

Correlations in random trees.

Piotr Białas

Faculty of Physics, Astronomy and Applied Computer Science
Jagellonian University

24 November 2010

P.B., A.K. Oleś, Phys. Rev. **E77** (2008) 36124, Phys. Rev. **E81** (2010) 41136.

Phase transition

- Two scales

$$L \sim V^{\frac{1}{d}}, \quad \xi \quad \langle s(0)s(r) \rangle_{conn.} \sim \frac{1}{r^{\eta-1}} f\left(\frac{r}{\xi}\right)$$

- Thermodynamic limit

$$L \longrightarrow \infty$$

- Phase transition

$$\xi \longrightarrow \infty$$

$$\chi \propto \sum_r \langle s(0)s(r) \rangle_{conn.} \sim \sum_r \frac{1}{r^{\eta-1}} f\left(\frac{r}{\xi}\right) \sim \xi^{2-\eta} \int dx \frac{1}{x^{\eta-1}} f(x)$$

Random Graphs

- Random Graphs

$$G \in \mathcal{G}, \quad P(G)$$

$$P(G) = Z^{-1} \rho(G), \quad Z = \sum_{G \in \mathcal{G}} \rho(G)$$

$$\rho(G) = \prod_{i \in G} \omega_{q_i}$$

- Ising model

$$P(G) = Z^{-1} \rho(C), \quad Z = \sum_C \rho(C)$$

$$\rho(C) = \prod_{\langle i,j \rangle} e^{\beta s_i s_j}$$

Plan

- Correlations.
- Trees.
- Connected graphs.

Correlations

$$\pi_{qr}(I) = \frac{\left\langle \sum_{i,j \in G} \delta_{q_i,q} \delta_{q_j,r} \delta_{I,d(i,j)} \right\rangle}{\left\langle \sum_{i,j \in G} \delta_{I,d(i,j)} \right\rangle} \quad \left\langle \frac{\sum_{i,j \in G} \delta_{q_i,q} \delta_{q_j,r} \delta_{I,d(i,j)}}{\sum_{i,j \in G} \delta_{I,d(i,j)}} \right\rangle$$

$$\pi_q(I) = \sum_r \pi_{qr}(I) = \frac{\left\langle \sum_{i,j \in G} \delta_{q_i,q} \delta_{I,d(i,j)} \right\rangle}{\left\langle \sum_{i,j \in G} \delta_{I,d(i,j)} \right\rangle}$$

$$\pi_{qr}^{conn.}(I) = \pi_{qr}(I) - \pi_q(I)\pi_r(I), \quad \text{zero if independent}$$

$$k_l(q) = \frac{\sum_r r \pi_{qr}(l)}{\pi_q(l)}$$

$$\pi_{\bar{q}\bar{r}}^{conn.}(l) = \sum_{qr} qr (\pi_{qr}(l) - \pi_q(l)\pi_r(l)) = \langle qr \rangle_l - \langle q \rangle_l^2$$

Random trees

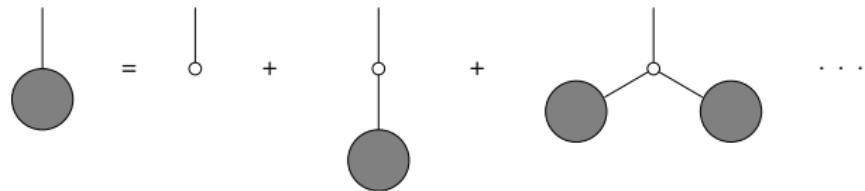
- Labeled random trees of size n .

$$T \in \mathcal{T}_n, \quad \rho(T) = \prod_{i \in T} \omega_{q_i}, \quad \omega_q \geq 0$$



$$Z_n = \sum_{T \in \mathcal{T}_n} \rho(T), \quad Z(\mu) = \sum_n e^{-\mu n} Z_n$$

