Emergence of a vertex of infinite degree in non–generic trees

Sigurður Örn Stefánsson, Nordita

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Random Geometry and Applications, workshop

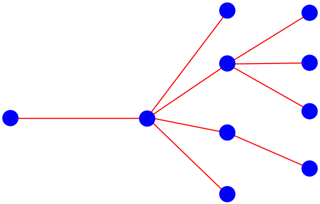
Outline

- Definition of the model
- Results in the generic phase (old and recent)
- Results in the non-generic phase (old and new)
- Conclusions

Rooted, planar trees

- A tree is a graph with no loops.
- Single out one vertex (take it to have degree 1) and call it the root (r).
- Planarity - trees are embedded in the plane. Edges are not allowed to cross.

  Two trees are the same if one can be deformed into the other without crossing edges.



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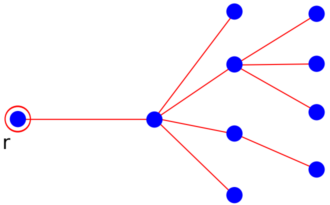

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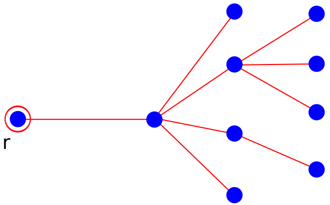

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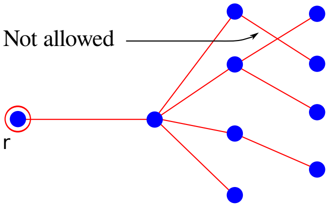

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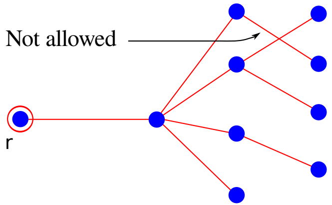

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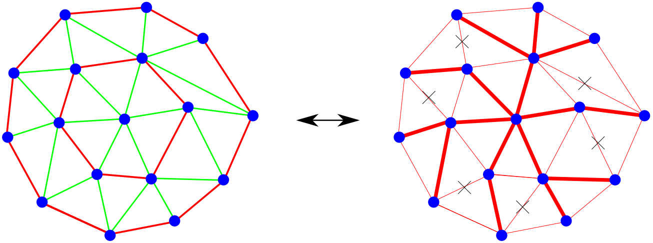
Physical motivation

Trees code information of surfaces

- 2D causal dynamical triangulations (Ambjørn and Loll)

  Planar trees



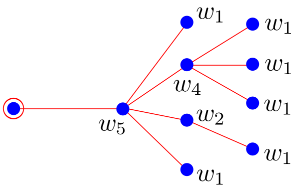
- A more general bijection exists between planar maps and well labelled trees.

A "tree phase" is observed in 2D quantum gravity interacting with conformal matter.

Equilibrium statistical mechanical (ESM) model

- Let $w_1, w_2, \ldots$ be nonnegative numbers - branching weights.
- Define the weight of a tree $\tau \in \Gamma_N$ by

$$w(\tau) = \prod_{v \in V(\tau) \setminus \{r\}} w_{\mathrm{deg(v)}}$$



$$w(\tau) = w_1^6 \; w_2 \; w_4 \; w_5$$

- Define a probability distribution ν_N on Γ_N by

$$\nu_N(\tau) = Z_N^{-1} w(\tau) \qquad \text{where} \quad Z_N = \sum_{\tau' \in \Gamma_N} w(\tau')$$

is a normalization - called the finite volume partition function.

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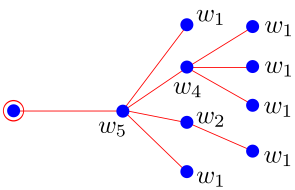

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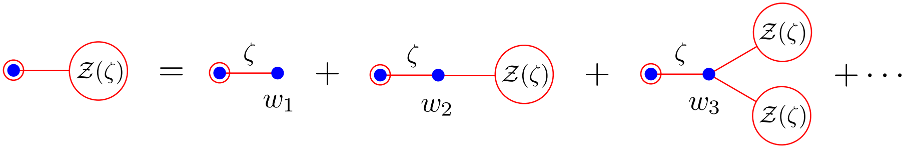
Generating functions

Define the generating functions

$$\mathcal{Z}(\zeta) = \sum_{N=1}^{\infty} Z_N \zeta^N \qquad \text{and} \quad g(z) = \sum_{n=1}^{\infty} w_n z^{n-1}$$

with radii of convergence ζ_0 and ρ, respectively. They obey the relation

$$\mathcal{Z}(\zeta) = \zeta g(\mathcal{Z}(\zeta))$$



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- $\mathcal{Z}_0 < \rho$: Generic, g analytic at $\mathcal{Z}_0$!
- $\mathcal{Z}_0 = \rho$: Non–generic.

