CELLULAR AUTOMATA
&
BRANCHING BALLISTIC ANNIHILATION

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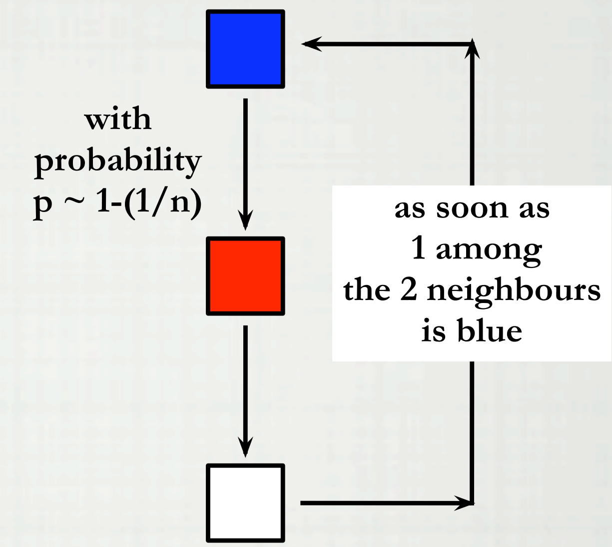
Random geometries, Nordita, 2010

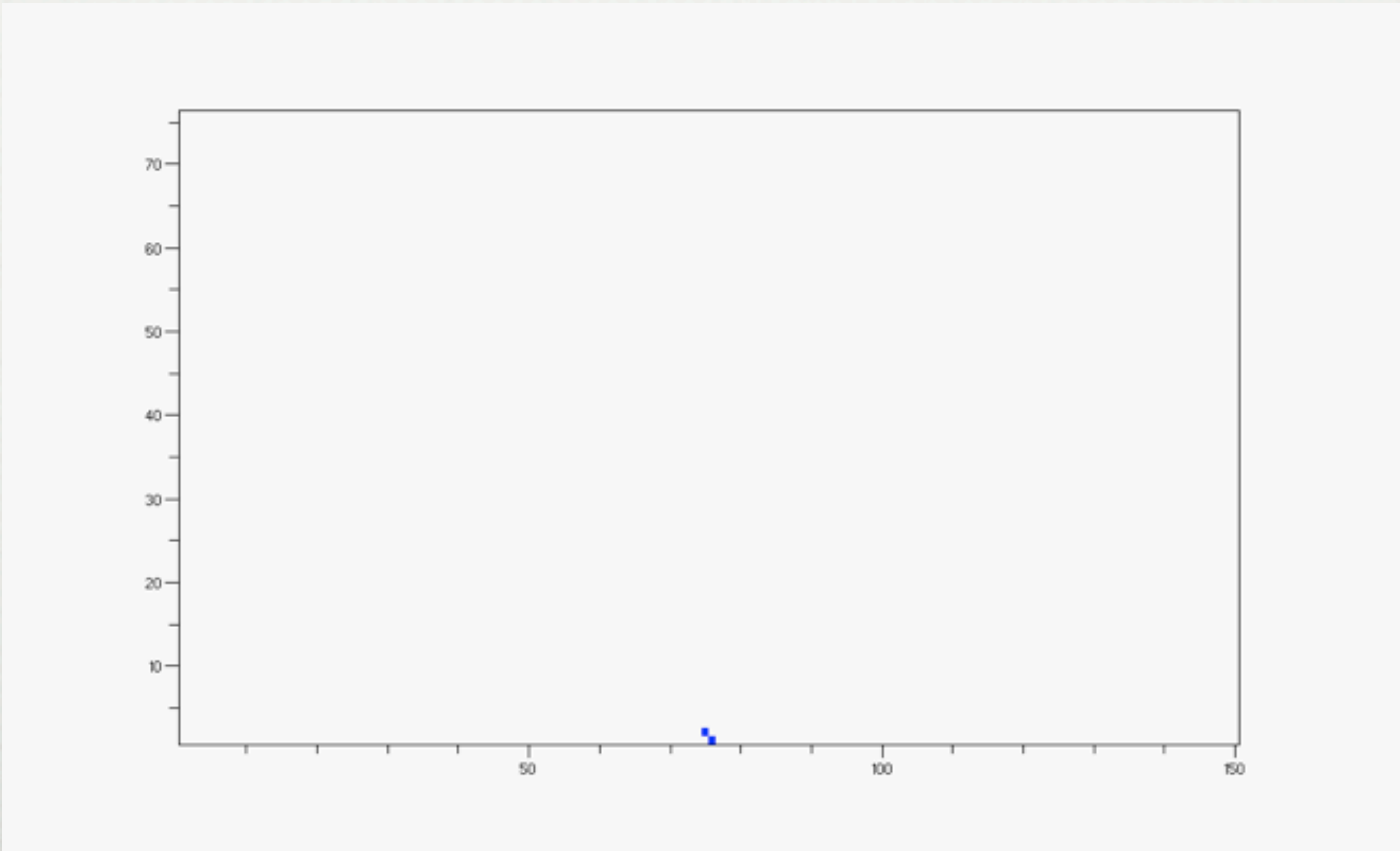
SIMPLE CELLULAR AUTOMATA #2

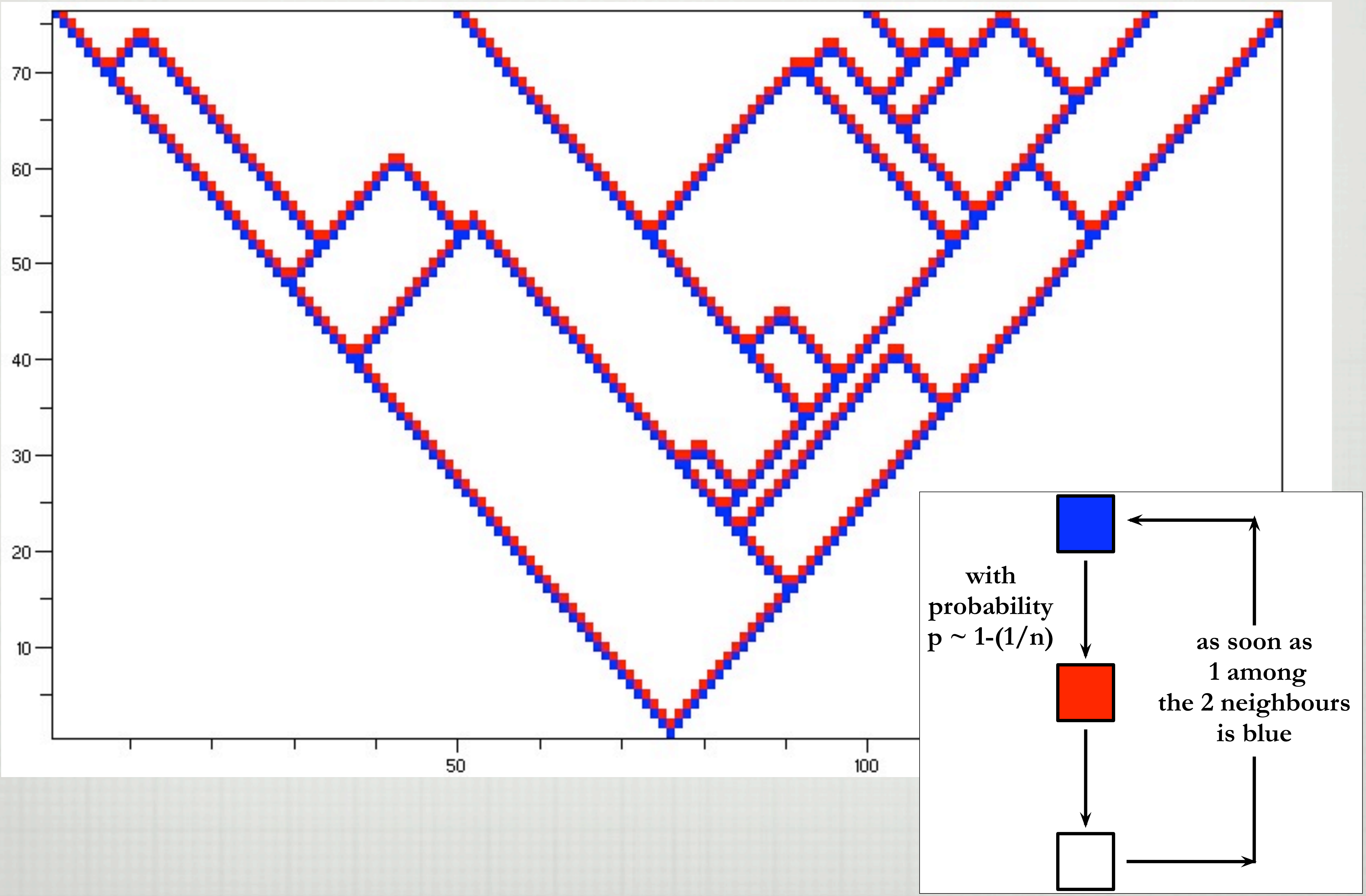
Let us see the set $\mathbb{Z}$ of integers as a set of cells with 3 possible colors