Generating functional



$$Z(\mu) = e^{-\mu} \sum_{q=1}^{\infty} \frac{\omega_q}{(q-1)!} Z^{q-1} = e^{-\mu} \frac{F(Z(\mu))}{Z(\mu)},$$

$$F(Z) \equiv \sum_{q=1}^{\infty} \frac{\omega_q}{(q-1)!} Z^q.$$

Generic trees

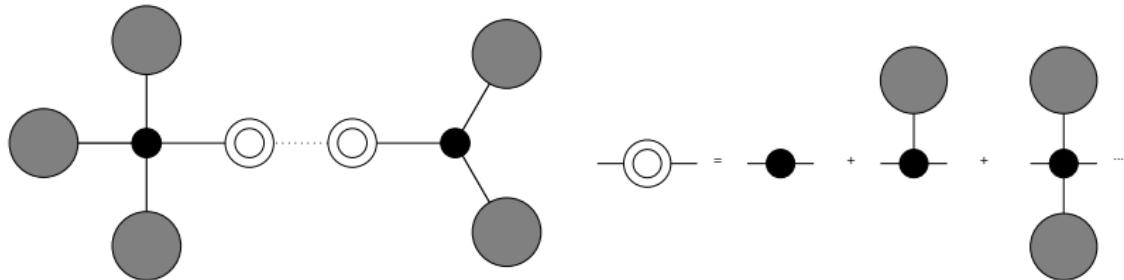
- For large class of weights

$$Z(\mu) \approx Z_0 - Z_1 \sqrt{\mu - \mu_s}$$

- Degree distribution in canonical ensemble ($n \rightarrow \infty$)

$$\pi_q = \frac{1}{F(Z_0)} \frac{\omega_q}{(q-1)!} Z_0^q$$

- Hausdorff dimension $d_H = 2$



$$\begin{aligned}\Omega_I(q, r; \mu) &= \frac{e^{-\mu} \omega_q}{(q-1)!} Z^{q-1}(\mu) \frac{e^{-\mu} \omega_r}{(r-1)!} Z^{r-1}(\mu) \\ &\times \left[e^{-\mu} \sum_{k=2} \frac{\omega_k}{(k-2)!} Z^{k-2} \right]^{l-1}\end{aligned}$$

$$\begin{aligned}\Omega_I(p, q; \mu) &\approx \frac{\omega_q}{(q-1)!} \frac{\omega_r}{(r-1)!} Z_0^{q+r-2} \\ &\times \left(1 - \frac{Z_1}{Z_0} \sqrt{\Delta\mu}\right)^{q+r-2} \left(1 - 2\frac{Z_0}{Z_1} \sqrt{\Delta\mu}\right)^{I-1}.\end{aligned}$$

$$\begin{aligned}\langle n_{qr} \rangle \propto \Omega_I(p, q; V) &\approx \frac{\omega_q}{(q-1)!} \frac{\omega_r}{(r-1)!} Z_0^{q+r-2} \\ &\times \left[\frac{Z_1}{Z_0} (q+r-2) + 2\frac{Z_0}{Z_1} (I-1) \right].\end{aligned}$$

$$\pi_{q,r}(l) = \pi_q \pi_r \frac{(q+r-2) + (\langle q^2 \rangle - 4)(l-1)}{2 + (\langle q^2 \rangle - 4)(l-1)},$$

$$\pi_q(l) = \pi_q \frac{q + (\langle q^2 \rangle - 4)(l-1)}{2 + (\langle q^2 \rangle - 4)(l-1)}.$$

$$\pi_{q,r}(1) = \pi_q \pi_r \frac{(q+r-2)}{2}, \quad \pi_q(1) = \pi_q \frac{q}{2}$$

$$\begin{aligned}\pi_{q,r}^{con}(l) &= \pi_{q,r}(l) - \pi_q(l)\pi_r(l) \\ &= -\frac{(q-2)(r-2)}{\left[2 + (\langle q^2 \rangle - 4)(l-1)\right]^2} \pi_q \pi_r.\end{aligned}$$

$$\pi_{q,r}^{con}(1) = -\frac{(q-2)(r-2)}{4} \pi_q \pi_r.$$

$$k_l(q) = 2 + \frac{\langle q^2 \rangle - 4}{q + (\langle q^2 \rangle - 4)(l-1)}.$$

$$k(q) = 2 + \frac{\langle q^2 \rangle - 4}{q}.$$

Erdos-Renyi trees.