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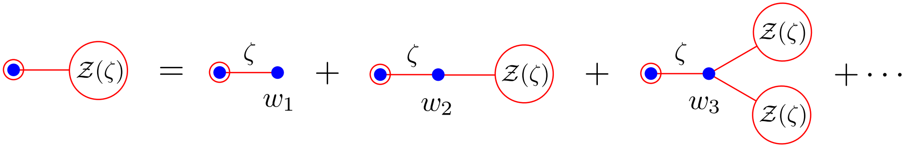

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The Galton–Watson branching process

$(p_n)_{n\geq 0}$ non–negative numbers such that $\sum_n p_n = 1$.

Generates a probability measure μ on the set of finite trees. Useful fact:

$$\nu_N(\tau) = \frac{\mu(\tau)}{\mu(\Gamma_N)}, \qquad \tau \in \Gamma_N, \qquad \text{with} \quad p_n = \zeta_0 w_{n+1} \mathcal{Z}_0^{n-1}$$

ESM on Γ_N can be viewed as a GW process conditioned on size N.

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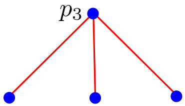

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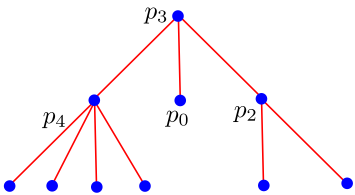

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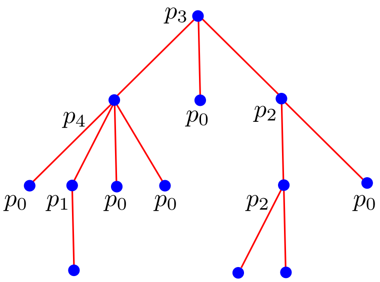

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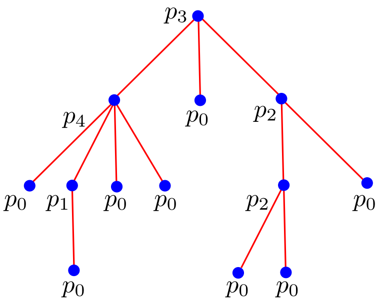

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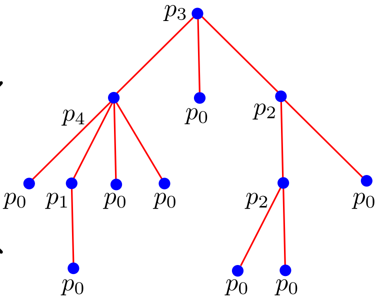

$$\Big) = p_0^8 \; p_1 \; p_2^2 \; p_3 \; p_4$$

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ESM on Γ_N can be viewed as a GW process conditioned on size N.

The Galton–Watson branching process

Define $m = \sum_{n=0}^{\infty} np_n$. m is the *mean offspring probability* of the GW process.

Galton Watson processes are divided into three categories according to the value of m:

- $m < 1$: Sub–critical. Dies out with probability one, "fast"
- $m = 1$: Critical. Dies out with probability one, "slower"
- $m > 1$: Super–critical. Survives forever with nonzero probability.