- blue (= infected),
- red (= healing),
- white (= healthy),

evolving according to the following synchronous rule :






70
60
50
40
30
20
10
50
100
with
probability
p ~ 1-(1/n)
as soon as
1 among
the 2 neighbours
is blue

MOTIVATIONS

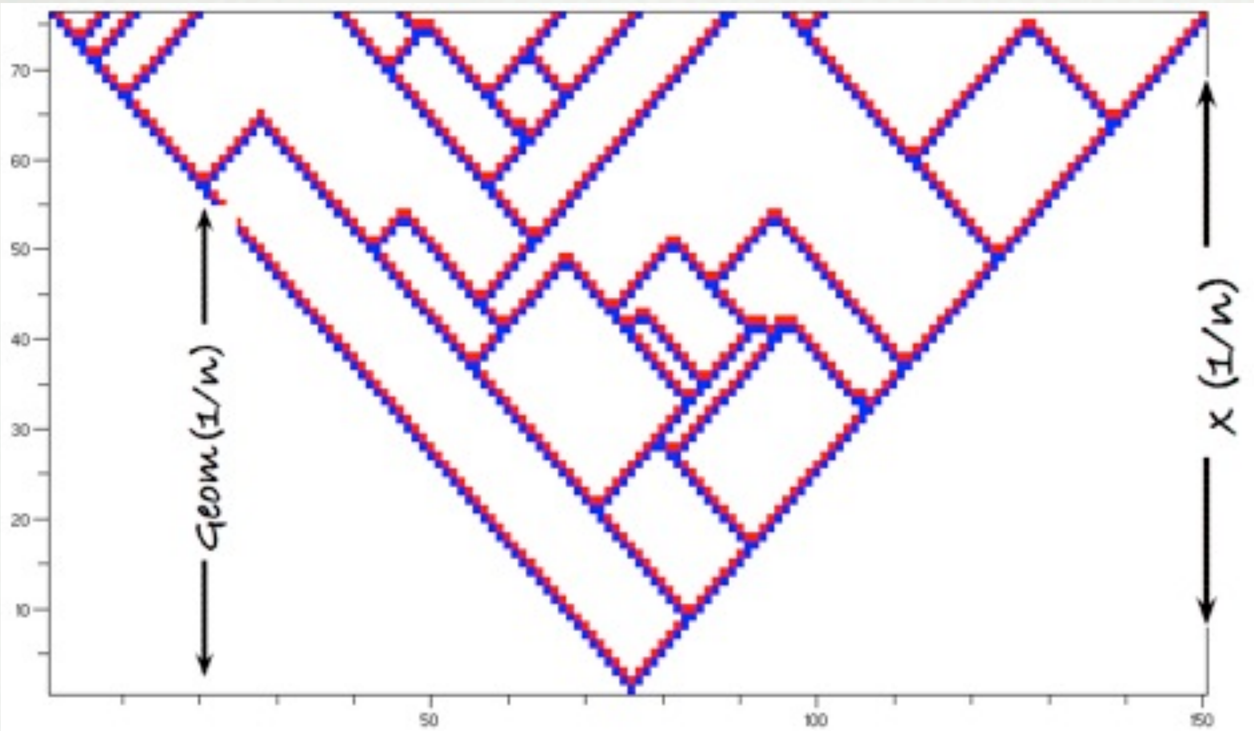
CELLULAR AUTOMATA IN GENERAL

- Computer scientist (Schabanel, Fates, Bournez e.g.) intend to study how to tune local rules to obtain a suitable global behaviour.
- Motivation : parallel computation, specially collaboration between agents (or processors) with a minimal cost devoted to sharing information (global interaction has a higher cost).
- 1-dimensional, 2-dimensional, synchronous, asynchronous automata exhibit very different behaviours : game of life, conflagration, etc ... Despite a small number of simple rules, rich and chaotic or complex behaviour.

THIS MODEL

- Ferrari, Belitsky, Blythe, Cafri, Evans, Cardy, ... use similar models for the modelisation of some chemical reactions, of highway traffic, etc ...

THE ASYMPTOTIC MODEL



- On the real axis, coexist two species of particles :
  - positive particles, that move at speed $+1$,
  - negative particles, that move at speed -1.
- When a collision occurs, it causes the annihilation of the two particles involved (ballistic annihilation).
- Branching : each particle gives birth to a particle of the other species, after an exponential time with parameter 1.
- The state of the system at time t,

