \\ &\times \exp \left[ -\frac{1}{2} (k_1^2 + k_2^2 + \cdots + k_N^2) \tau \right] \end{aligned}$$

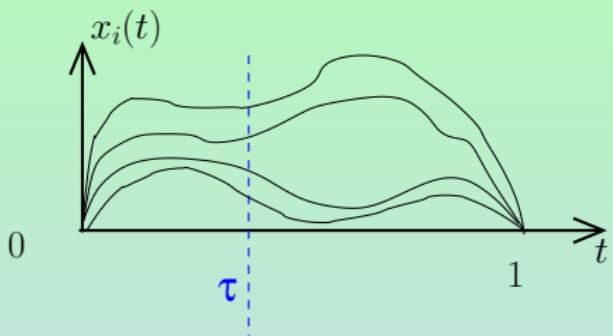
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$$P_{\text{joint}}(\mathbf{x}, \tau) = Z_N^{-1} \sigma(\tau)^{-N^2} \prod_{i < j=1}^N (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^N x_i^2}, \quad \sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

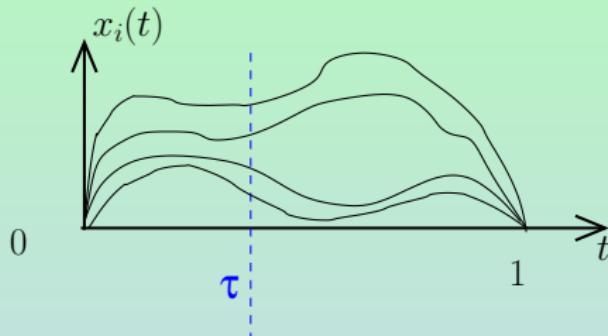
# Joint probability for watermelons with a wall



$$P_{\text{joint}}(\mathbf{x}, \tau) = \lim_{\epsilon_i \rightarrow 0} \frac{\langle \epsilon | e^{-\tau \hat{H}_1} | \mathbf{x} \rangle \langle \mathbf{x} | e^{-(1-\tau) \hat{H}_1} | \epsilon \rangle}{\langle \epsilon | e^{-\hat{H}_1} | \epsilon \rangle}, \quad \hat{H}_1 = - \sum_{i=1}^N \frac{1}{2} \frac{\partial^2}{\partial x_i^2} + V(x_i)$$

$$V(x) = \begin{cases} 0, & x > 0 \\ \infty, & x < 0 \end{cases}$$

# Joint probability for watermelons with a wall



$$P_{\text{joint}}(\mathbf{x}, \tau) = Z_N'^{-1} \sigma(\tau)^{-N(2N+1)} \prod_{i=1}^N x_i^2 \prod_{1 \leq i < j \leq N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

# Outline

1 Vicious walkers and random matrices

2 Path integral approach to vicious walkers problem

- Transition probability
- A regularization procedure for watermelons
- Joint probability distribution for watermelons

3 Distribution of the maximal height

- Watermelons without wall
- Watermelons with a wall

4 Connection with Yang-Mills theory on the sphere

- Partition function of 2d Yang-Mills theory
- Large  $N$  limit and Tracy-Widom distribution

5 Conclusion

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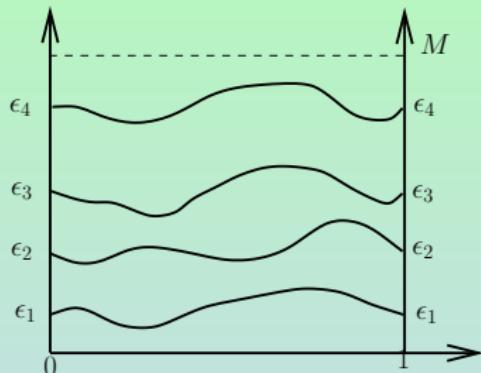
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# Distribution of the maximal height : without wall

- Cumulative distribution of the maximal height

$$F_N(M) = \Pr [x_N(\tau) \leq M, \forall 0 \leq \tau \leq 1]$$



- Path integral for free fermions

$$F_N(M) = \lim_{\epsilon_i \rightarrow 0} \frac{\mathcal{N}(\epsilon, M)}{\mathcal{N}(\epsilon, M \rightarrow \infty)}, \quad \mathcal{N}(\epsilon, M) = \langle \epsilon | e^{-\hat{H}_M} | \epsilon \rangle$$

$$\hat{H}_M = \sum_{i=1}^N \frac{-1}{2} \frac{\partial^2}{\partial x_i^2} + V_M(x_i), \quad V_M(x) = \begin{cases} 0, & x < M \\ \infty, & x > M \end{cases}$$