The ESM model corresponds to either size–conditioned sub–critical GW processes or size–conditioned critical GW processes with

$$m = \mathcal{Z}_0 \frac{g'(\mathcal{Z}_0)}{g(\mathcal{Z}_0)} \leq 1.$$

Follows from

$$\mathcal{Z}(\zeta) = \zeta g(\mathcal{Z}(\zeta))$$

Things to do

Identify different phases. Bialas and Burda, 1996.	Generic (G) - long trees Nongeneric (NG) - crumpled trees
Calculate Z_N for N large.	G - Meir and Moon, 1978. G/NG - Janson, 2006. Flajolet and Sedgewick, 2009 (AC). NG - Bialas and Burda, 1996. With assumption that scaling exponent exists. Jonsson, Stefánsson, 2010.
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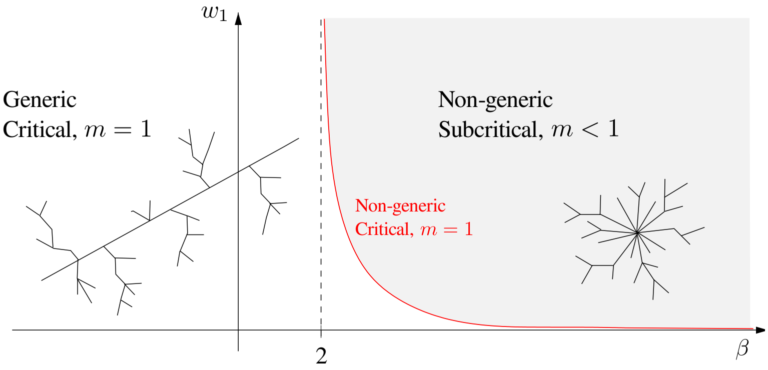
The phase structure

For simplicity choose

- w_1 - as a free parameter, and
- $w_n \sim n^{-\beta}$ - with $\beta \in \mathbb{R}$ a free parameter.

If $w_n = 0$ for all $n > d$ then always generic - no vertex of large deg. appears.



The generic phase

Theorem. (Meir and Moon, '78) $Z_N = \left(\frac{g(\mathcal{Z}_0)}{2\pi g''(\mathcal{Z}_0)}\right)^{\frac{1}{2}} N^{-\frac{3}{2}} \zeta_0^{-N}(1 + O(N^{-1}))$.

Proof follows rather easily from the fact that g is analytic at $\mathcal{Z}_0$.

Let Γ be the set of all trees, finite and infinite.

Theorem. (Durhuus, Jonsson and Wheater, 2007)
The measures ν_N, viewed as probability measures on Γ, converge weakly as $N \to \infty$ to a probability measure ν which is concentrated on the set of trees which have exactly one simple path from the root to infinity (a spine). The number of left and right branches i and j, from a vertex on the spine are independently distributed by

$$\phi(i,j) = \zeta_0 w_{i+j+2} \mathcal{Z}_0^{i+j}.$$

The branches attached to the spine are i.i.d. critical Galton–Watson processes with branching weights

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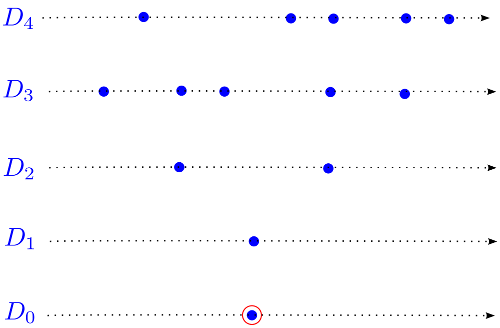
Definition of Γ

$(D_R)_{R\geq 0}$ a sequence of **finite**, ordered sets. D_0 and D_1 have one element. If $D_S = \emptyset$ for some S then $D_R = \emptyset$ for all $R > S$.

$(\phi_R)_{R\geq 1}$ a sequence of **order preserving** maps $\phi_R : D_R \to D_{R-1}$.

Γ is defined as the set of all pairs of such sequences (modulo sequences of order isomorphisms which are consistent with the maps ϕ_R).



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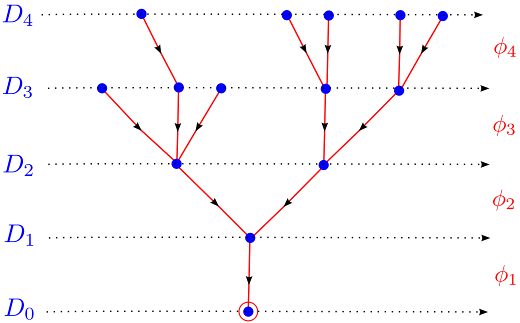


