# A new scaling limit for matrix models and its geometric interpretation

J. Ambjørn<sup>1, 2</sup>

(Collaborators: R. Loll, Y. Watabiki, W. Westra, S. Zohren)

<sup>1</sup>Niels Bohr Institute, Copenhagen, Denmark

<sup>2</sup>University of Utrecht, The Netherlands

Nordita, November 2010

・ロ・・ (日・・ ほ・・ (日・)

# Hermitian Matrix Models

Recall the Hermitian matrix model, defined as a formal power series in  $\tilde{g}$ 

$$Z(\tilde{g}) = \int d\phi \, \mathrm{e}^{-N\mathrm{tr}\left(\frac{1}{2}\phi^2 - \frac{\tilde{g}}{3}\phi^3\right)}$$
  
=  $\sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \, \mathrm{e}^{-\frac{1}{2}N\mathrm{tr}\left(\phi^2\right)} \, \left(\frac{N\tilde{g}}{3}\mathrm{tr}\,\phi^3\right)^k,$ 

$${oldsymbol d} \phi = \prod_{lpha \leq eta} {oldsymbol d} \operatorname{Re} \phi_{lphaeta} \prod_{lpha < eta} {oldsymbol d} \operatorname{Im} \phi_{lphaeta}.$$

◆□ → ◆□ → ◆三 → ◆三 → ◆○ ◆

The integral can be evaluated in the standard way by doing all possible Wick contractions of  $(\operatorname{tr} \phi^3)^k$  and using

$$\left\langle \phi_{\alpha\beta}\phi_{\alpha'\beta'}\right\rangle = \mathbf{C}\int \mathbf{d}\phi \ \mathbf{e}^{-\frac{1}{2}\sum_{\alpha\beta}|\phi_{\alpha\beta}|^2}\phi_{\alpha\beta}\phi_{\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\beta\alpha'},$$

#### Geometric interpretation:



< ロ > < 同 > < 臣 > < 臣 > -

3

$$\mathrm{d}\phi \; \mathrm{e}^{-\frac{N}{g_{\mathrm{s}}}\mathrm{tr}\; V(\phi)} \propto \mathrm{d}U(N) \prod_{i=1}^{N} \mathrm{d}\ell_{i} \; \mathrm{e}^{-\frac{N}{g_{\mathrm{s}}}V(\ell_{i})} \prod_{i < j} |\ell_{i} - \ell_{j}|^{2}$$

The "classical" limit is obtained for  $g_s \rightarrow 0$ , where all eigenvalues are lumped together at  $\ell_0$ , where

 $V'(\ell_0)=0.$ 

However, for  $g_s > 0$  the integration over the non-diagonal matrix elements produces the Vandermonde determinant, which acts as a "quantum" correction, a repulsion between different eigenvalues. Result: eigenvalues are smeared out over an interval around  $\ell_0$ , even in the large *N* limit.

(ロ) (同) (E) (E) (E)





Ambjørn New scaling limit

Large N saddelpoint equation:

$$w(z) := \left\langle \frac{1}{N} \operatorname{tr} \frac{1}{z - \phi} \right\rangle = \frac{1}{Z} \int \mathrm{d}\phi \; \frac{1}{N} \frac{1}{z - \phi} \; \mathrm{e}^{-\frac{N}{g_{\mathrm{s}}} \operatorname{tr} \; V(\phi)}$$

For  $V(\phi) = -g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3$  one has

$$w(z)=\frac{1}{2g_s}\left(V'(z)+g(z-b)\sqrt{(z-c)(z-d)}\right)$$

The constants *b*, *c* and *d* are determined by the requirement that  $w(z) \rightarrow 1/z$  for  $z \rightarrow \infty$ .

In the large *N* expansion one has to any order the same square root structure.

◆□ → ◆□ → ◆三 → ◆三 → ◆○ ◆

## The conventional scaling limit

The usual scaling limit of the matrix model is obtained (for fixed  $g_s$ ) by adjusting g such that b(g) = c(g). At this point the analytic structure of w(z) changes from  $(z - c(g))^{1/2} \rightarrow (z - c(g))^{3/2}$ , and this change can only be accommodated by invoking arbitrary high k in the sum

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \, \mathrm{e}^{-\frac{N}{2g_{s}} \mathrm{tr} \left(\phi^{2}\right)} \left(\frac{Ng}{3g_{s}} \mathrm{tr} \, \phi^{3}\right)^{k},$$

This is why one geometrically can imagine a "continuum" limit where the size of each triangle shrinks to zero while the continuum size of the surface stays constant.