# Distribution of the maximal height : without wall

- After some algebra...

$$F_N(M) = \frac{B_N}{M^{N^2}} \int_0^\infty dy_1 \cdots \int_0^\infty dy_N \exp \left[ -\frac{y^2}{2M^2} \right] \left( \det_{1 \leq i,j \leq N} y_i^{j-1} \cos \left( y_i + j \frac{\pi}{2} \right) \right)^2$$

and manipulations

$$F_N(M) = \frac{2^{-\binom{N}{2}}}{\prod_{j=0}^{N-1} j!} \det_{1 \leq i,j \leq N} \left[ (-1)^{i-1} H_{i+j-2}(0) - H_{i+j-2}(\sqrt{2}M) e^{-2M^2} \right]$$

where  $H_n(M) \equiv$  Hermite Polynomials

see also T. Feierl '08

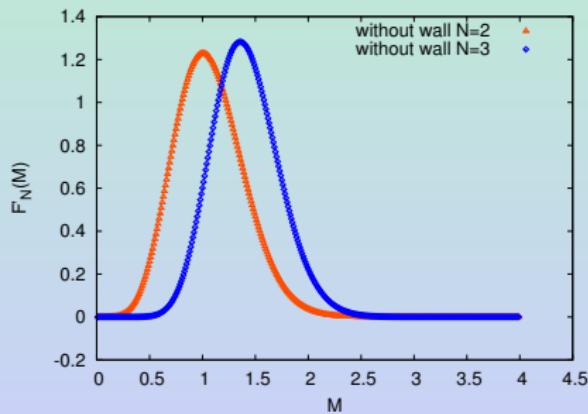
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- Shape and asymptotic behavior

$$F_N(M) \propto M^{N^2+N}, \quad M \rightarrow 0$$

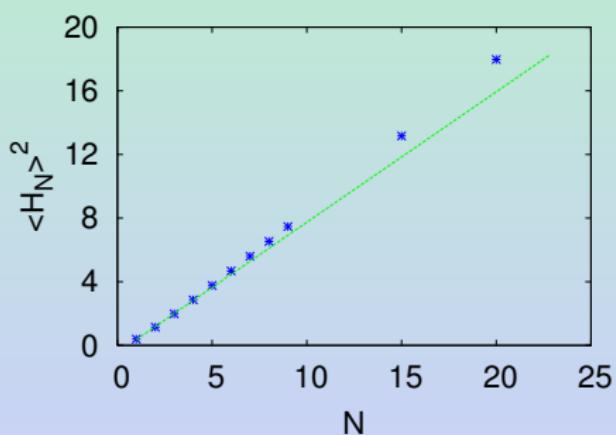
$$1 - F_N(M) \sim e^{-2M^2}, \quad M \rightarrow \infty$$



# Comparison with numerics by Bonichon & Mosbah

Numerical estimate by Bonichon & Mosbah

$$\langle H_N \rangle_{\text{num}}^2 \sim 0.82N$$



Exact behavior at large  $N$

$$\langle H_N \rangle^2 \sim N$$

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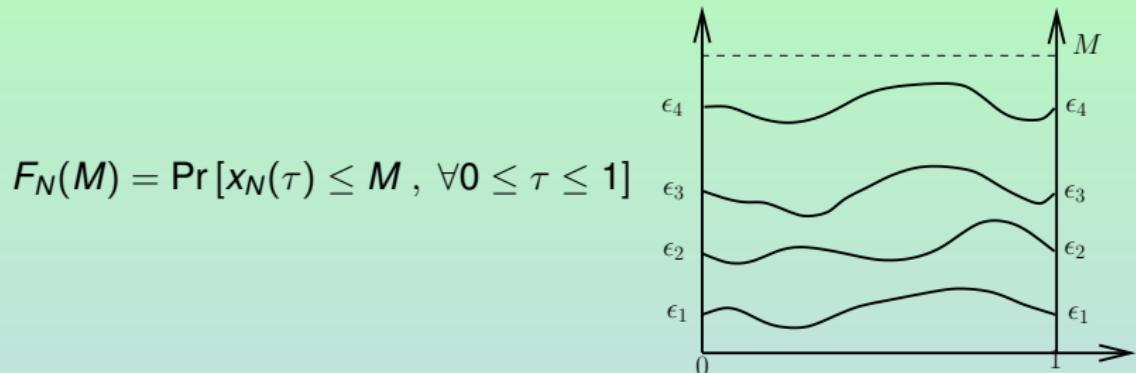
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# Distribution of the maximal height : with a wall