Weak convergence

In order for this to make sense we need a topology on Γ. Define a metric d on Γ by

$$d(\tau, \tau') = \inf \left\{ \frac{1}{R} : B_R(\tau) = B_R(\tau') \right\}$$

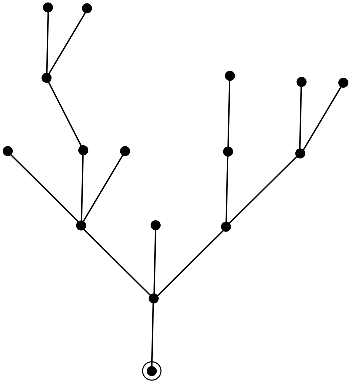
where $B_R(\tau)$ is a graph ball of radius R centered at the root of τ. That the measures ν_N converge weakle to a measure ν means that for any bounded, continuos (w.r.t. d) function f on Γ,

$$\int_\Gamma f d\nu_N \to \int_\Gamma f d\nu$$

as $N \to \infty$.

Definition of the metric d

τ



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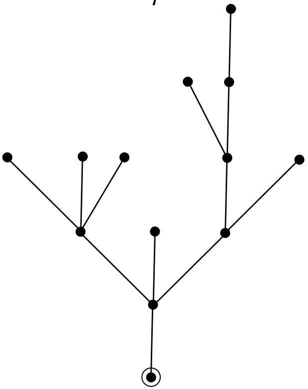

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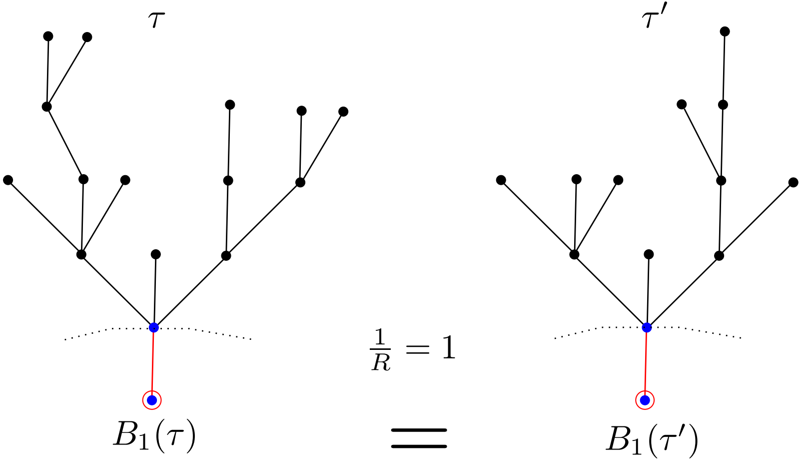

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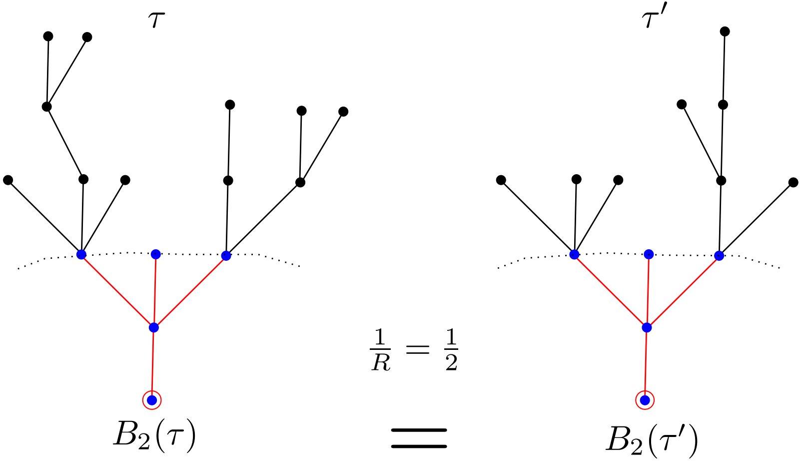

