

Real-Space Condensation and Extreme Value Statistics

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November 20, 2010

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Plan:

I Factorised Steady States in Mass Transport Models

- Zero-range process (**ZRP**) as a fundamental model
- Factorised Steady State (**FSS**)
- General Mass Transport Model

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- Nature of the condensate ; physical examples
- Connection to extreme value statistics and large deviations
- Condensation in polydisperse hard spheres

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References:

- T Hanney and M.R. Evans, J. Phys. A 2005
M.R.Evans, S.N. Majumdar, RKP Zia, J. Phys. A 2004, Phys. Rev. Lett.
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M.R.Evans, S.N. Majumdar, I Pagonabarraga, E Trizac J. Chem. Phys 2010

Mass Transport Models

Conserved quantity “**Mass**” transferred **stochastically** from site to site of some lattice or network according to (local) dynamical rules

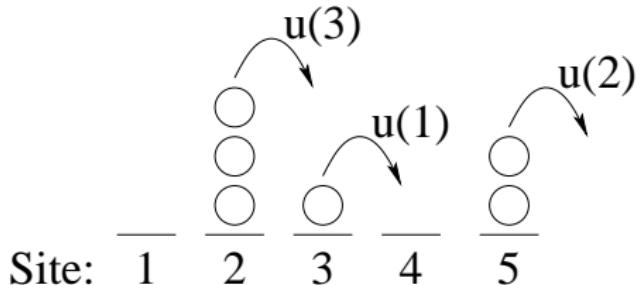
Examples

- Traffic — vehicular, granular, biological
- Rewiring networks — links transferred between nodes
- Econophysics — wealth transferred between agents

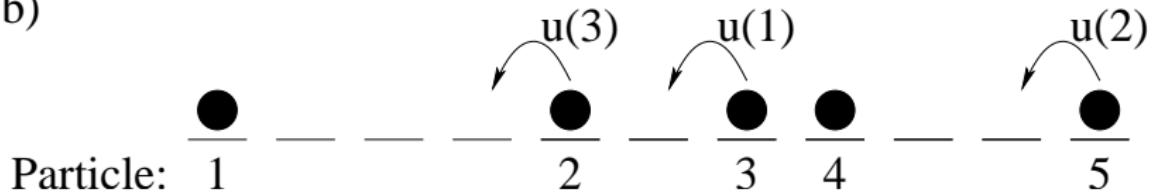
Dynamics leads to **non-equilibrium stationary state**

Zero Range Process

a)



b)



- a) “balls-in-boxes” picture
- b) “Exclusion Process” picture

Motivation for ZRP

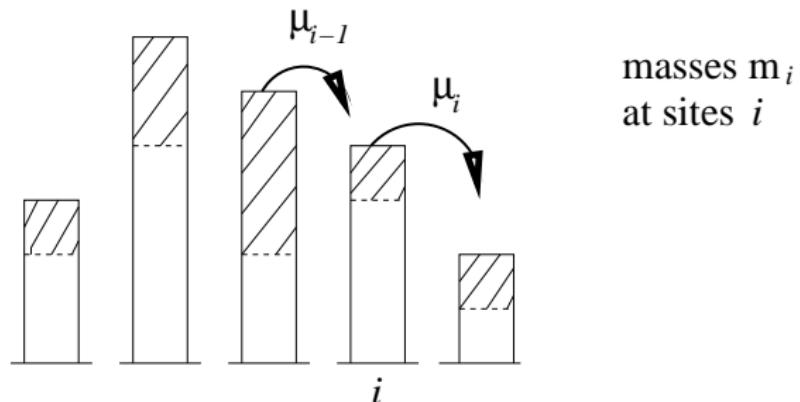
- Specific physical systems map onto ZRP
 - e.g. polymer dynamics, sandpile dynamics, traffic flow
- Effective description of dynamics involving exchange between domains
 - e.g. phase separation dynamics
- Factorised Steady State

$$P[m_1, \dots, m_L] = \frac{1}{Z_{N,L}} f(m_1) \dots f(m_L) \delta\left(\sum_i m_i - N\right)$$