- Cumulative distribution of the maximal height



- Path integral for free fermions

$$F_N(M) = \lim_{\epsilon_i \rightarrow 0} \frac{\mathcal{N}(\epsilon, M)}{\mathcal{N}(\epsilon, M \rightarrow \infty)}, \quad \mathcal{N}(\epsilon, M) = \langle \epsilon | e^{-\hat{H}_M} | \epsilon \rangle$$

$$\hat{H}_M = \sum_{i=1}^N \frac{-1}{2} \frac{\partial^2}{\partial x_i^2} + V_M(x_i), \quad V_M(x) = \begin{cases} 0, & 0 < x < M \\ \infty, & x < 0 \text{ \& } x > M \end{cases}$$

# Distribution of the maximal height : with a wall

- After some algebra...

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1, \dots, n_N=0}^{+\infty} \prod_{i=1}^N n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2} \sum_{i=1}^N n_i^2}$$
$$A_N = \frac{\pi^{2N^2+N}}{2^{N^2-N/2} \prod_{j=0}^{N-1} \Gamma(2+j) \Gamma(\frac{3}{2}+j)} \text{ cf Selberg integral}$$

# Distribution of the maximal height : with a wall

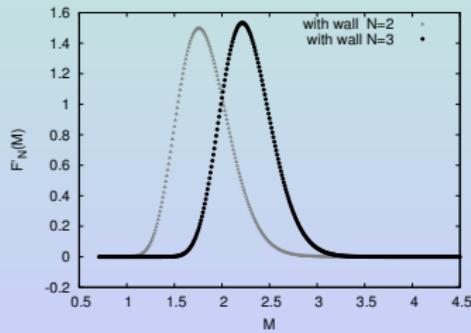
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- Shape and asymptotic behavior

$$F_N(M) \sim \frac{\alpha_N}{M^{2N^2+N}} e^{-\frac{\pi^2}{12M^2} N(N+1)(2N+1)}, M \rightarrow 0$$

$$1 - F_N(M) \sim e^{-2M^2}, M \rightarrow \infty$$



# Distribution of the maximal height : with a wall

- After some algebra...

$$F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1, \dots, n_N=0}^{+\infty} \prod_{i=1}^N n_i^2 \prod_{1 \leq j < k \leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2} \sum_{i=1}^N n_i^2}$$
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What about the asymptotic behavior of  $F_N(M)$  for  $N \rightarrow \infty$  ?

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# Partition function of YM<sub>2</sub> in 2d

- Partition function of Yang-Mills theory on a 2d manifold  $\mathcal{M}$  with a gauge group  $G$ , described by a gauge field  $A_\mu(x) \equiv A_\mu^a(x) T^a$

$$\mathcal{Z}_{\mathcal{M}} = \int [D A_\mu] e^{-\frac{1}{4\lambda^2} \int \text{Tr}[F^{\mu\nu} F_{\mu\nu}] \sqrt{g} d^2x}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

Ex:  $G \equiv SU(2)$  : electro-weak interaction,  $G \equiv SU(3)$  : chromodynamics

# Partition function of YM<sub>2</sub> in 2d

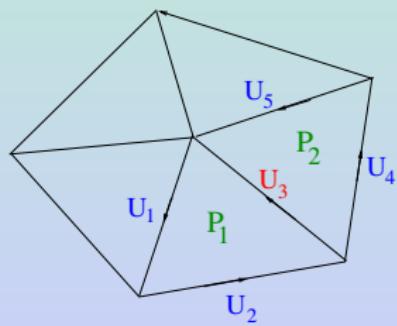
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- Regularization on the lattice, e.g.  $G = U(N)$