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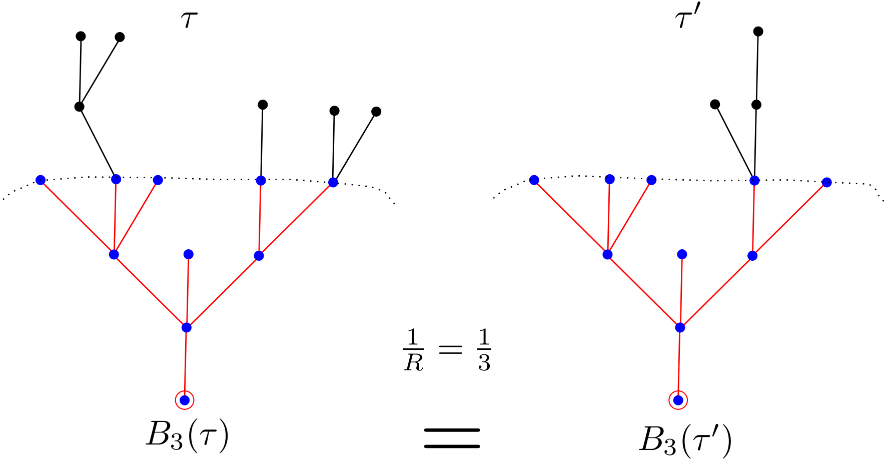

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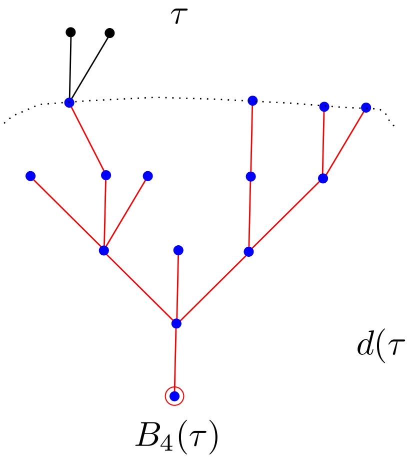
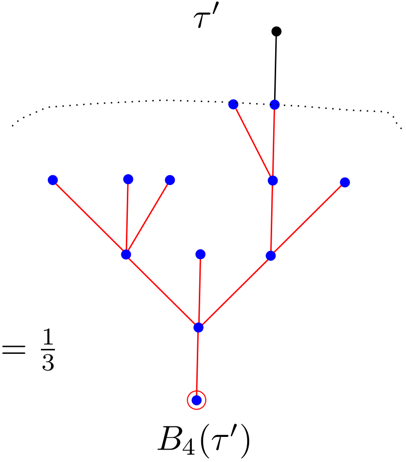


$$d(\tau, \tau') = \frac{1}{3}$$

$$\neq$$



The metric space (Γ, d) has some nice properties and to prove week convergence of ν_N we only need to prove the following:

- For any $R \geq 1$ and every tree τ_0 of height R the sequence

$$\nu_N(\{\tau \in \Gamma : B_R(\tau) = \tau_0\})$$

  is convergent.

- Tightness. For any $\epsilon > 0$ there exists a compact set $K \subset \Gamma$ such that

$$\nu_N(\Gamma \setminus K) < \epsilon \qquad \text{for all } N.$$

  If Γ is compact this condition is obviously fulfilled. In the present case Γ is not compact and proving this amounts to showing that very large vertices are unlikely.

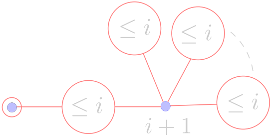
Non–generic phase - calculation of Z_N

Theorem. (Jonsson, Stefánsson, 2010. Confirms Bialas and Burda, 1996)
For the NG branching weights $w_n \sim n^{-\beta}$ *which satisfy* $m < 1$ *it holds that*

$$Z_N = (1-m)^{-\beta} N^{-\beta} \zeta_0^{1-N} \left(1 + o(1)\right).$$

Idea of proof: $\quad Z_N = Z_{1,N} + E_N$.

$$Z_{1,N} = \text{contribution from} \sum_{i=0}^{N-1}$$



Can write down an exact expression for $Z_{1,N}$ using truncated versions of $\mathcal{Z}(\zeta)$.

Using Lagrange's Inversion formula we can get estimates of $Z_{1,N}$ in terms of probalities of the sum of i.i.d. random variables.