$$X_t = (X_t^+, X_t^-),$$

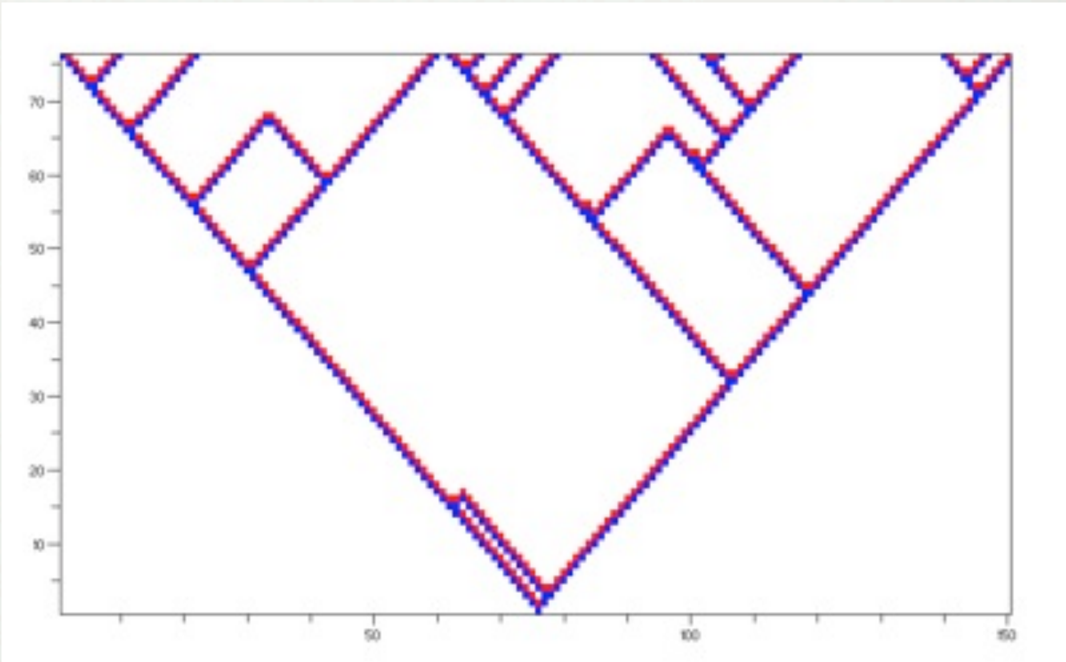
is a couple of (multi)sets.
- Here $X_0 = (\{0\}, \{0\})$.
- Law of X_t ?
- Asymptotic behaviour ?

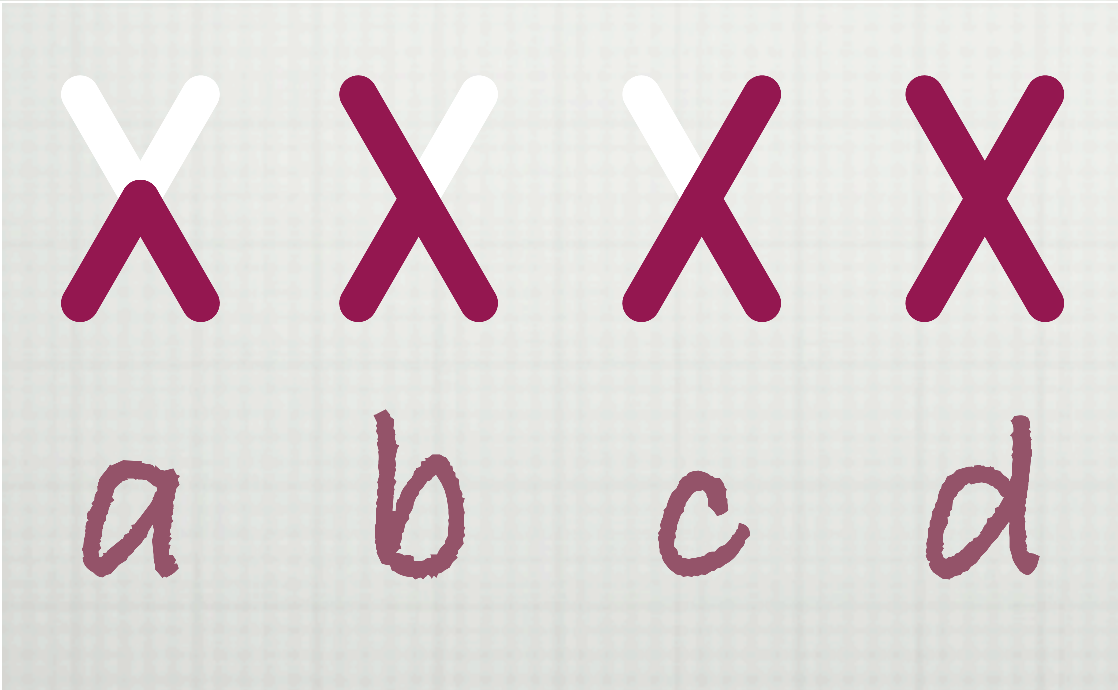
BRANCHING BALLISTIC ANNIHILATION

- Ballistic variation of a model introduced by Cardy & Täuber (1996)
- (1,0,0,0)-variant of a more general (a,b,c,d)-model (Blythe et al. 2000)

$$a+b+c+d=1,$$

in which no birth, but 4 kinds of behaviour are considered after a collision :





- We shall consider 3 cases
  - pure branching: (0,0,0,1),
  - symmetric: (0,0.5,0.5,0),
  - annihilation: (1,0,0,0).

PURE BRANCHING : (0,0,0,1)

Assume the system starts with 1 positive particle at 0, and set :

$$\mathbb{E}\left[\langle X_t, f\rangle\right] = \mathbb{E}\left[\sum_{\xi\in X_t} f(\xi)\right] = \langle \delta_t, f\rangle + \langle \psi(t,\cdot), f\rangle$$

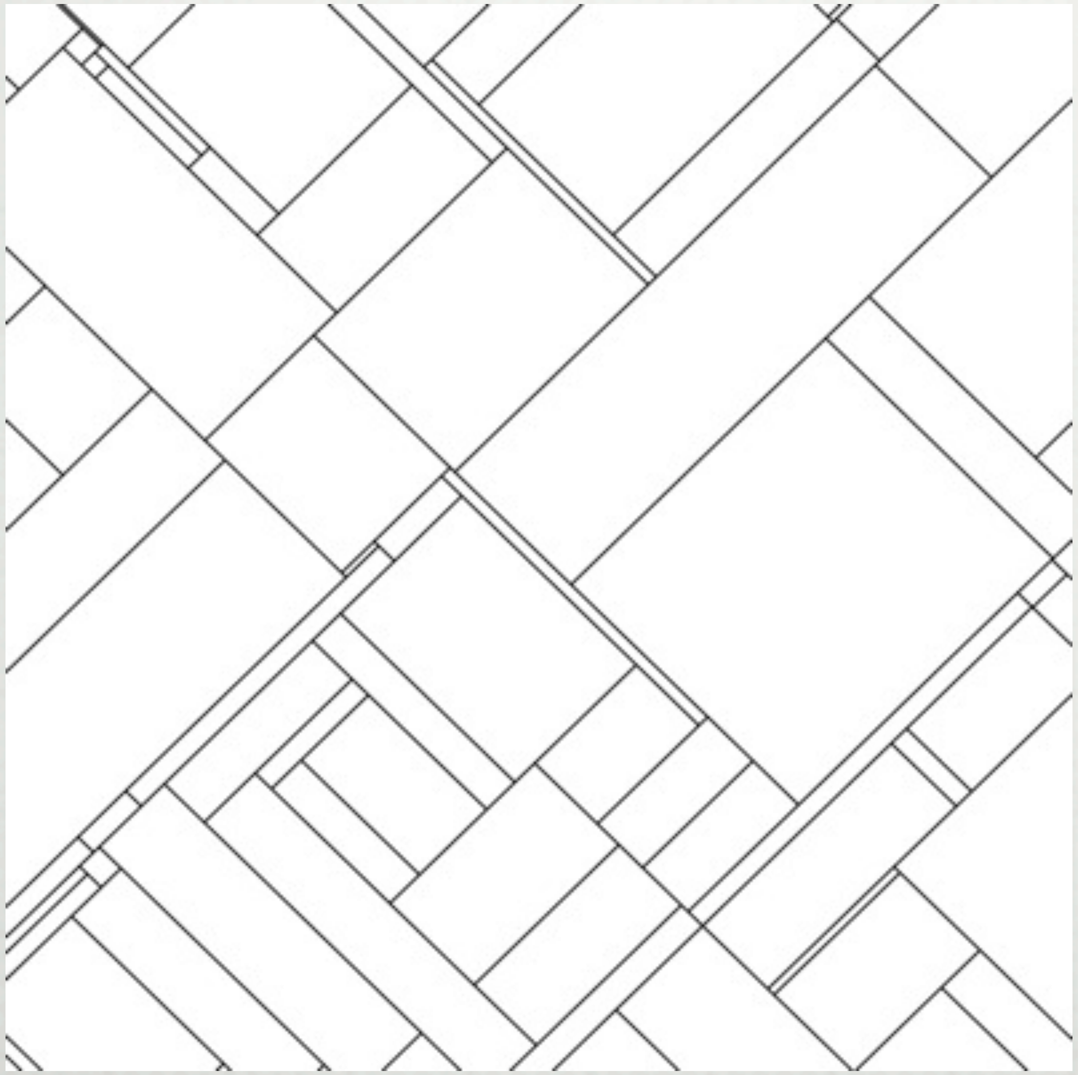
Theorem (M. Krikun).