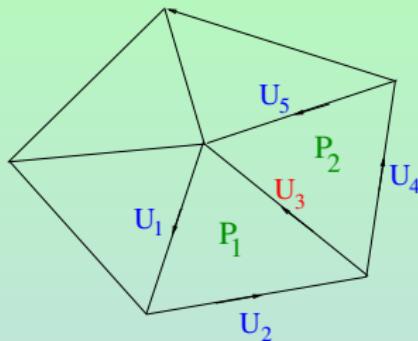
$$\mathcal{Z}_{\mathcal{M}} = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$
$$U_P = \prod_{L \in \text{plaquette}} U_L$$



# Heat-kernel action

$$\mathcal{Z}_M = \int \prod_L dU_L \prod_{\text{plaquettes}} Z_P[U_P]$$

$$U_P = \prod_{L \in \text{plaquette}} U_L$$



- A common choice : Wilson's action

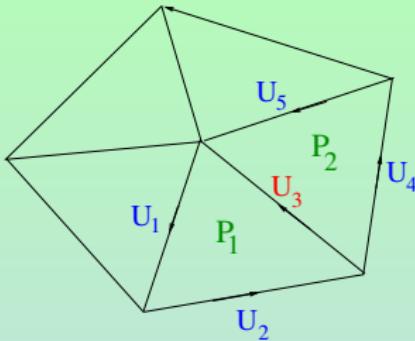
Wilson'74

$$Z_P(U_P) = \exp \left[ bN \text{Tr}(U_P + U_P^\dagger) \right]$$

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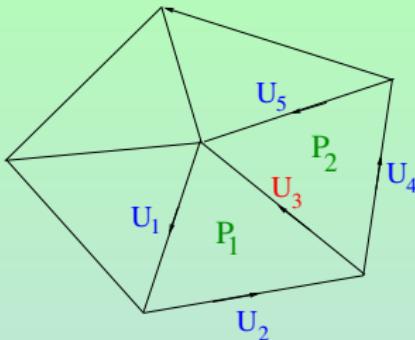
- Alternative choice : invariance under decimation  $\Rightarrow$  Migdal's recursion relation

$$\int dU_3 Z_{P_1}(U_1 U_2 U_3) Z_{P_2}(U_4 U_5 U_3^\dagger) = Z'_{P_1+P_2}(U_1 U_2 U_4 U_5)$$

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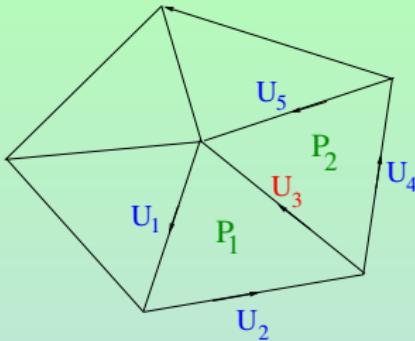
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$$Z_P = \sum_R d_R \chi_R(U_P) \exp \left[ -\frac{A_P}{2N} C_2(R) \right]$$

Migdal'75, Rusakov'90

# Partition function of YM<sub>2</sub> on the sphere

- Exact formula for the partition function computed with the heat-kernel action

$$\mathcal{Z}_{\mathcal{M}} = \sum_R d_R^2 \exp \left[ -\frac{A}{2N} C_2(R) \right]$$

- Irreducible representations  $R$  of  $G$  are labelled by the lengths of the Young diagrams:

- If  $G = \mathrm{U}(N)$

$$\mathcal{Z}_{\mathcal{M}} = c_N e^{-A \frac{N^2-1}{24}} \sum_{n_1, \dots, n_N=0}^{\infty} \prod_{i < j} (n_i - n_j)^2 e^{-\frac{A}{2N} \sum_{j=1}^N n_j^2}$$

- If  $G = \mathrm{Sp}(2N)$

$$\mathcal{Z}_{\mathcal{M}} = \hat{c}_N e^{A(N+\frac{1}{2}) \frac{N+1}{12}} \sum_{n_1, \dots, n_N=0}^{\infty} \left( \prod_{j=1}^N n_j^2 \right) \prod_{i < j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}$$