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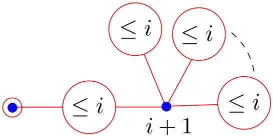

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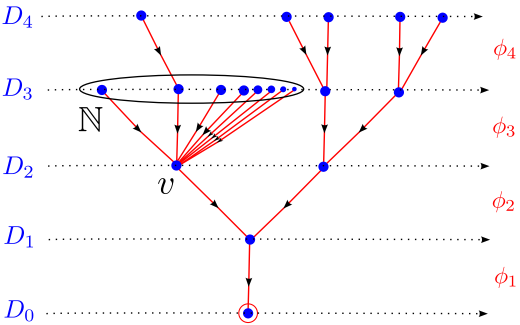
Non–generic phase - weak convergence of ν_N

Γ is not a good space any more - need vertices of infinite degree.

$(D_R)_{R\geq 0}$ a sequence of **countable**, ordered sets. $(\phi_R)_{R\geq 1}$ a sequence of **order preserving** maps $\phi_R : D_R \to D_{R-1}$. If $|\phi_R^{-1}(v)| = \infty, v \in D_R$ then $\phi_R^{-1}(v)$ is ordered as $\mathbb{N}$.

Define $\bar{\Gamma}$ as the set of all pairs of such sequences (modulo sequences of order isomorphisms which are consistent with the maps ϕ_R).



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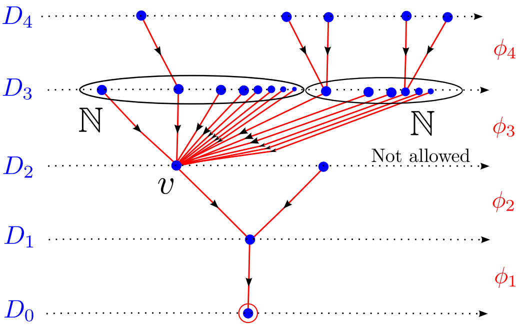

Non–generic phase - weak convergence of ν_N

Γ is not a good space any more - need vertices of infinite degree.

$(D_R)_{R\geq 0}$ a sequence of **countable**, ordered sets. $(\phi_R)_{R\geq 1}$ a sequence of **order preserving** maps $\phi_R : D_R \to D_{R-1}$. If $|\phi_R^{-1}(v)| = \infty, v \in D_R$ then $\phi_R^{-1}(v)$ is ordered as $\mathbb{N}$.

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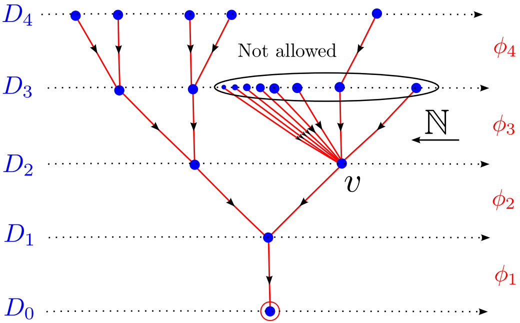
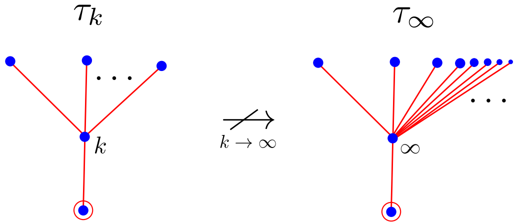


A new metric

The metric d is no longer good. Look at the following example:



$$d(\tau_k, \tau_\infty) = 1, \;\; \forall k$$

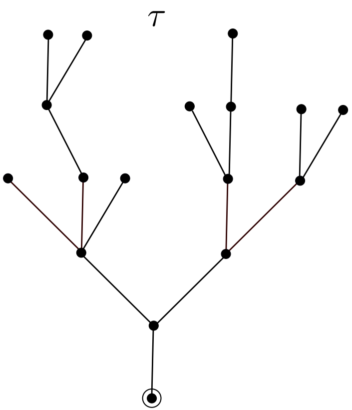
We can never approach graphs having vertices of infinite degree with finite graphs. Therefore, we don't expect the measures ν_N to converge.