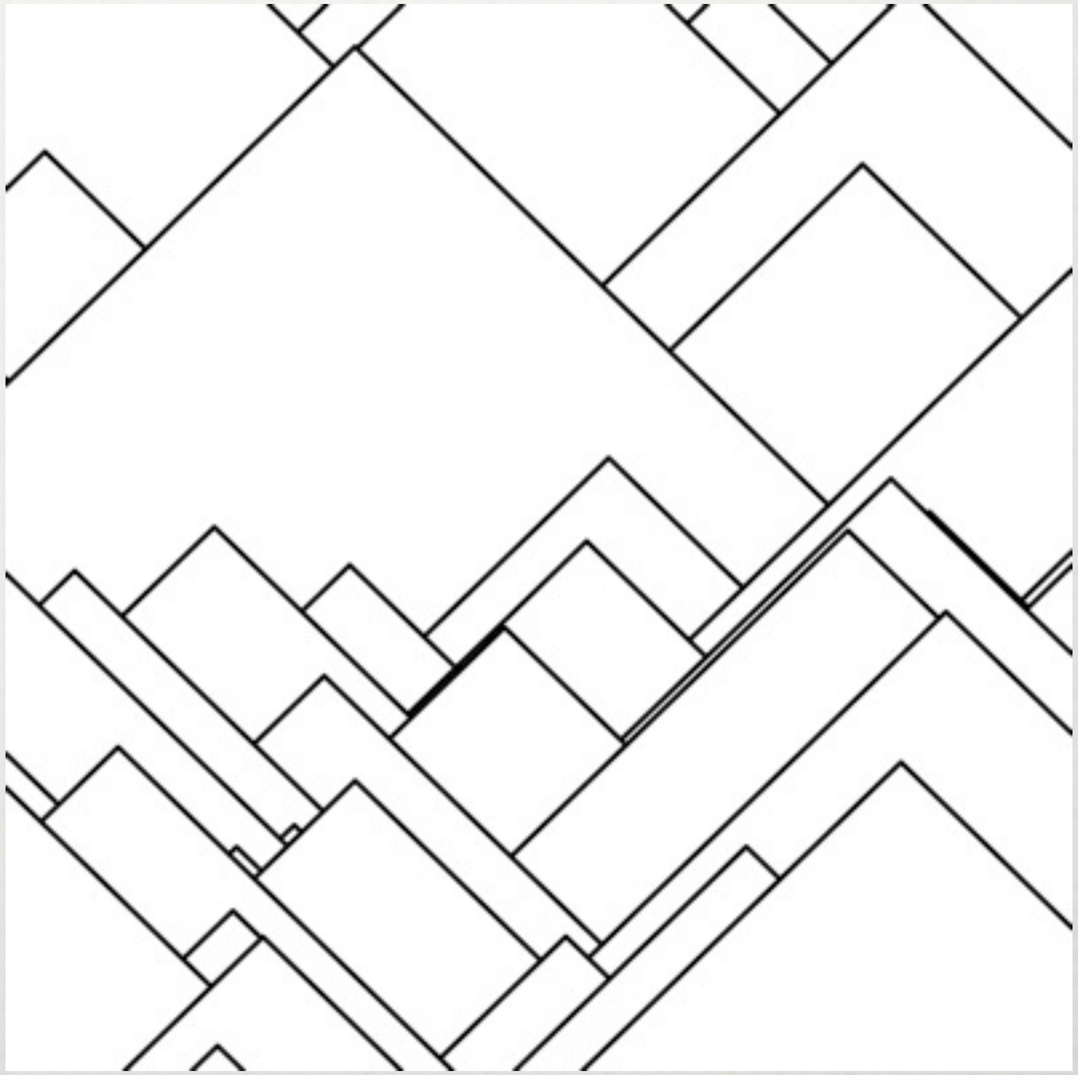
$$\lim_{t\to\infty} \psi(t, z\sqrt{t})\, e^{-t}\sqrt{t} = \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}$$

- The total mass is e^t.
- Most of it is supported by an $O(\sqrt{t})$-wide interval around 0.