# Correspondence between YM<sub>2</sub> on the sphere and watermelons

- Partition function of YM<sub>2</sub> on the sphere with gauge group Sp(2N)

$$\mathcal{Z}_M = \mathcal{Z}(A; \text{Sp}(2N))$$

$$\mathcal{Z}(A; \text{Sp}(2N)) = \hat{c}_N e^{A(N+\frac{1}{2})\frac{N+1}{12}} \sum_{n_1, \dots, n_N=0}^{\infty} \left( \prod_{j=1}^N n_j^2 \right) \prod_{i < j} (n_i^2 - n_j^2)^2 e^{-\frac{A}{4N} \sum_{j=1}^N n_j^2}$$

- Cumulative distribution of the maximal height of watermelons with a wall

$$F_p(M) = \frac{A_p}{M^{2p^2+p}} \sum_{n_1, \dots, n_p=0}^{+\infty} \left( \prod_{j=1}^p n_j^2 \right) \prod_{i < j} (n_i^2 - n_j^2)^2 e^{-\frac{\pi^2}{2M^2} \sum_{j=1}^p n_j^2}$$
$$\propto \mathcal{Z}\left(A = \frac{2\pi^2 N}{M^2}; \text{Sp}(2N)\right)$$

P. J. Forrester, S. N. Majumdar, G.S. '10

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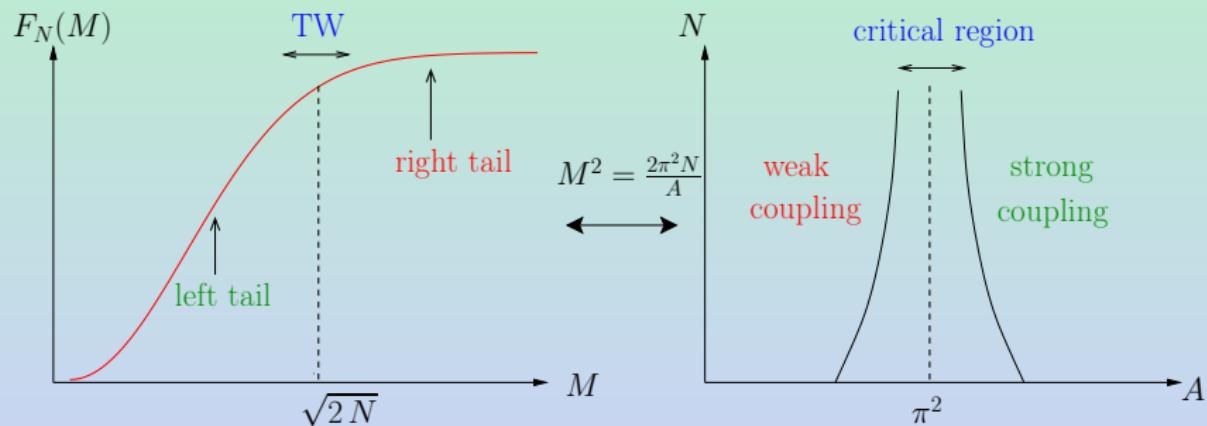
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# Large $N$ limit of YM<sub>2</sub> and consequences for $F_N(M)$

- Weak-strong coupling transition in YM<sub>2</sub> Durhuus-Olesen '81,  
Douglas-Kazakov'93



# Large $N$ limit of YM<sub>2</sub> and consequences for $F_N(M)$

- In the critical regime, "double-scaling limit", the method of orthogonal polynomials (Gross-Matysin '94, Crescimanno-Naculich-Schnitzer '96) shows

$$\frac{d^2}{dt^2} \log F_N\left(\sqrt{2N}(1 + t/(2^{7/3}N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t) + q'(t)\right)$$
$$q''(t) = 2q^3(t) + tq(t), \quad q(t) \sim \text{Ai}(t), \quad t \rightarrow \infty$$

i.e.

$$F_p(M) \rightarrow \mathcal{F}_1\left(2^{11/6}p^{1/6} \left|M - \sqrt{2p}\right|\right)$$
$$\mathcal{F}_1(t) = \exp\left(-\frac{1}{2} \int_t^\infty ((s-t)q^2(s) - q(s)) ds\right)$$
$$\equiv \text{Tracy-Widom distribution for } \beta = 1$$