(ロ) (同) (E) (E) (E)

 $g = g_c(1 - \Lambda a^2), \quad z = c(g_c) + aZ, \quad a \to 0$ 

$$w(z) = \frac{1}{2} \left( V'(z) + g \sqrt{c(g_c) - d(g_c)} a^{3/2} W_E(Z, \Lambda) \right)$$

$$W_E(Z,\Lambda) = (Z - \sqrt{2\Lambda/3})\sqrt{Z + 2\sqrt{2\Lambda/3}}$$

**a** has the interpretation as the length of the side of the triangles (polygons) which appear in  $V(\phi)$ .

Notice the non-scaling part V'(z)/2 which dominates when  $a \rightarrow 0$  and renders the average number of polygons present in the ensemble with partition function w(z) finite, even at the critical point. This somewhat embarrassing fact can be circumvented by differentiating w(z) a sufficient number of times with respect to g and z, after which these "non-universal" contributions vanish.

・ロト・日本・モン・モン モー りへの

# The new scaling limit

Two limits: "classical limit":  $g_s = 0$ and conventional scaling limit:  $g_s > 0$  and  $g \rightarrow g_c(g_s)$ .

Is it possible to find a new, non-trivial scaling limit, closer to the classical limit when  $g_s \rightarrow 0$  ?

Close to criticallity  $b(g_c) = c(g_c)$ :

$$g_c(g_s) = rac{1}{2}(1 - rac{3}{2}g_s^{2/3} + O(g_s^{4/3})),$$

 $z_c(g_s) = c(g_c, g_s) = 1 + g_s^{1/3} + O(g_s^{2/3}),$ 

$$c(g_c) - d(g_c) = 4g_s^{1/3} + 0(g_s^{2/3})$$

### A non-trivial scaling is obtained for $g_s = G_s a^3$ .

Again the scaling parameter *a* can be given the geometric interpretation as the link lengths of the polygons in  $V(\phi)$ . Note that the length of the cut goes to zero as  $a \rightarrow 0$ , thus we are closer to the "classical" limit. However, it will survive in the continuum limit:

< □ > < □ > < 亘 > < 亘 > < 亘 > < □ > < □ > <

$$g = g_c(g_s)(1 - a^2 \Lambda) = \bar{g}(1 - a^2 \Lambda_{cdt} + O(a^4))$$
  
 $z = z_c + aZ = \bar{z} + aZ_{cdt} + O(a^2)$   
 $\Lambda_{cdt} \equiv \Lambda + \frac{3}{2}G_s^{2/3}, \quad \bar{g} = \frac{1}{2}, \quad Z_{cdt} \equiv Z + G_s^{1/3}, \quad \bar{z} = 1.$ 

Using these definitions one computes in the limit  $a \rightarrow 0$  that

$$w(z) = rac{1}{a} rac{\Lambda_{
m cdt} - rac{1}{2}Z_{
m cdt}^2 + rac{1}{2}(Z_{
m cdt} - H)\sqrt{(Z_{
m cdt} + H)^2 - rac{4G_s}{H}}}{2G_s}.$$

$$h^3-h+rac{2G_s}{(2\Lambda_{
m cdt})^{3/2}}=0, \hspace{1em} h=H/\sqrt{2\Lambda_{
m cdt}}$$

$$w(z) = \frac{1}{a} W_{\rm cdt}(Z_{\rm cdt}, \Lambda_{\rm cdt}, G_{\rm s})$$

DQR

$$\begin{split} G_{s} &\rightarrow 0: \qquad W_{cdt}(Z_{cdt},\Lambda_{cdt},G_{s}) \rightarrow \frac{1}{Z_{cdt} + \sqrt{2\Lambda_{cdt}}} \\ G_{s} &\rightarrow \infty: \qquad \frac{(Z_{cdt}-H)\sqrt{(Z_{cdt}+H)^{2}-\frac{4G_{s}}{H}}}{2G_{s}} \rightarrow \\ G_{s}^{-5/6} \ \left(Z-\sqrt{2\Lambda/3}\right)\sqrt{Z+2\sqrt{2\Lambda/3}} \end{split}$$

In the first case the cut disappear and one can say that this result is "classical" in a way that will be made precise shortly.

In the second case one recover the standard scaling  $W_E(Z, \Lambda)$ . However, the part not related to the square root will not scale, in accordance with the previous discussion of standard scaling.

・ロ・・ (日・・ ほ・・ (日・)

## The matrix model

The new scaling can be obtained by a change of variables:

 $\phi \rightarrow \bar{z}\,\hat{l} + a\Phi + O(a^2)$ 

Up to a  $\phi$  independent term we then have:

$$V(\phi) = \bar{V}(\Phi), \quad \bar{V}(\Phi) \equiv rac{\Lambda_{
m cdt}\Phi - rac{1}{6}\Phi^3}{2G_s}$$

 $Z(g,g_s) = a^{N^2} Z(\Lambda_{\mathrm{cdt}},G_s), \quad Z(\Lambda_{\mathrm{cdt}},G_s) = \int d\Phi \; \mathrm{e}^{-N\mathrm{tr}\, ar{V}(\Phi)}$ 

$$\frac{1}{z-\phi} = \frac{1}{a} \frac{1}{Z_{\rm cdt}-\Phi} \quad \Rightarrow \quad w(z) = \frac{1}{a} W_{\rm cdt}(Z_{\rm cdt},\Lambda_{\rm cdt},G_{\rm s})$$

Relation correct to all order in *N*. The new scaling limit is itself a matrix model defined by  $\overline{V}(\Phi)$ .