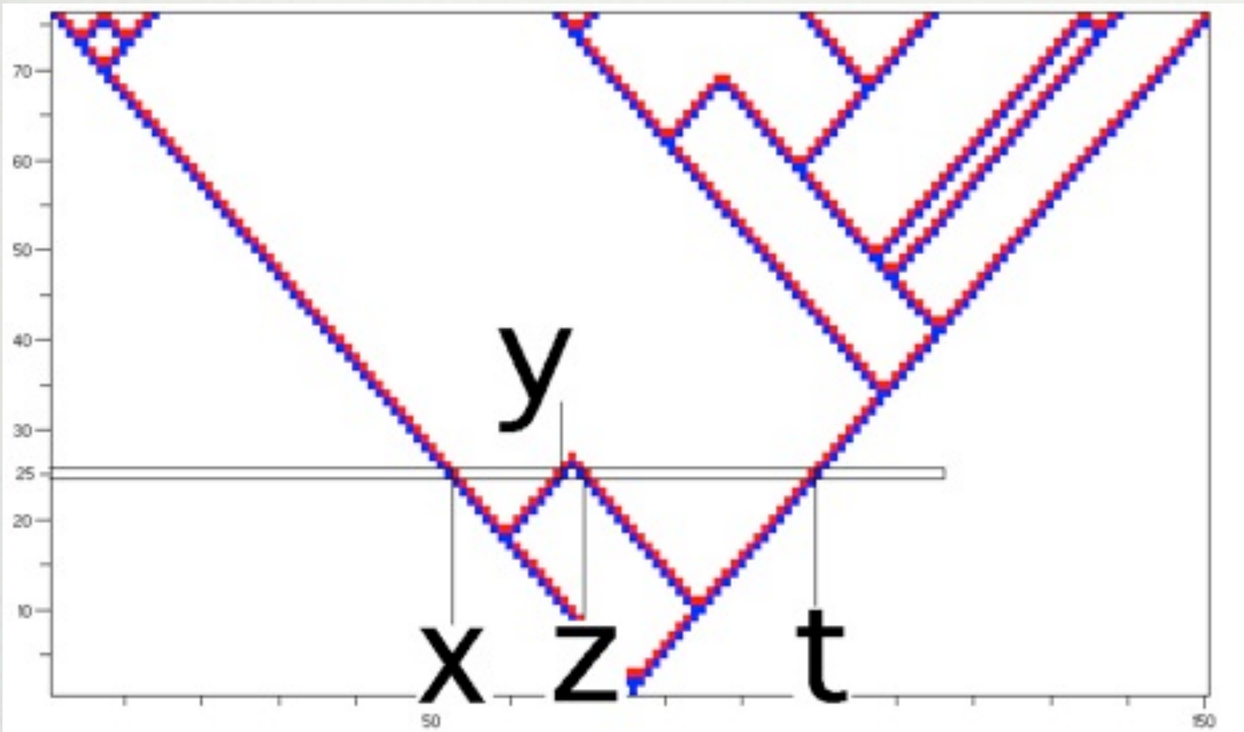
SIMULATIONS

(0,0.5,0.5,0) | (1,0,0,0)





GLOSSARY: POINT PROCESSES ON THE LINE



- multiset = counting measure
  $\pi = \{x,x,x,y,z,z,t\} = 3\,\delta_x + \delta_y + 2\,\delta_z + \delta_t$
- state of the system at time $t (=25)$
  $(\delta_y + \delta_t, \delta_x + \delta_z) = (\pi^+, \pi^-) = X_t$
- notation for the integral against π
  $\langle \pi, \phi \rangle = 3\,\phi(x) + \phi(y) + 2\,\phi(z) + \phi(t)$
  $\langle \pi, 1_A \rangle = \#(\pi \cap A)$

- locally finite: has a finite intersection with any bounded interval.
- good multiset: locally finite and unbounded in both directions.
- point process: random multiset.
- Laplace transform at ϕ of the point process Π:

$$(\mathcal{L}\Pi)(\varphi) = \mathbb{E}\left[e^{\langle \varphi, \Pi \rangle}\right]$$

$$(\mathcal{L}\Pi)(a 1_A) = \mathbb{E}\left[e^{a\,\#(\Pi \cap A)}\right]$$

POISSON POINT PROCESSES

- Π is a Poisson point process with intensity μ if the master formula holds :

$$(\mathcal{L}\Pi)(\varphi) = e^{\int (e^{\varphi}-1)d\mu}$$

$$\begin{aligned}(\mathcal{L}\Pi)(a1_A) &= e^{\mu(A)(e^a-1)} \\ &= \mathbb{E}\left[e^{a\,\#(\Pi\cap A)}\right]\end{aligned}$$

$$\begin{aligned}(\mathcal{L}\Pi)(a1_A + b1_B) &= e^{\mu(A)(e^a-1)+\mu(B)(e^b-1)} \\ &= \mathbb{E}\left[e^{a\,\#(\Pi\cap A)+b\,\#(\Pi\cap B)}\right]\end{aligned}$$

- If $A\cap B=\varnothing$, then $\#(\Pi\cap A)$ and $\#(\Pi\cap B)$ are independent Poisson random variables with respective expectations $\mu(A)$ and $\mu(B)$.

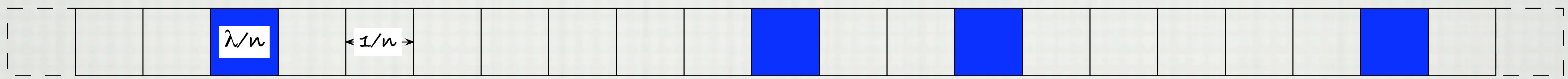
POISSON PROCESSES ON THE LINE

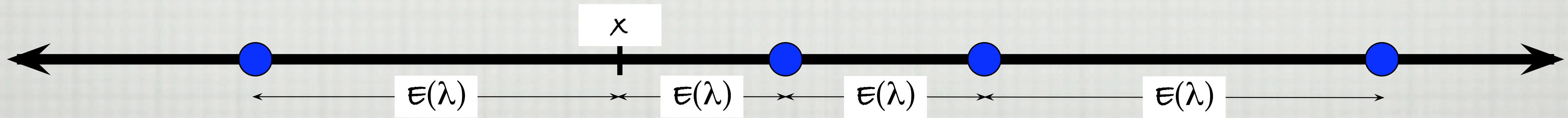
- Π is a Poisson process on the line with intensity $\lambda>0$ if

$$\mu = \lambda \times \text{(Lebesgue measure)}$$

- If $A\cap B=\emptyset$, then $\#(\Pi\cap A)$ and $\#(\Pi\cap B)$ are independent Poisson random variables with respective expectations $\lambda|A|$ and $\lambda|B|$.

- Other descriptions :





THE TILTED PROCESS

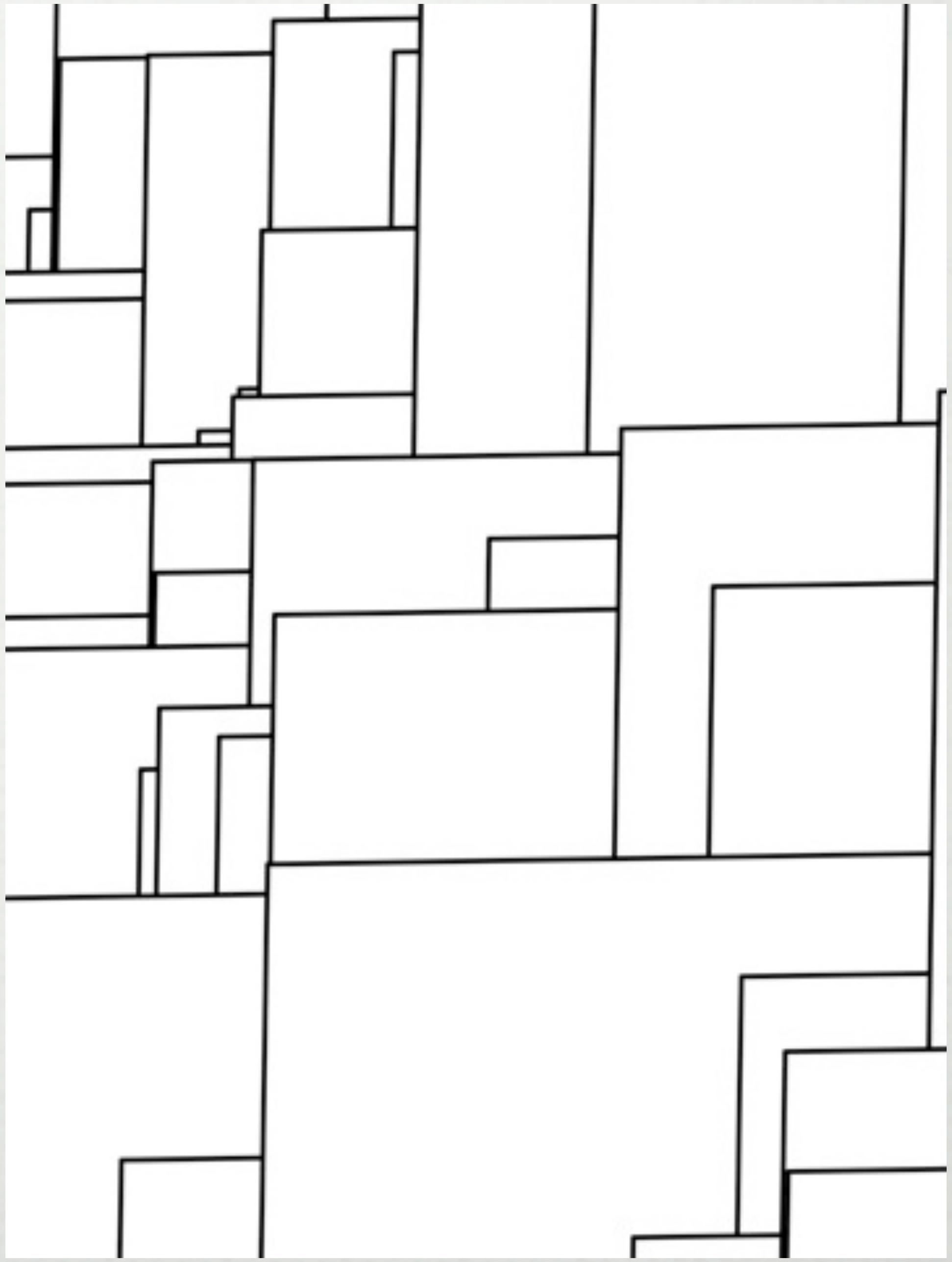
Branching ballistic annihilation
(1,0,0,0)