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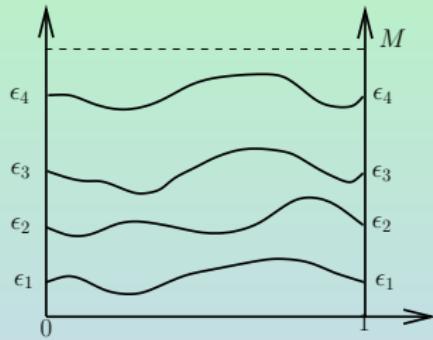
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- Also interesting results for large deviations

# What about other gauge groups ?

- Ratio of reunion probabilities for  $N$  vicious walkers on the segment  $[0, M]$  with absorbing boundary conditions



$$F_N(M) = \text{Proba}[x_N(\tau) \leq M, \forall \tau \in [0, 1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

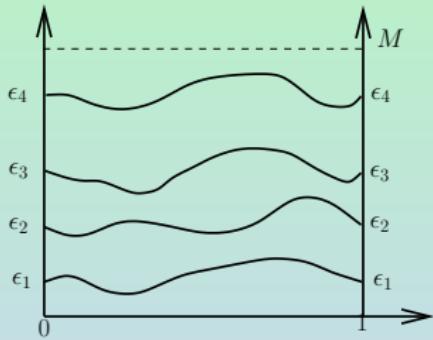
$R_M(1) \equiv$  proba. that  $N$  walkers return to their initial positions at  $\tau = 1$

Related to YM<sub>2</sub> on the sphere with gauge group Sp( $2N$ )

$$F_N(M) \propto \mathcal{Z} \left( A = \frac{2\pi^2 N}{M^2}; \text{Sp}(2N) \right)$$

# What about other gauge groups ?

- Ratio of reunion probabilities for  $N$  vicious walkers on the segment  $[0, M]$  with **periodic boundary conditions**



$$F_N(M) = \text{Proba}[x_N(\tau) \leq M, \forall \tau \in [0, 1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

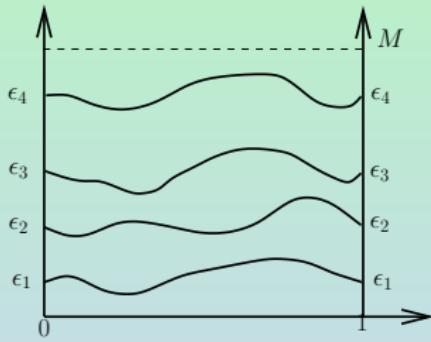
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Related to YM<sub>2</sub> on the sphere with **gauge group U(N)**

$$F_N(M) \propto \mathcal{Z} \left( A = \frac{4\pi^2 N}{M^2}; \text{U}(N) \right)$$

# What about other gauge groups ?

- Ratio of reunion probabilities for  $N$  vicious walkers on the segment  $[0, M]$  with reflecting boundary conditions



$$F_N(M) = \text{Proba}[x_N(\tau) \leq M, \forall \tau \in [0, 1]]$$

$$F_N(M) = \frac{R_M(1)}{R_\infty(1)}$$

$R_M(1) \equiv$  proba. that  $N$  walkers return to their initial positions at  $\tau = 1$

Related to YM<sub>2</sub> on the sphere with gauge group SO(2N)

$$F_N(M) \propto \mathcal{Z} \left( A = \frac{4\pi^2 N}{M^2}; \text{SO}(2N) \right)$$

# Outline

1 Vicious walkers and random matrices

2 Path integral approach to vicious walkers problem

- Transition probability
- A regularization procedure for watermelons
- Joint probability distribution for watermelons

3 Distribution of the maximal height

- Watermelons without wall
- Watermelons with a wall

4 Connection with Yang-Mills theory on the sphere

- Partition function of 2d Yang-Mills theory
- Large  $N$  limit and Tracy-Widom distribution

5 Conclusion

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- Exact results for  $\langle H_N \rangle$  for large  $N$
- Exact result for distribution of the maximal height  $H_N$  using path integral techniques
- Connection with the partition function of  $YM_2$  on the sphere with gauge group  $Sp(2N)$

$$F_N(M) \rightarrow \mathcal{F}_1 \left( 2^{11/6} N^{1/6} \left| M - \sqrt{2N} \right| \right), \quad N \rightarrow \infty$$

see also Johansson'03

- Relation between boundary conditions in vicious walkers problem and gauge group in the  $YM_2$

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Any deep reason behind this ?