where

$$f(m) = \prod_{n=1}^m \frac{1}{u(n)}$$

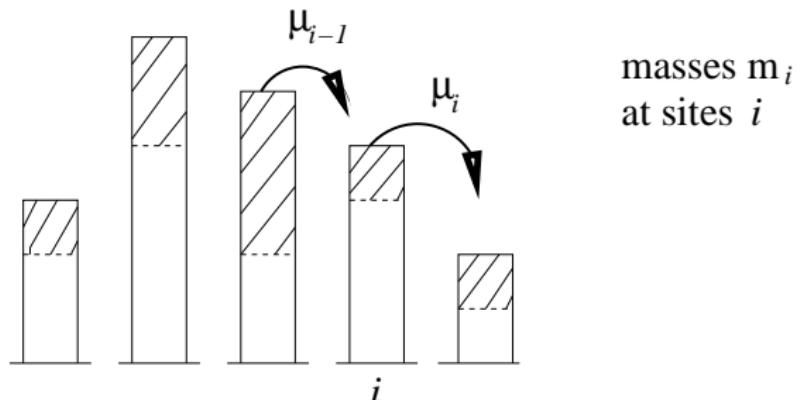
General Mass Transport Model



In a timestep: $m_i(T+1) = m_i(T) - \mu_i(T) + \mu_{i-1}(T)$

$\phi(\mu|m)$ is the distribution of μ , the mass transferred

General Mass Transport Model



masses m_i
at sites i

$$\text{In a timestep: } m_i(T+1) = m_i(T) - \mu_i(T) + \mu_{i-1}(T)$$

$\phi(\mu|m)$ is the distribution of μ , the mass transferred

A **necessary and sufficient** condition for a **factorised steady state** is
(EMZ J. Phys. A 2004)

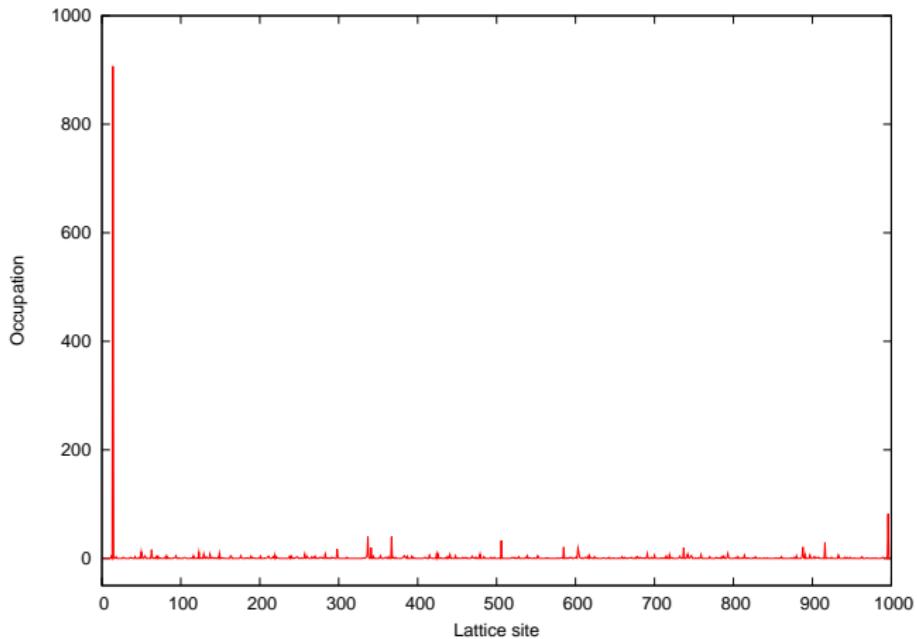
$$\phi(\mu|m) = \frac{v(\mu)\omega(m-\mu)}{[v * \omega](m)}$$