Start with the line $x+t=a\sqrt{2}$, with a locally finite population X_a of positive particles.

Evolution

- Each particle, with rate $1/\sqrt{2}$, shoots and kills its nearest neighbour on the left ;

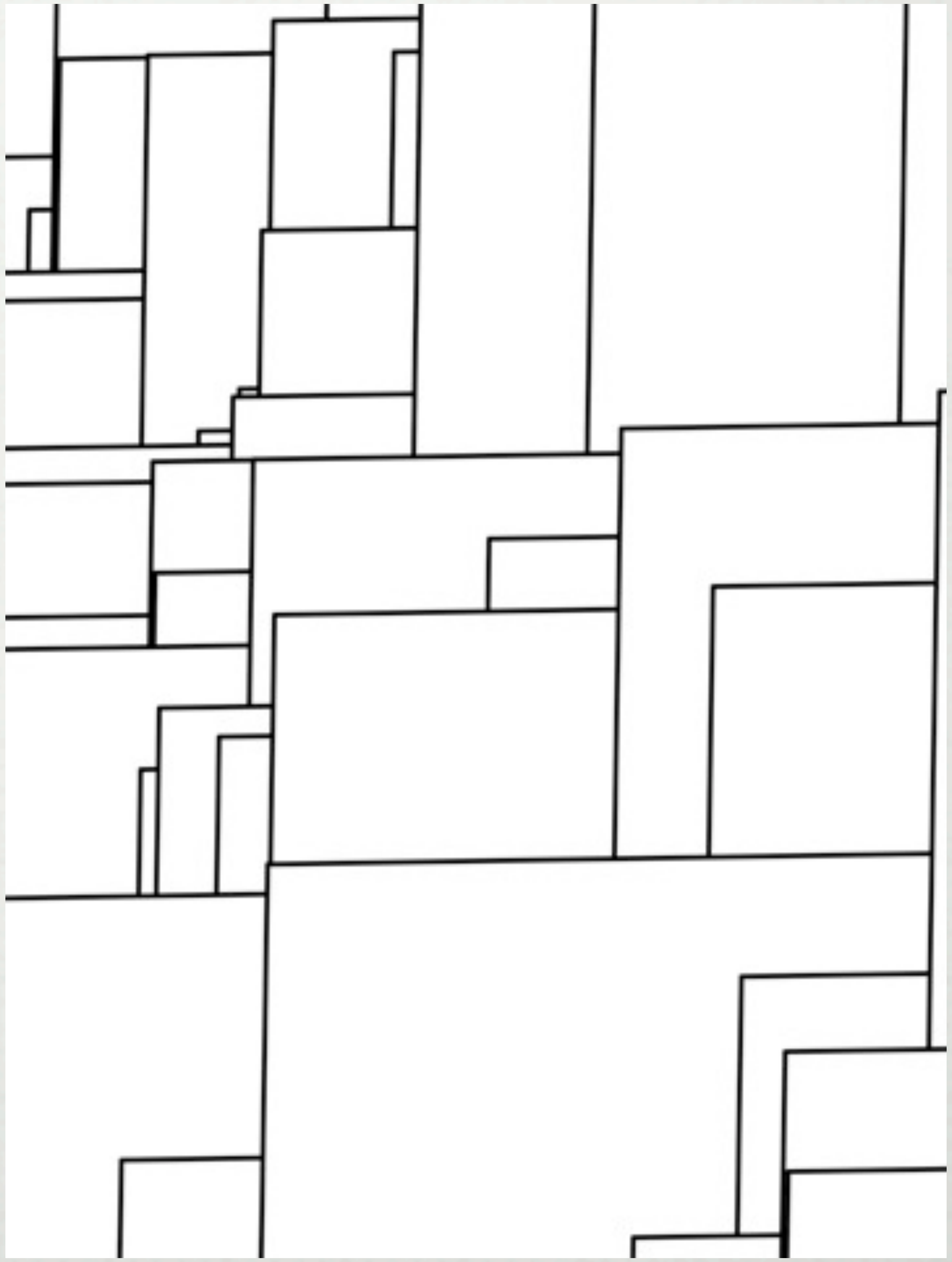
- The interval between the two particles is filled with an independent copy of a PPP($1/\sqrt{2}$).



STATIONARY MEASURE OF TILTED PROCESSES #1

* **Theorem.** For the annihilation case — parameter (1,0,0,0) — **PPP** ($1/\sqrt{2}$) is the **unique stationary probability measure** supported by **good multisets** (infinite in both directions and locally finite) ;