$$\omega_q = 1, \quad \pi_q = \frac{1}{e} \frac{1}{(q-1)!}, \quad \langle q^2 \rangle = \sum_{q=1}^{\infty} \frac{1}{e} \frac{q^2}{(q-1)!} = 5.$$

$$\pi_q(l) = \pi_q \frac{q+l-1}{1+l},$$

$$\pi_{q,r}^{conn.}(l) = -\frac{1}{e^2} \frac{(q-2)(r-2)}{(l+1)^2} \frac{1}{(q-1)!} \frac{1}{(r-1)!},$$

$$\pi_{\bar{q},\bar{r}}^{con}(l) = -\frac{1}{(l+1)^2}, \quad k_l(q) = 2 + \frac{1}{q+l-1}.$$

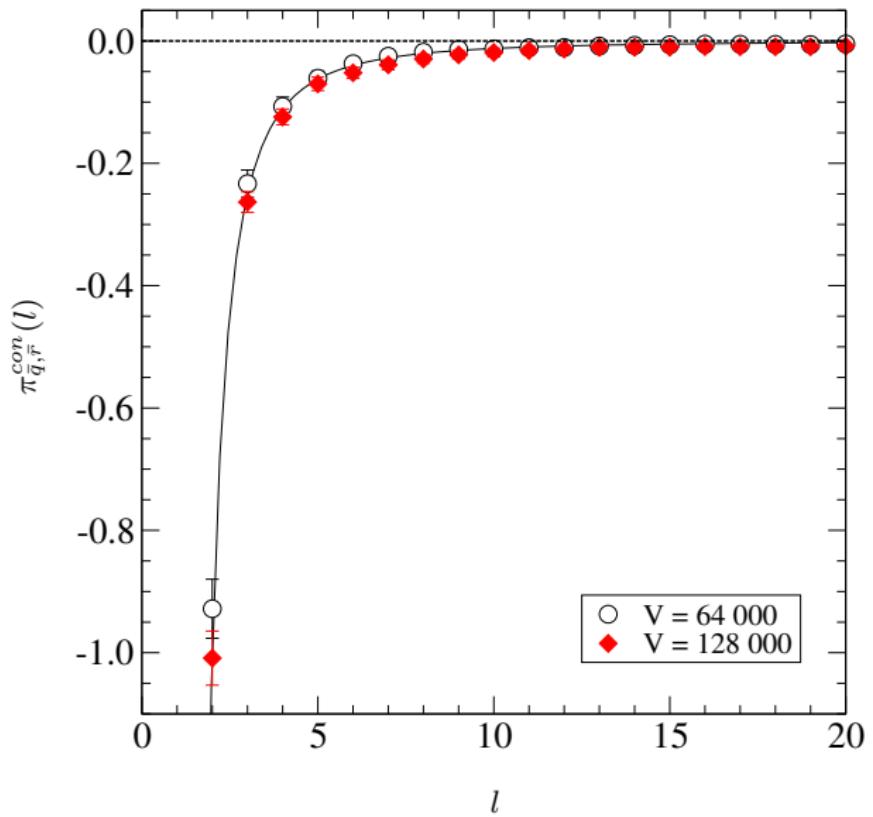
Scale-free trees

$$\omega_q = q^{-\beta}(q-1), \quad F(Z) = \sum_{q=1}^{\infty} \frac{Z^q}{q^\beta} \equiv \text{Li}_\beta(Z)$$