Then

$$f(m) = [v * \omega](m) = \int_0^m d\mu v(\mu)\omega(m-\mu)$$

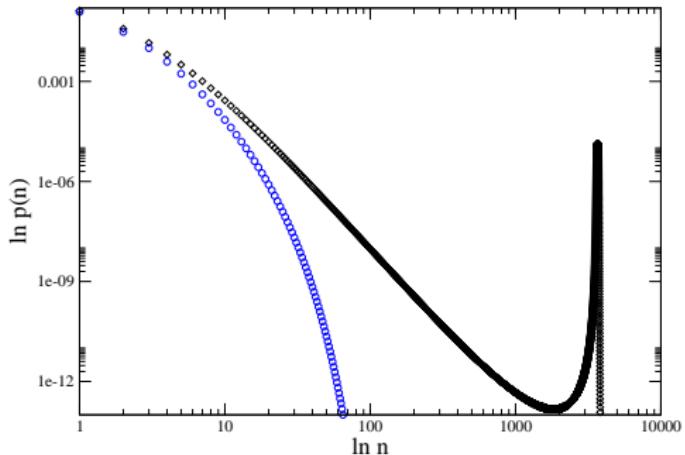
Real Space Condensation

Snapshot of ZRP $u(m) = 1 + \frac{3}{m}$



Real Space Condensation

Single-site mass distribution in ZRP $u(m) = 1 + \frac{5}{m}$



below critical density
above critical density

Real Space Condensation in ZRP

Grand Canonical Ensemble: $p(m) \sim z^m f(m)$ $z < 1$ z is fugacity

Constraint: $\sum_{m=0}^{\infty} mp(m) = \rho$ i.e. density ρ as function of z

If $u(m) = 1 + \frac{\gamma}{m}$ \Rightarrow

$$f(m) \sim m^{-\gamma}$$

Then $z \rightarrow 1$ gives the max allowed value of density

$$\rho \rightarrow \infty \quad \text{if } \gamma \leq 2$$

$$\rho \rightarrow \rho_c < \infty \quad \text{if } \gamma > 2$$

Thus for $\gamma > 2$ we have condensation if $\rho > \rho_c$

Bialas, Burda and Johnston 1997

Summary of Real Space Condensation

For $u(m) = 1 + \frac{\gamma}{m}$ we have $f(m) \sim m^{-\gamma}$.

Then for $\gamma > 2$

- At low density $\rho < \rho_c$ the system is in **fluid phase**, within GCE

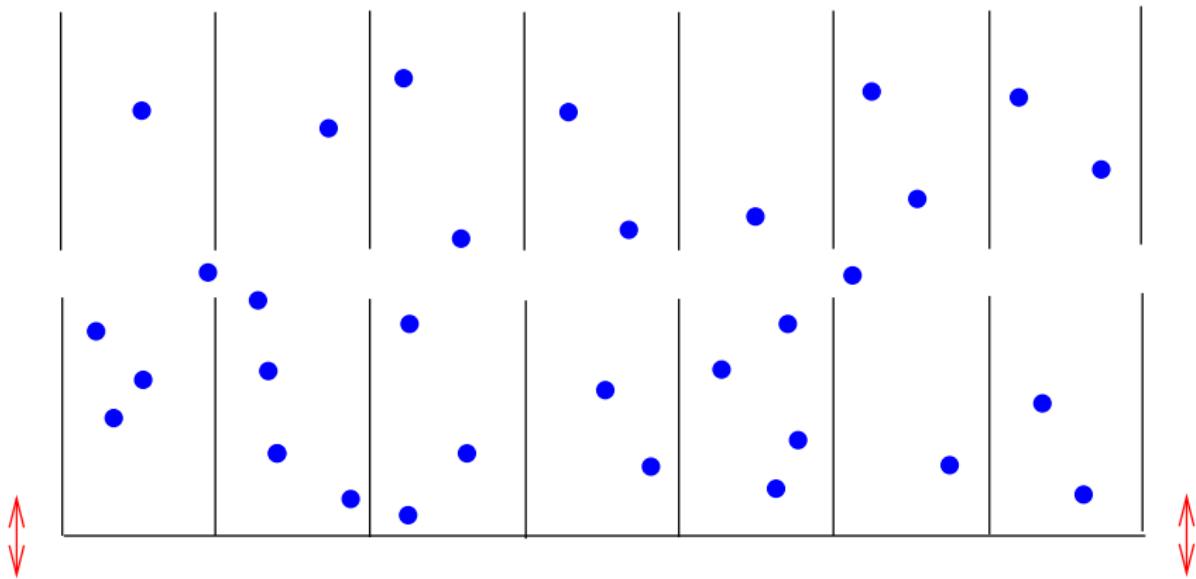
$$p(m) \sim z^m f(m) \quad z < 1$$

- at critical density $\rho = \rho_c$, $p(m) \sim f(m) \sim m^{-\gamma}$ critical fluid
- for $\rho > \rho_c$ we have **condensate coexisting with critical fluid**
(need to analyse within CE)