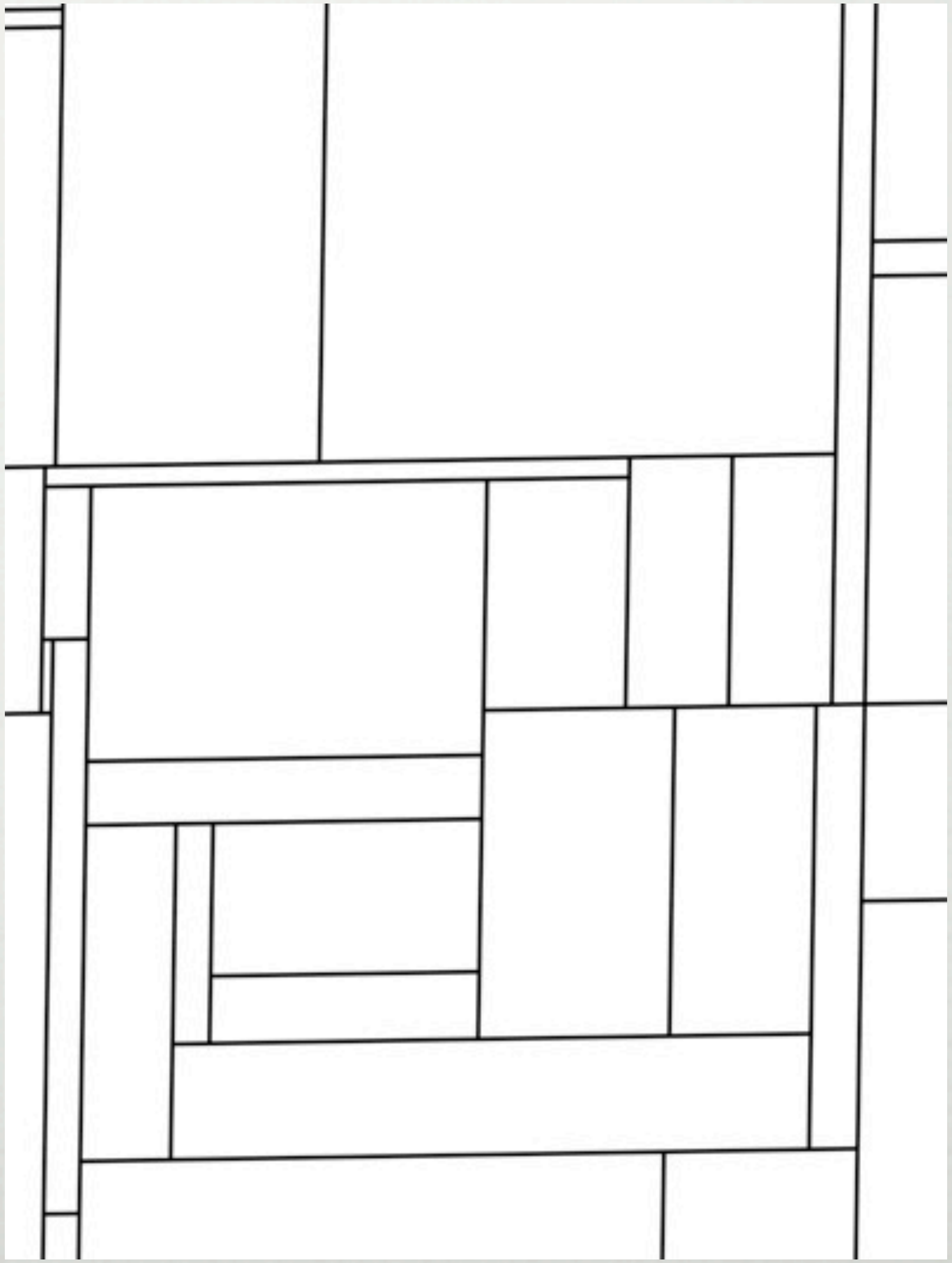
* **Consequence** : one may define a stationary process on the whole plane.



STATIONARY MEASURE OF TILTED PROCESSES #2

✳ **Theorem.** For the symmetric case, PPP($\sqrt{2}$) is the unique stationary probability measure supported by good multisets.

✳ **Consequence** : one may define a stationary process on the whole plane.



SYMMETRIC CASE (0,0.5,0.5,0): SYMMETRY GROUP

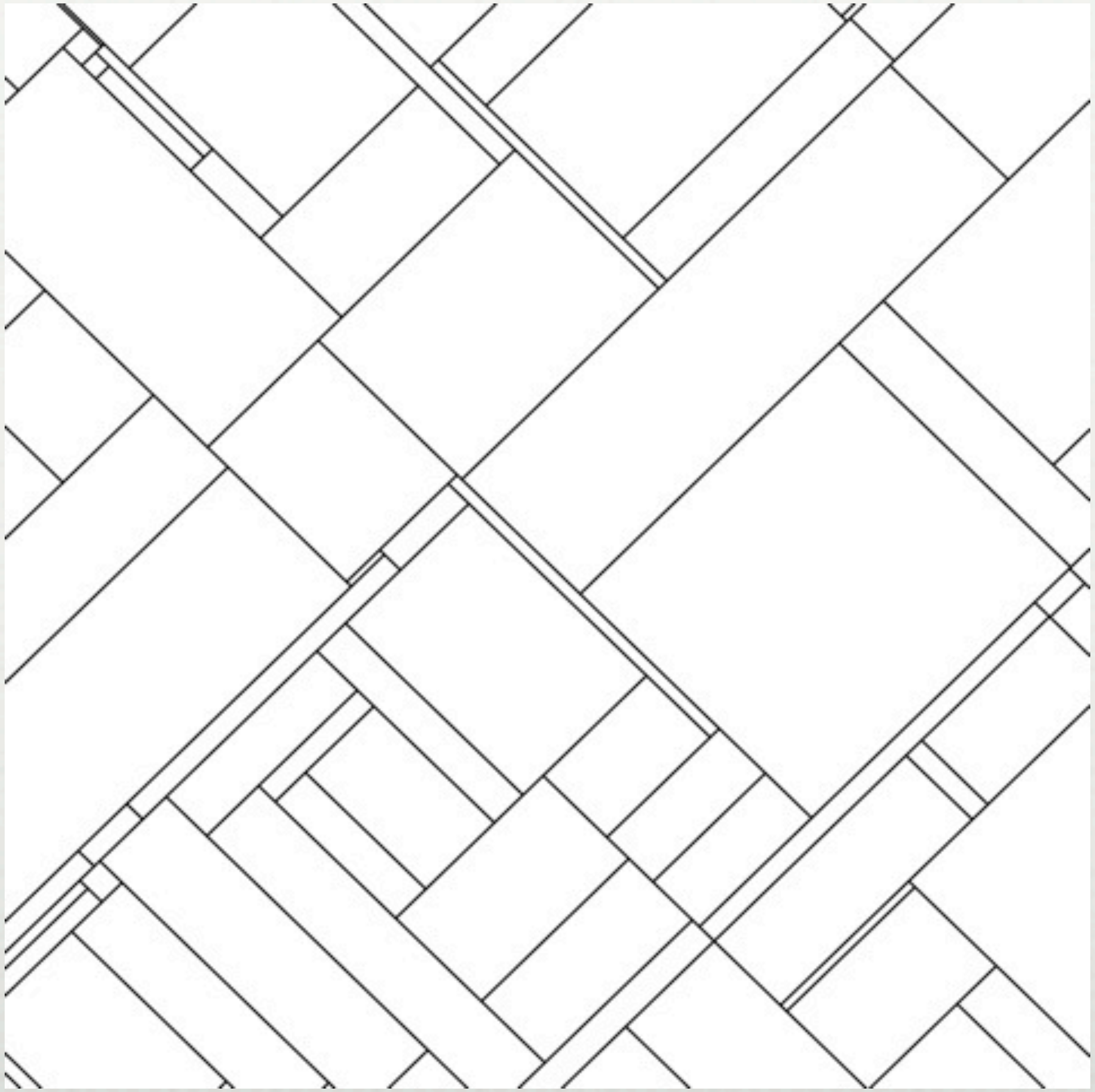
The process, once defined on the whole plane, can be seen as a random tessellation.

Theorem.

- The symmetry group of the random tessellation contains the diedral group of the square and the translations of the plane.
- In particular, the tilted processes (by any angle) are Markov, stationary and reversible.

SYMMETRIC CASE (0,0.5,0.5,0): SYMMETRY GROUP

"Proof" (empiric ...)