Thus the "classical" limit of the matrix integral with potential  $\overline{V}(\Phi)$ , when only the minimum plays a role, leads to the following expectation value:

(ロ) (同) (E) (E) (E)

$$\frac{1}{N}\left\langle \operatorname{tr} \frac{1}{Z_{\operatorname{cdt}} - \Phi} \right\rangle = \frac{1}{Z_{\operatorname{cdt}} + \sqrt{2\Lambda_{\operatorname{cdt}}}} = \lim_{G_{\operatorname{s}} \to 0} W_{\operatorname{cdt}}(Z_{\operatorname{cdt}}, \Lambda_{\operatorname{cdt}}, G_{\operatorname{s}}).$$

### Geometric interpretation (trivial topology)

Disk amplitude:  $W_{\lambda,G_s}(\ell)$  and  $W_{\lambda,G_s}(x)$  where

$$W_{\lambda,G_s}(x) = \int_0^\infty \mathrm{d}\ell \; \mathrm{e}^{-x\ell} \; W_{\lambda,G_s}(\ell)$$

"Propagator" of a boundary as a function of the geodesic distance from the boundary:  $G_{\lambda,G_s}(\ell_1,\ell_2;t)$  and  $G_{\lambda,G_s}(x,y;t)$  where:

$$G_{\lambda,G_s}(x,y;t) = \int_0^\infty \mathrm{d}\ell_1 \int_0^\infty \mathrm{d}\ell_2 \; \mathrm{e}^{-x\ell_1-y\ell_2} \; G_{\lambda,G_s}(\ell_1,\ell_2;t).$$

(日) (同) (E) (E) (E) (E)

Propagator for  $G_s = 0$ :



$$G_{\lambda}^{0}(x, y; t) = \frac{4\lambda e^{-2\sqrt{\lambda}t}}{(\sqrt{\lambda} + x) + e^{-2\sqrt{\lambda}t}(\sqrt{\lambda} - x)}$$

$$\times \frac{1}{(\sqrt{\lambda} + x)(\sqrt{\lambda} + y) - e^{-2\sqrt{\lambda}t}(\sqrt{\lambda} - x)(\sqrt{\lambda} - y)}$$

◆□ → ◆□ → ◆臣 → ◆臣 → ○三 ・ のへで

$$G_{\lambda}^{(0)}(l_1, l_2; t) = \frac{e^{-[\coth\sqrt{\lambda}t]\sqrt{\lambda}(l_1+l_2)}}{\sinh\sqrt{\lambda}t} \frac{\sqrt{\lambda l_1 l_2}}{l_2} l_1\left(\frac{2\sqrt{\lambda l_1 l_2}}{\sinh\sqrt{\lambda}t}\right)$$

The following geometric picture couples W and G and leads to consistency relations for the scaling of W:

$$-\frac{\partial W_{\lambda,G_s}(x)}{\partial \lambda} = \int_0^\infty \mathsf{d}t \int_0^\infty \mathsf{d}\ell \ G_{\lambda,G_s}(x,\ell;t) \,\ell W_{\lambda,G_s}(\ell).$$



$$egin{array}{lll} \mathcal{W}_{reg} & \xrightarrow[a o 0]{} a^\eta \ \mathcal{W}_\lambda(x), & \eta < 0, \ t_{reg} & \xrightarrow[a o 0]{} t/a^arepsilon, & arepsilon = 1. \ \mathcal{W}_{reg} & \xrightarrow[a o 0]{} const. + a^\eta \ \mathcal{W}_\lambda(x), & \eta = 3/2 \ t_{reg} & \xrightarrow[a o 0]{} t/a^arepsilon, & arepsilon = 1/2, \ \end{array}$$

(日) (同) (E) (E) (E) (E)

The propagator satisfies the following (integral) equation:



Shaded parts of graphs represent the full,  $G_s$ -dependent propagator and disc amplitude, and non-shaded parts the CDT propagator where  $G_s = 0$ . In all four graphs, the geodesic distance from the final to the initial loop is given by *t*. Differentiating the integral equation, corresponding to the figure with respect to *t* leads to ( $g_s = G_s a^3$ )

$$a^{\varepsilon} \frac{\partial G_{\lambda,G_{s}}(x,y;t)}{\partial t} = -\frac{\partial}{\partial x} \Big[ \Big( a(x^{2} - \lambda) + 2g_{s} a^{\eta - 1} W_{\lambda,G_{s}}(x) \Big) G_{\lambda,G_{s}}(x,y;t) \Big]$$