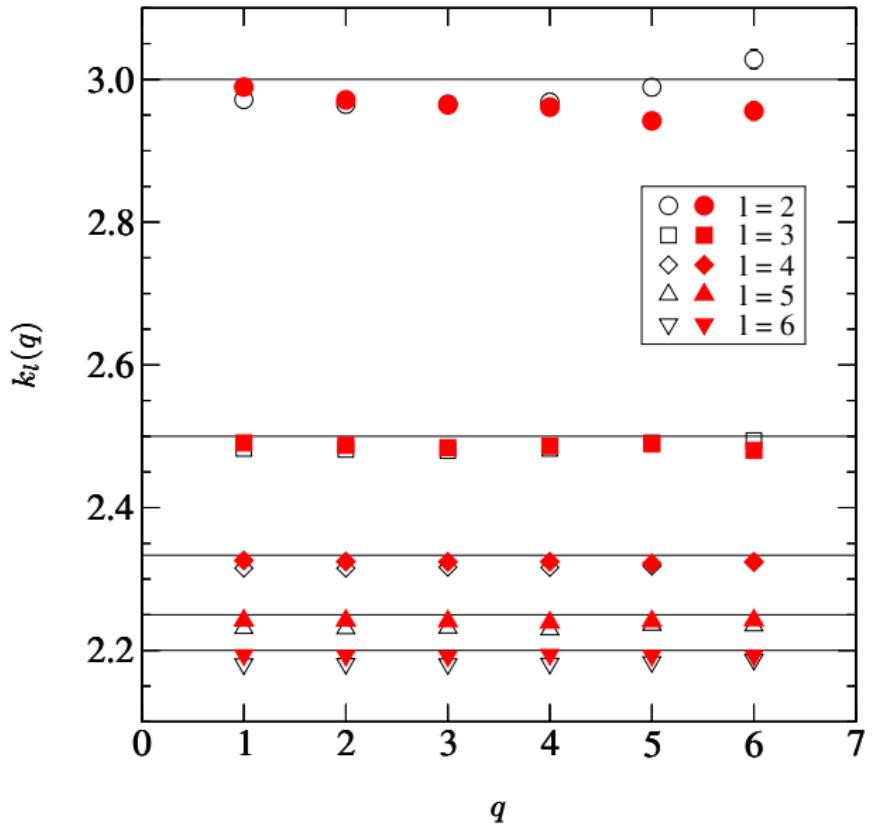
$$\pi(q) = \frac{q^{-\beta} Z_0^q}{F(Z_0)}$$

$$\beta \longrightarrow \beta_c \approx 2.4788 \quad \quad Z_0 \longrightarrow 1, \quad \quad \langle q^2 \rangle \longrightarrow \infty$$

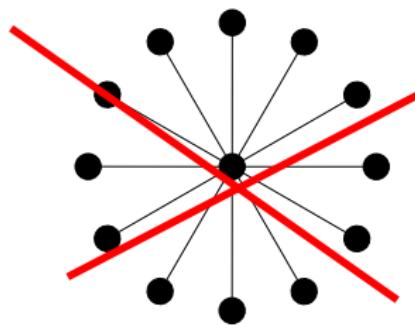
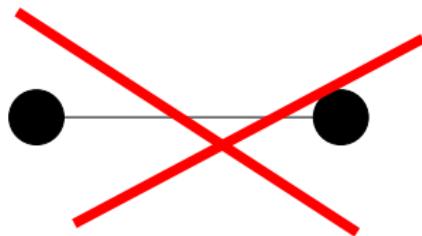
$$\begin{aligned}
 \pi_{q,r}^{con}(l) &= \pi_{q,r}(l) - \pi_q(l)\pi_r(l) \\
 &= -\frac{(q-2)(r-2)}{\left[2 + (\langle q^2 \rangle - 4)(l-1)\right]^2} \pi_q \pi_r \longrightarrow 0 \\
 \sum_{qr} qr \pi_{q,r}^{con}(l) &\longrightarrow -\frac{1}{(l-1)^2}
 \end{aligned}$$



$$\bar{k}_l(q) = 2 + \frac{\langle q^2 \rangle - 4}{q + (\langle q^2 \rangle - 4)(l-1)} \longrightarrow 2 + \frac{1}{l-1}$$



The mechanism ?



$$0 = \pi_{11}(1) \neq \pi_1(1)\pi_1(1) \neq 0$$

Maximal entropy graphs

- Maximal entropy graphs (maximally random) with a given degree distribution π_q are uncorrelated if π_q does not have long tails.
- What about maximal entropy *connected* graphs?

Giant connected component

- M. Molloy, B. Reed, Combinatorics, Probab. Comput. **7** (1998) 295.
- M.E.J. Newman, S.H. Strogatz, D. J. Watts,
Phys. Rev. **E64** (2001) 026118
- F. Chung, L. Lu, Ann. of Comb. 6 (2002) 125.
- A. Fronczak, P. Fronczak, J. Hołyst,
AIP Conf. Proc. **776**(2005) 52.
- M.E.J. Newman, arXiv:0707.0080

- In large n limit all the finite connected components are trees.
- A single giant connected component can arise.