Define a metric $\bar{d}$ on $\bar{\Gamma}$ by

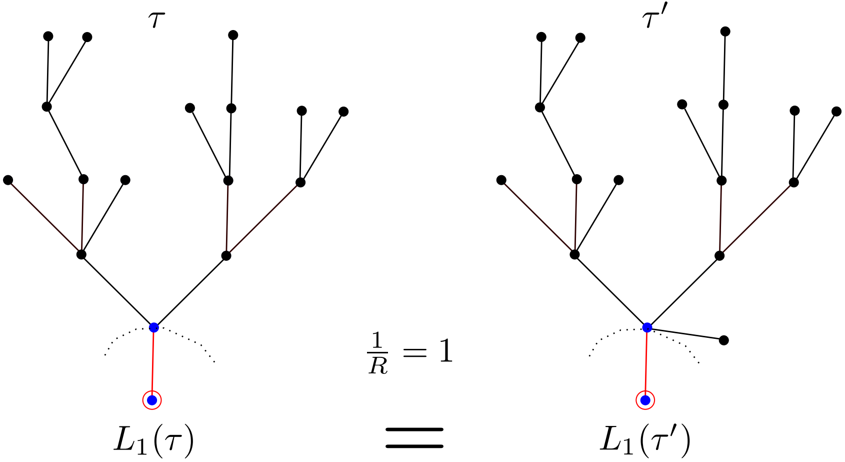
$$\bar{d}(\tau, \tau') = \inf \left\{ \frac{1}{R} : L_R(\tau) = L_R(\tau') \right\}$$

$L_R(\tau) \subset B_R(\tau)$ is the "left ball" of graph radius R (explain soon...)