(口)

When  $\eta = 3/2$  and  $\varepsilon = 1/2$  we can take  $g_s > 0$  for  $a \to 0$  (corresponding to  $G_s \to \infty$ ):

$$\frac{\partial G_{\lambda,G_s}(x,y;t)}{\partial t} = -\frac{\partial}{\partial x} \Big[ 2g_s W_{\lambda,G_s}(x) G_{\lambda,G_s}(x,y;t) \Big],$$

Alternatively, for  $\eta = -1$ , we have

$$\frac{\partial G_{\lambda,G_s}(x,y;t)}{\partial t} = -\frac{\partial}{\partial x} \Big[ \hat{W}_{\lambda,G_s}(x) G_{\lambda,G_s}(x,y;t) \Big],$$

$$\hat{W}_{\lambda,G_s}(\boldsymbol{x}) = (\boldsymbol{x}^2 - \lambda) + 2\boldsymbol{G}_s W_{\lambda,G_s}(\boldsymbol{x})$$

◆□ → ◆□ → ◆三 → ◆三 → ◆○ ◆

When  $\eta < 0$ , i.e. when the disk amplitude scales, it satisfies the following equation:



Shaded parts represent the full disc amplitude, unshaded parts the CDT disc amplitude and the CDT propagator.  $G_s$  is associeted with branching.

• • • • • • • • • • •

$$W_{\lambda,G_{s}}(x) = W_{\lambda}^{(0)}(x) + G_{s} \int_{0}^{\infty} dt \int_{0}^{\infty} d\ell_{1} d\ell_{2} (\ell_{1} + \ell_{2}) G_{\lambda}^{(0)}(x,\ell_{1} + \ell_{2};t) W_{\lambda,G_{s}}(\ell_{1}) W_{\lambda,G_{s}}(\ell_{2})$$

The branching of baby-universes in the ordinary scaling limit is the reason that "proper time" scales as  $a^{1/2}$  and not as a. We have an infinite number of baby universes in the continuum limit.



(ロ) (同) (E) (E) (E)

In the new scaling limit the number of baby universes is finite for a universe with finite volume (i.e. area in 2d).



< 3 > <

# String field theory

$$\begin{aligned} \mathbf{G}_{\lambda}^{(0)}(\mathbf{I}_{1},\mathbf{I}_{2};t) &= \langle \mathbf{I}_{2} | \mathbf{e}^{-t\mathcal{H}_{0}(I)} | \mathbf{I}_{1} \rangle \\ \mathbf{H}_{0}(I) &= -I \frac{\partial^{2}}{\partial I^{2}} + \lambda I. \end{aligned}$$

Second quantization or third quantization

 $\begin{aligned} [\Psi(I), \Psi^{\dagger}(I')] &= I\delta(I - I') \\ \Psi(I)|0\rangle &= \langle 0|\Psi^{\dagger}(I) = 0, \qquad \Psi^{\dagger}(I)|0\rangle = |I\rangle. \\ G_{\lambda}^{(0)}(I_{1}, I_{2}; t) &= \langle 0|\Psi(I_{2})e^{-t\hat{H}_{0}}\Psi^{\dagger}(I_{1})|0\rangle \\ \hat{H}_{0} &= \int_{0}^{\infty} \frac{dI}{I} \Psi^{\dagger}(I)H_{0}(I)\Psi(I) \end{aligned}$ 



$$\hat{H} = \hat{H}_0 - G_s \int dl_1 \int dl_2 \Psi^{\dagger}(l_1) \Psi^{\dagger}(l_2) \Psi(l_1 + l_2)$$

$$-\alpha G_s \int dl_1 \int dl_2 \Psi^{\dagger}(l_1 + l_2) \Psi(l_2) \Psi(l_1) - \int \frac{dl}{l} \rho(l) \Psi(l),$$

Generalized disk amplitudes (Hartle-Hawking wave functions):

$$w(\ell_1,\ldots,\ell_n) \equiv \lim_{t\to\infty} \langle 0|e^{-t\hat{H}}\Psi^{\dagger}(\ell_1)\cdots\Psi^{\dagger}(\ell_n)|0\rangle_{connect}$$

(日) (四) (三) (三) (三) (三)

## Unfinished stuff

For the ordinary matrix models we have a description of conformal matter coupled to 2d quantum gravity by multicritical one-matrix models and by two-matrix models.

It would be interesting to define the new scaling limit in these cases provided it is simple. If the exponents are the same as in flat space, it would the provide a simple realization of these critical systems.

・ロ・・ (日・・ ほ・・ (日・)