Summary of equations



$$G_1(x) = \frac{1}{Z} G'_0(x), \quad G_0(x) = \sum_{q=0}^{\infty} \pi_q x^q$$

$$H_1(x) = x G_1(H_1(x)) \quad H_0(x) = x G_0(H_1(x))$$



$$u = H_1(1) = \sum_q P_1(q), \quad h = H_0(1) = G_0(u) = \sum_q P_0(q)$$

Giant connected component



$$u = H_1(1) = \sum_s P_1(s), \quad u = G_1(u)$$

is the probability that a directed link leads to a finite component



$$u^2 = H_1^2(1) = \sum_s P_{1,1}(s)$$

probability that a link belongs to the finite component



$$h \equiv H_0(1) = \sum_s P_0(s), \quad h = G_0(u);$$

Probability that a random vertex belongs to a finite connected component.

Average degree of the giant component



$$V^{(g)} = (1 - h)V$$

Number of vertices in the giant component



$$L^{(g)} = (1 - u^2)L$$

Number of links in the giant component



$$z^{(g)} = \frac{2L^{(g)}}{V^{(g)}} = z \frac{1 - u^2}{1 - h}$$

Degree distribution

- One can show that

$$p_q^{(f)} = p_q \frac{u^q}{h},$$

$$p_q^{(g)} = p_q \frac{1 - u^q}{1 - h}$$

M. Bauer, D. Bernard, cond-mat/0206150

P.B., A.K. Oleś arXiv:0710.3319

Correlations in giant connected component

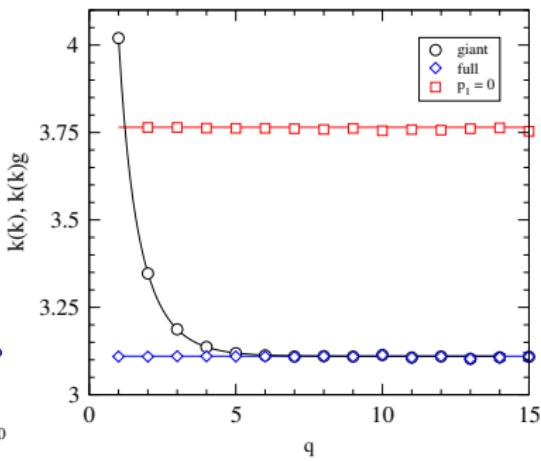
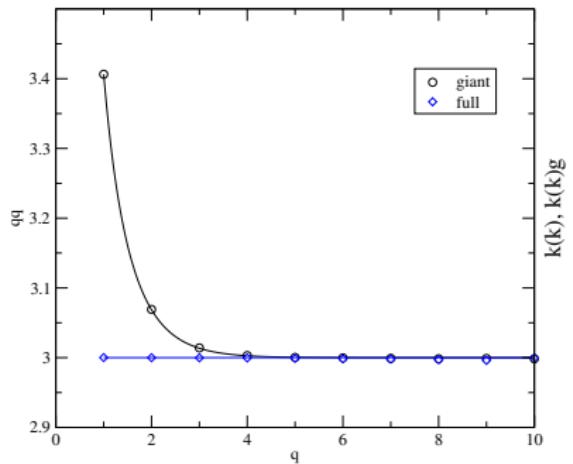
$$n_{qr} = n_{qr}^{(g)} + n_{qr}^{(f)}, \quad n_{qr} = \frac{qn_q r n_r}{2L}$$

$$p_{q,r}^{(g)} = \frac{qp_q rp_r}{z^2} \frac{1}{1-u^2} \left(1 - \frac{u^q u^r}{u^2} \right)$$

$$\bar{q}^{(g)}(q) = \frac{\langle q^2 \rangle}{z} \frac{1}{1-u^q} \left(1 - \frac{\langle q^2 \rangle^{(f)}}{z^{(f)}} \frac{z}{\langle q^2 \rangle} u^q \right)$$

When $u = 0$ (only one connected component) then vertices are uncorrelated, $u = 0$ if and only if $p_1 = 0$

ER and Exponential graphs



Giant connected components are disassortative.