SYMMETRIC CASE (0,0.5,0.5,0): STATIONARY DISTRIBUTION

Theorem.

- The stationary distribution of the horizontal process is that of a couple of independent PPP(1).

- The stationary distribution of the a-tilted process is that of a couple of independent PPP() with respective parameters :

  - $|\cos a+\sin a|$ and $|\cos a-\sin a|$.

ANNIHILATION (1,0,0,0): STATIONARY DISTRIBUTION

✳ **Theorem.**

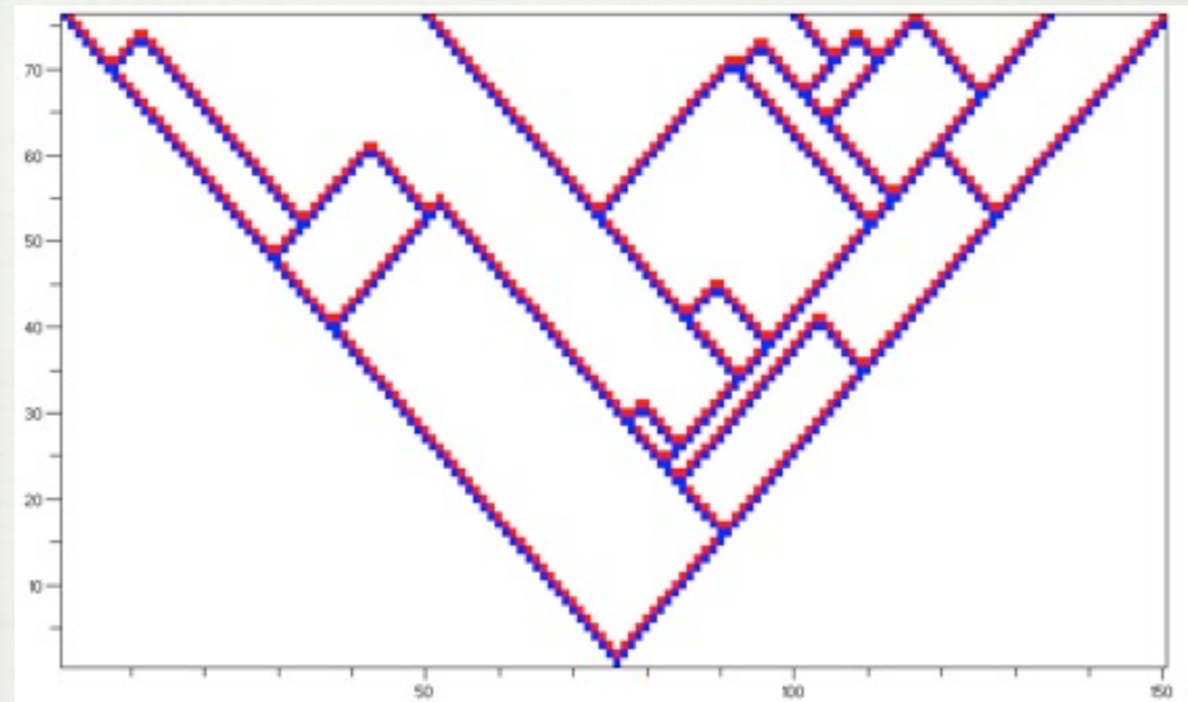
- ☐ The stationary distribution of the horizontal process of positive (resp. negative) particles is a PPP (1/2).

- ☐ But the two PPP are dependent.

✳ **Open problem:** to describe the full stationary distribution, i.e. the joint law of positions of positive and negative particles.

ANNIHILATION (1,0,0,0): BACK TO THE 1ST QUESTION

✳**Theorem.** The stationary distribution of the horizontal process of positive (resp. negative) particles is a PPP(1/2).



✳**Theorem.** The distribution in the quarter-plane is the restriction of the stationary distribution in the plane.

STATIONARY MEASURE : ANALYTIC APPROACH

Measure-valued Markov process :

Let X_s be the set of positions of positive particles on the line $\{x+t=s\sqrt{2}\}$. Assume the initial state $X_0=x$ to be a good multiset. Then, for ϕ positive, continuous, and with compact support,

$$G\left(e^{-\langle\varphi,\cdot\rangle}\right)(x) \ := \ \lim_{s\downarrow 0} \ s^{-1}\left(\mathbb{E}\left[e^{-\langle\varphi,X_s\rangle}\right]-e^{-\langle\varphi,x\rangle}\right)$$

is well defined.

Conjecture : If furthermore, a random multiset X (a.s. good) satisfies, for every ϕ positive, continuous, and with compact support,

$$\mathbb{E}\left[G\left(e^{-\langle\varphi,\cdot\rangle}\right)(X)\right]=0,$$

then the law of X is a stationary measure of the Markov process. Result well known when the state space is locally compact, but here it is not.

SYMMETRIC CASE : SYMMETRY GROUP

Theorem.

- The symmetry group of the random tessellation contains the diedral group of the square and the translations.
- In particular, the tilted processes by any angle are Markov, stationary and reversible*.

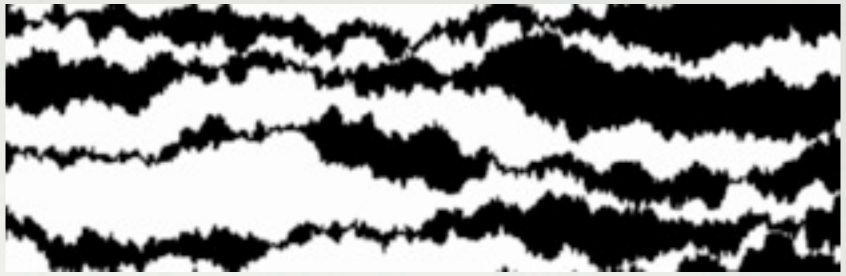
Proof. The generator of the tilted process is given by

$$Ge^{\langle \phi,.\rangle}(\pi) = e^{\langle \phi,\pi\rangle} \sum_{\{y,x\}\subset\pi,\, y<x} 2^{-\#[y,x)\cap\pi} \left(e^{-\langle \phi \mathbf{1}_{(y,x)},\pi\rangle + \mu \int_y^x (e^{\phi(u)}-1)du} - 1\right).$$

One has to check that for any symmetry T of the real axis (of the form T(x)=a-x) :

$$Ge^{\langle \phi\circ T,.\rangle}(\pi) = Ge^{\langle \phi,.\rangle} \circ T(\pi).$$

RELATED RESULTS



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THANK YOU !

CONSTRUCTION OF THE PROCESS

Theorem . For each 1-Lipshitz fonction f and for each locally finite set of points on

$$\partial f := \{(x,t) : t = f(x)\},$$

the process in the half plane

$$f_+ := \{(x,t) : t > f(x)\},$$

is well defined.

Idea :

- start with a pure branching process -> infinite trees

- cut wisely chosen branches

- the only problem arises when $f(x)$ has slope ± 1 on an infinite interval : the past of a point of the half plane can contain infinitely many branching points.

MARKOV PROPERTY

- **Property.** Consider the process built from a configuration on a border ∂g. Then, for a border ∂f (s.t. $g<f$), the parts of the process below and under ∂f are independent, conditionally, given the process on ∂f.

- **Conjecture** : true for the 2 components inside and outside a Jordan curve, conditionally, given the process on the curve.