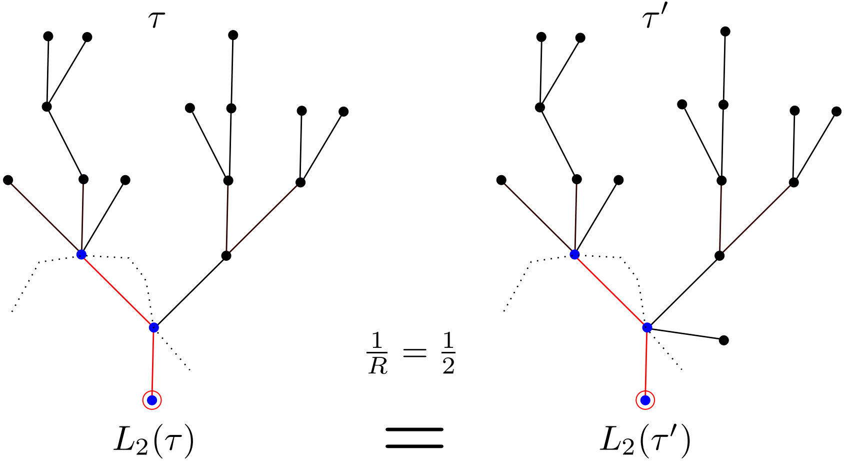
The new metric



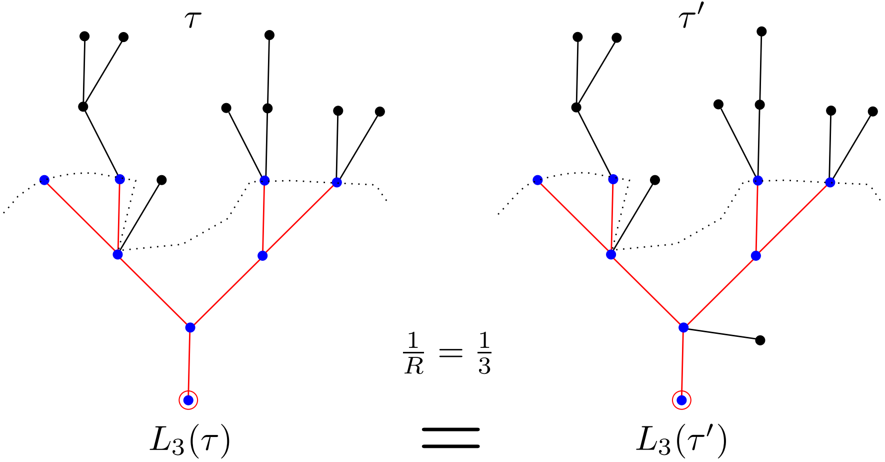
The new metric



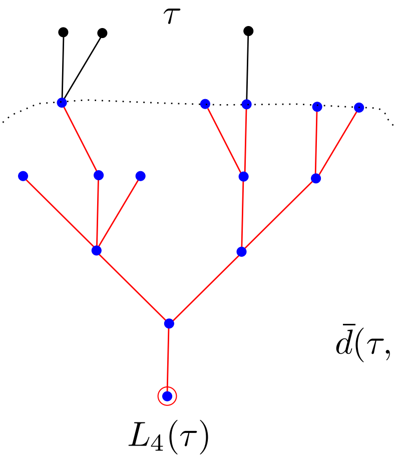
The new metric



The new metric

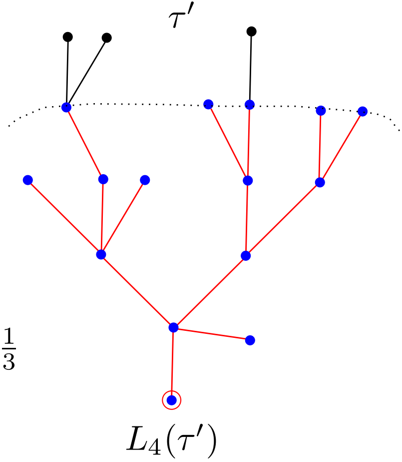


The new metric



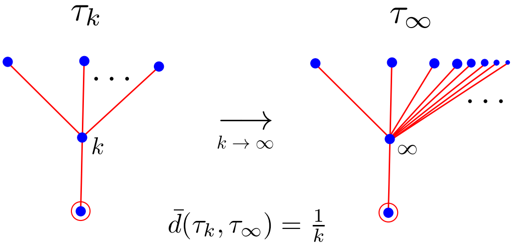
$$\bar{d}(\tau, \tau') = \frac{1}{3}$$

$$\not\approx$$



The new metric

Just to be sure:



The measure space $(\bar{\Gamma}, \bar{d})$ is compact -> don't have to prove tightness. Only have to show that

- For any $R \geq 1$ and every tree τ_0 of maximum height R and with maximum vertex degree R the sequence $\nu_N(\{\tau \in \Gamma : L_R(\tau) = \tau_0\})$ is convergent.

Works for both phases and critical line -> simplifies proof of the generic case.

Weak convergence of ν_N

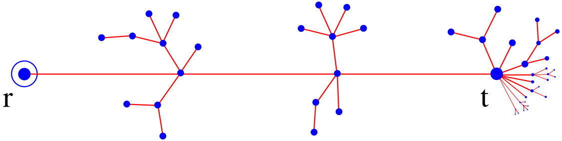
Theorem

For the NG branching weights $w_n \sim n^{-\beta}$ which satisfy $m < 1$ the measures ν_N, viewed as measures on $\bar{\Gamma}$, converge in a weak sense to a measure ν which is concentrated on the set of trees with exactly one vertex of infinite degree which we denote by t.

The length ℓ of the path (r, t) is distributed by $\psi(\ell) = (1 - m)m^{\ell-1}$.

The outgrowths from the path (r, t) are finite, independent, subcritical Galton–Watson trees defined by the offspring probabilities $p_n = \zeta_0 w_{n+1}$.

The numbers i and j of left and right outgrowths from a vertex $v \in (r, t), v \neq t$ are independently distributed by $\phi(i, j) = 1/m\zeta_0 w_{i+j+2}$.



Conclusions

- Have proven weak convergence of the finite volume measures ν_N for NG trees. The new method applies to both phases and simplifies the proof in the generic phase.
- Can also prove convergence on the critical line. Use results (with mild generalizations) of Janson, 2006 and Flajolet and Sedgewick, 2009 about behaviour of Z_N on the critical line. Get same results as in the generic case (single spine having finite i.i.d. GW outgrowths). However the GW outgrowths can have $g''(\mathcal{Z}_0) = \infty$ -> different properties, Hausdorff dimension from 2 to ∞, spectral dimension from $4/3$ to 2 (scaling assumptions).
- Calculation of the spectral dimension d_s - dimension seen by a random walker travelling on the graph. Due to the vertex of infinite degree d_s is a. s. infinite. However, defined in turns of ensemble average, it takes the value $2(\beta - 1)$, $\beta > 2$ ($w_n \sim n^{-\beta}$). Different from the value 2 which was previously obtained using scaling assumptions (Correia and Wheater, 1998).
- This phenomenon of "condensation" appears in other models - ESM of caterpillars, zero–range process, simplicial gravity...