Thus a finite fraction of the 'mass' condenses at a single spatial position
The position of the condensate is spontaneously selected

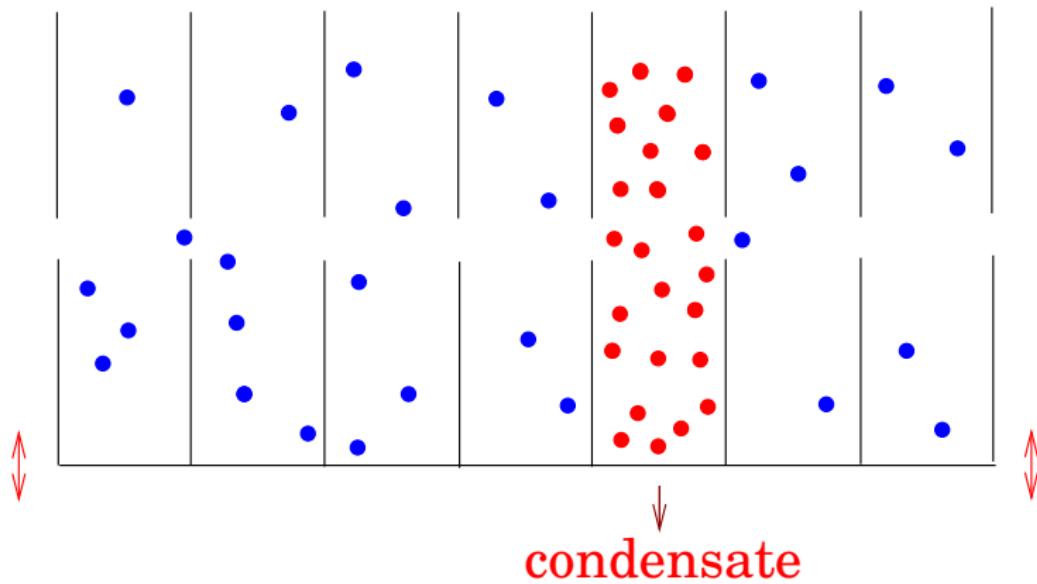
Shaken Granular System

Shaken granular system: ([van der Meer et. al., 2007](#))



Condensation in Shaken Granular System

Shaken granular system: (van der Meer et. al., 2007)



Physical Systems with Real-space Condensation:

- Traffic and Granular flow (O'Loan, Evans, Cates, '98)
- Cluster Aggregation and Fragmentation (Majumdar et al 1998)
- Granular clustering (van der Meer et al, 2000)
- Phase separation in driven systems (Kafri et al, 2002).
- Socio-economic contexts: company formation, city formation, wealth condensation etc. (Burda et al, 2002)
- Networks (Dorogovstev & Mendes, 2003,...)
- ...

Nature of the Condensate

Canonical partition function:

$$Z(N, L) = \int_0^\infty \left[\prod_i^L dm_i f(m_i) \right] \delta \left(\sum_j^L m_j - N \right)$$

let $\int_0^\infty dm f(m) = 1$ then

$Z_{L,N}$ = probability that a walker taking positive jumps with distribution $f(m)$ reaches N after L jumps

Consider $\int_0^\infty dm m f(m) \equiv \mu_1 < \infty$ i.e. $f(m) \sim m^{-\gamma}$ with $\gamma > 2$ then

if $L\mu_1 > N$ jumps all typically $O(1)$ Fluid $\rho < \mu_1$

if $L\mu_1 < N$ one jump is $O(N - L\mu_1)$ Condensate $\rho > \mu_1$

Condensate shows up in a large deviation of a sum of random variables

Why only one condensate?

Consider $f(m) \simeq Am^{-\gamma}$ and total mass $N = L\mu_1 + M_{ex}$ with $M_{ex} = O(L)$.