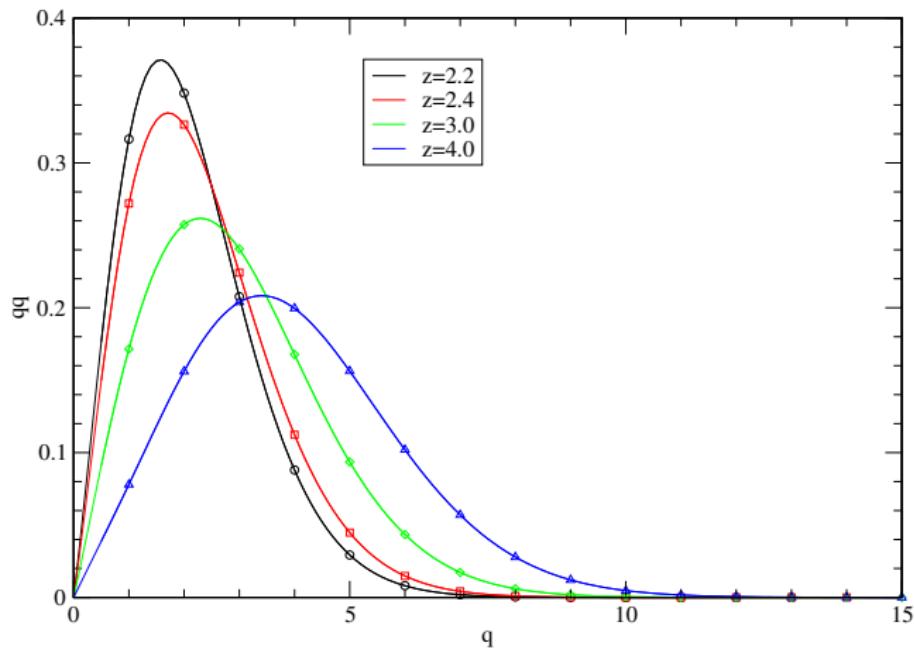
Connected ER graphs

- We have connected ER graph with average degree $z^{(g)}$
- We look for ER graphs whose giant connected component has average degree $z^{(g)}$.
- We find z by solving

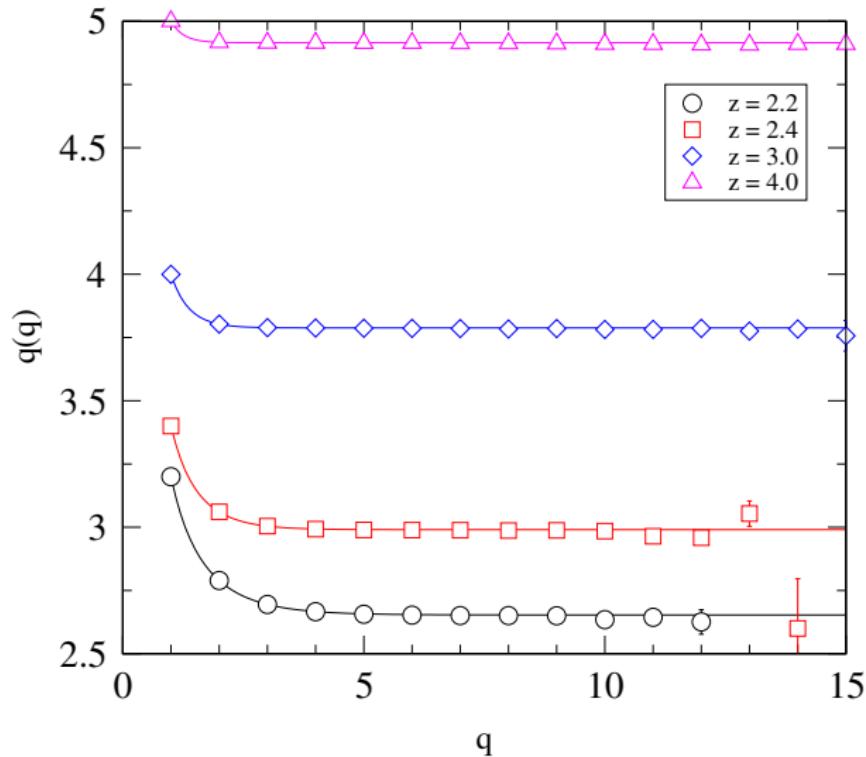
$$z^{(g)} = z(1 + h(z))$$

- Properties of giant component of ER graph with average degree z should be the same as the properties of maximal entropy connected graph with average degree $z^{(g)}$