Weight of one-condensate configurations:

$$Lf(M_{ex}) \sim L^{1-\gamma}$$

Weight of two-condensate configurations:

$$\binom{L}{2} \int dm f(m)f(M_{ex} - m) \sim L^{3-2\gamma}$$

For $1 - \gamma > 3 - 2\gamma \Rightarrow \gamma > 2$ one-condensate configurations dominate

Canonical calculation of p_{cond}

$$Z(N, L) = \int_0^\infty \left[\prod_i^L dm_i f(m_i) \right] \delta \left(\sum_j^L m_j - N \right)$$

$$p(m) = f(m) \frac{Z(N-m, L-1)}{Z(N, L)}$$

Take Laplace Transform

$$Z(s, L) = \int_0^\infty dN e^{-sN} Z(N, L) = [g(s)]^L$$

where

$$g(s) = \int_0^\infty dm e^{-sm} f(m)$$

So need to invert Laplace transform

$$Z(N, L) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds g(s)^L e^{sN}$$

Canonical calculation II

Saddle point exists if $\rho \leq \mu_1$ in which case we recover grand canonical results

Otherwise need to evaluate contour integral

$$Z(N, L) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \exp(L \ln g(s) + sN)$$

Dominate contributions comes from $s \simeq 0$ so expand

$$g(s) \simeq 1 - \mu_1 s + A\Gamma(1 - \gamma)s^{\gamma-1} \quad 3 > \gamma > 2 \dots$$

$$g(s) \simeq 1 - \mu_1 s + \frac{\mu_2}{2}s^2 + \dots + A\Gamma(1 - \gamma)s^{\gamma-1} + \dots \quad \gamma > 3$$

Position of non-analytic term in the expansion gives different behaviours for $3 > \gamma > 2$ and $\gamma > 3$

Results for condensate bump scaling laws

$$3 > \gamma > 2$$

$$p_{\text{cond}} \simeq \frac{1}{L} \frac{1}{L^{1/(\gamma-1)}} V_\gamma(z) \quad z = \frac{(m - M_{\text{ex}})}{L^{1/(\gamma-1)}}$$

$$V_\gamma = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \exp(-zs + A\Gamma(1-\gamma)s^{\gamma-1})$$

strongly asymmetric

$$\gamma > 3$$

$$p_{\text{cond}} \simeq \frac{1}{L} \frac{1}{\sqrt{2\pi\Delta^2 L}} \exp\left(-\frac{z^2}{2\Delta^2}\right) \quad z = \frac{(m - M_{\text{ex}})}{L^{1/2}}$$

gaussian

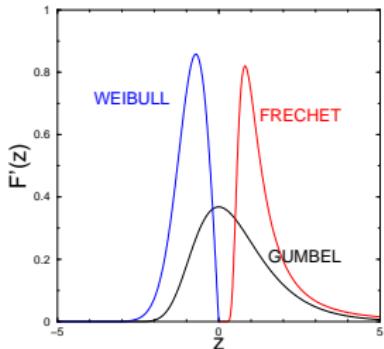
Condensation and Extreme Value Statistics (EVS)

Recall EVS for *independent* random variables

Q. what is the distribution $Q_N(x)$ of the largest variable amongst N drawn from a distribution $f(m)$?

A. For N and x large $Q_N(x) \rightarrow F[(x - a_N)/b_N]$ where there are 3 universal forms for $F(z)$:

Gumbel when $f(m)$ decays faster than a power law; **Frechet** when $f(m)$ decays as a power law; **Weibull** distribution when m is bounded



Extreme value statistics in a factorised steady state

For a **factorised steady state** the random site masses m_i are *correlated* random variables due to the global constraint of conserved total mass

What is the distribution of largest mass?

- for $\rho < \rho_c$ **Gumbel** distribution
- at criticality $\rho = \rho_c$ **Frechet** distribution
- condensed phase $\rho > \rho_c$ **different** distributions $p_{\text{cond}}(m)$ - the condensate 'bump'

Generalisations

- Pair-Factorised Steady States

$$P(\{m_i\}) \propto \prod_i g(m_i, m_{i+1}) \delta\left(\sum_i m_i - M\right)$$

can lead to **spatially extended condensates**

(Evans, Hanney, Majumdar Phys. Rev. Lett. 2006; Waclaw et al Phys. Rev. Lett. 2009

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- **Multiple Condensates** - suppress a single condensate by non-conserving dynamics or non-monotonic hop rates e.g.

$$u(m) = 1 + \frac{b}{m} + c \left(\frac{m}{L}\right)^k$$

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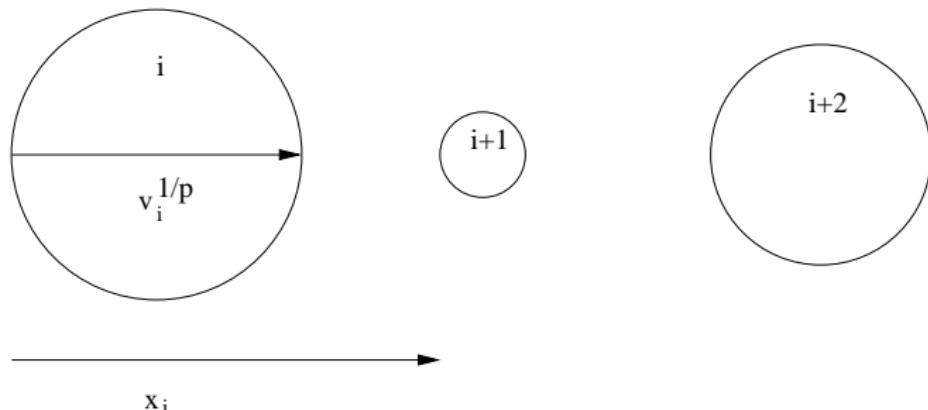
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- **Entropy driven condensation** in polydisperse hard spheres - when spheres diffuse and exchange volume the formation of a condensate (one large sphere) maximises the entropy
(Evans, Majumdar, Pagonabarraga, Trizac J. Chem. Phys. 2010)

Polydisperse hard p-spheres



'p-sphere on a line' volume $v = \ell^p$

rods $p = 1$

discs $p = 2$

⋮ ⋮

Tonks gas $p \rightarrow \infty$ ($\ell \rightarrow 1 \ \forall i$)

Dynamics: Diffusion and volume exchange.

Constraints: $\sum_i x_i = L$ $\sum_i v_i = V$ $x_i \geq v_i^{1/p}$

Steady State: All configurations equally likely

Entropy Driven Condensation

Grand Canonical

$$p(x, v) = \frac{e^{-\lambda x - sv} \theta(x - v^{1/p})}{G(\lambda, s)}$$

$$G(\lambda, s) = \frac{1}{\lambda} \int_0^\infty dv e^{-\lambda sv - \lambda v^{1/p}}$$

Marginals

$$p(v) = \frac{e^{-sv - \lambda v^{1/p}}}{\lambda G(\lambda, s)}$$

$$p(x) = \frac{e^{-\lambda x} [1 - e^{-sx^p}]}{sG(\lambda, s)}$$

Constraints:

$$\frac{1}{\rho} = \frac{\bar{L}}{N} \quad \text{length per particle}$$

$$\phi = \frac{\bar{V}}{N} \quad \text{volume per particle}$$

Entropy Driven Condensation

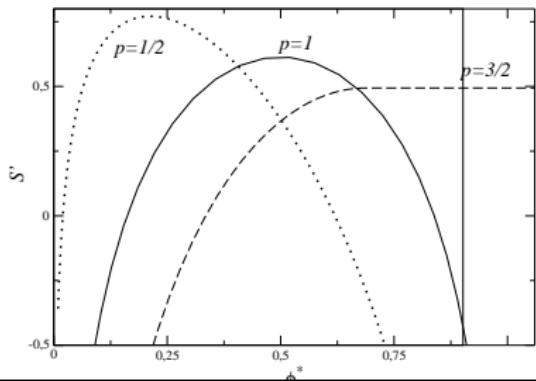
For $p \leq 1$ all volume fractions ϕ can be achieved

For $p > 1$, for given density ρ maximum volume fraction ϕ^* is achieved when $s \rightarrow 0$ then

$$p(v) \rightarrow \frac{\lambda^p}{\Gamma[1+p]} e^{-\lambda v^{1/p}} \quad \text{where} \quad \lambda = \rho(1+p)$$
$$\Rightarrow \phi_{\max} = \frac{1}{2} \langle x^p \rangle$$

Condensation when volume per particle = half available volume.

By forming a condensate, the entropy remains maximal as ϕ increases



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- Understanding in terms of large deviations and extreme value statistics