Connected ER graphs – Degree distribution



Connected ER graphs – $k(q)$



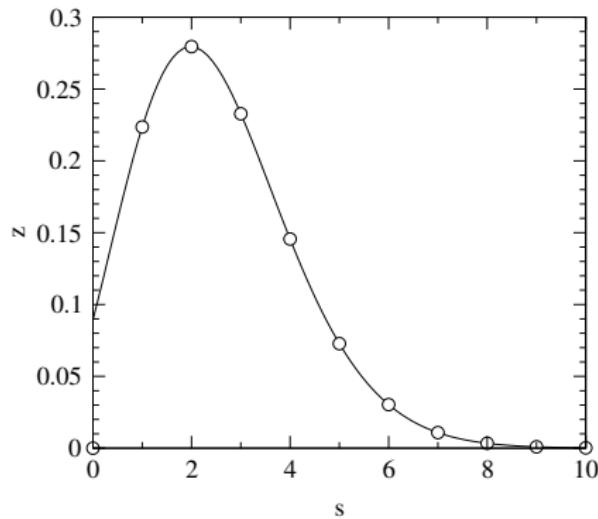
Arbitrary degree distribution

$$p_k^{(g)} = p_k \frac{1 - u^k}{1 - h}$$

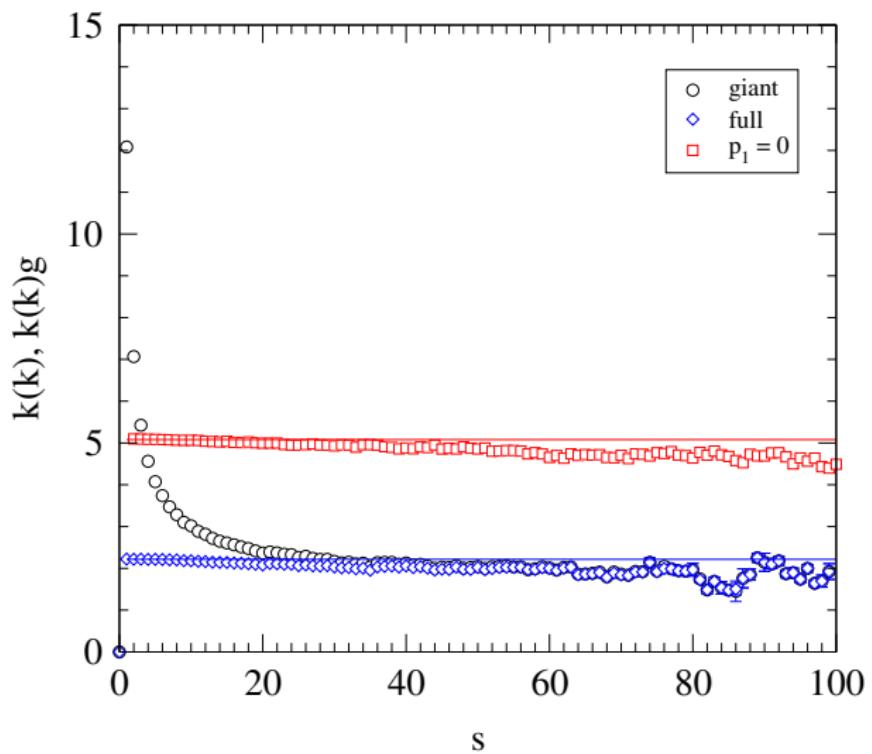
$$p_k = (1 - h) \frac{p_k^{(g)}}{1 - u^k}, \quad p_0 = 0, \quad u = \frac{\sum_{k=1}^{\infty} p_k^{(g)} \frac{k u^{k-1}}{1 - u^k}}{\sum_{k=1}^{\infty} p_k^{(g)} \frac{k}{1 - u^k}}$$

has a solution if

$$\sum_k k p_k^{(g)} \geq 2$$



$$p_k^{(g)} = e^{-z^{(g)}} \frac{(z^{(g)})^k}{k!}, p_0 = 0$$



Summary

- Random trees are correlated.
- Those correlations are long range.
- Those trees are non-critical.
- This happens only in canonical (fixed n) ensemble.
- Connectedness imply disassortative correlations.
- Vertices in maximum entropy connected random graphs are correlated if and only if $p_1 \neq 0$.