Long-range percolation on the hierarchical lattice

Pieter Trapman

Department of Mathematics, Stockholm University

Nordita Workshop,
Random Graphs and Networks in Mathematics and Physics,
Stockholm, 1 November 2010

joint work with
Slavik Koval (Utrecht University)
Ronald Meester (VU University Amsterdam)

arXiv:1004.1251v1

Epidemiological "justification"

- Consider a well organized population, where everybody lives in a household of N individuals,
- Every family lives in a N-floors apartment building, at each floor are N apartments,
- N of those apartment buildings are served by one supermarket
- etc.
- The frequency that people in the same household meet is higher than the frequency that people on the same floor, but not in the same household meet,
  which in turn is higher than the frequency at which people from the same building but from different floors meet etc.

Model

The vertices are "the leaves of an infinite regular N-tree", and the distance between vertices is the distance to their "most recent common ancestor".

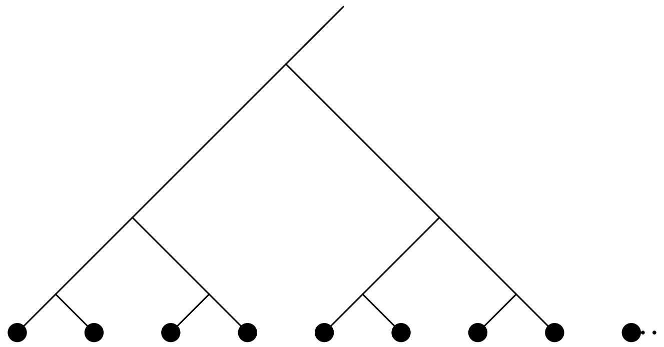


Figure: Hierarchical lattice of order 2 (the ultimate points) with the metric generating tree attached.

- Formal definition of hierarchical lattice of order N:

$$\Omega_N := \left\{ \mathbf{x} = (x_1, x_2, \dots) : x_i \in \{0, 1, \dots, N-1\}, \sum_{i=1}^{\infty} x_i < \infty \right\}$$

- Labeling by non-negative integers: $f(x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i N^{i-1}$
- Distance on Ω_N

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\} & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$$

For $x \in \Omega_N$, define $\mathcal{B}_r(x)$ to be the ball of radius r around x.
Some properties:

1. (Ω_N, d) is ultrametric: It satisfies the strengthened version of the triangle inequality
$$d(x, y) \leq \max(d(x, z), d(z, y))$$
for any triple $x, y, z \in \Omega_N$
2. $\mathcal{B}_r(x)$ contains N^r vertices for any x
3. For every $x \in \Omega_N$ there are $(N-1)N^{k-1}$ vertices at distance k
4. If $y \in \mathcal{B}_r(x)$ then $\mathcal{B}_r(x) = \mathcal{B}_r(y)$
5. Either $\mathcal{B}_r(x) = \mathcal{B}_r(y)$ or $\mathcal{B}_r(x) \cap \mathcal{B}_r(y) = \emptyset$

- SIR epidemic with fixed infectious period of length 1 on this network
- After infectious period infectious individual recovers and stay immune forever
- If an infectious person meets a susceptible person, the susceptible one becomes infectious: They share an edge in the *infection graph*
- Individuals at distance k meet according to Poisson process with intensity $\lambda(k) = \alpha/\beta^k$.
- Cluster of ultimately infected individuals is distributed as the cluster around the initial infected individual (the origin) in "long-range percolation" with $p(k) = 1 - e^{-\lambda(k)}$.

- Hierarchical lattice of order N, denoted by Ω_N
- Presence of absence of (undirected) edges between different pairs of vertices are independent
- Connection probability of vertices at distance k is $p_k = 1 - \exp[-\alpha/\beta^k]$ ($\approx \alpha/\beta^k$ for large k)
- $\mathcal{C}(x)$ is the cluster (connected component) of vertex x and $|\mathcal{C}(x)|$ is its size
- All vertices are "the same", so we may consider $\mathcal{C} = \mathcal{C}(0)$, without loss of generality
- $\mathbb{P}_{\alpha,\beta}$ is the probability measure corresponding to long-range percolation with parameters α and β
- $\theta(\alpha, \beta) := \mathbb{P}_{\alpha,\beta}(|\mathcal{C}| = \infty)$
- For $\mathcal{S}_1, \mathcal{S}_2 \subset \Omega_N$, $\mathcal{S}_1 \leftrightarrow \mathcal{S}_2$ denotes the presence of an edge between the two sets

Regime $\beta \leq N$

Theorem

If $\beta \leq N$, then $\theta(\alpha, \beta) = 1$.

Almost trivial: By

$$\sum_{k=1}^{\infty} (N-1)N^{k-1}(1 - \exp[-\alpha/\beta^k]) = \infty$$

the origin is almost surely connected to infinitely many vertices

Regime $\beta \geq N^2$

Theorem

If $\beta \geq N^2$, then $\theta(\alpha, \beta) = 0$.

Proof for $\beta = N^2$:
Proof relies on the fact that for each k, the probability that "ball of diameter k around 0 is not connected to its complement" is bounded away from 0.

$$\begin{aligned} \mathbb{P}(\mathcal{B}_k(0) \not\leftrightarrow \overline{\mathcal{B}_k(0)}) &= \exp\left(-\alpha N^k \frac{(N-1)}{N^2} \sum_{j=1}^{\infty} \frac{N^{j+k-1}}{N^{2(j+k-1)}}\right) \\ &= \exp\left(-\frac{\alpha}{N}\right) > 0, \end{aligned}$$

So, this event will eventually happen and therefore $\theta(\alpha, \beta) = 0$

Hierarchical lattice Results: Regimes Results: Uniqueness Results: Continuity

Regime $N < \beta < N^2$

Theorem

If $N < \beta < N^2$, then $0 < \alpha_c(\beta) := \inf\{\alpha; \theta(\alpha, \beta) > 0\} < \infty$

Lower bound follows by coupling with branching process
Upper bound: brute force and renormalization:

Pieter Trapman Stockholm University
Long-range percolation on the hierarchical lattice

Upper bound: brute force and renormalization:

- Choose η and K such that $\sqrt{\beta} < \eta \leq (N^K - 1)^{1/K}$ this is possible since $\sqrt{\beta} < N$
- A ball of radius nK is *good* if the largest connected component contained in it (say $\mathcal{C}^m_{nK}$) has size at least η^{nK}

$$s_n := \mathbb{P}\left(|\mathcal{C}^m_{nK}| \geq \eta^{nK}\right)$$

- Probability that two good clusters of radius nK at distance $(n+1)K$ share an edge is at least

$$1 - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)$$

- s_{n+1} is bounded below by the probability that $N^k - 1$ out of the N^k balls of radius nK in $\mathcal{C}_{(n+1)K}$ are good, and the good clusters are all connected to the the first of the good clusters
- This probability is larger than

$$1 - \binom{N^K}{2}\left(1 - s_n - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$$

- That is: $1 - s_{n+1} < \binom{N^K}{2}\left(1 - s_n - \exp\left(-\frac{\alpha}{\beta^K}\left(\frac{\eta^2}{\beta}\right)^{nK}\right)\right)^2$
- By induction we can show that for α large enough and all $n > 1$, $1 - s_n < \gamma^n$, for $0 < \gamma$ small
- This implies that a large ball is with high probability good
- Proving that with positive probability the origin is contained in a large good cluster for all large enough n can be done along the same lines

Uniqueness of infinite component

Theorem

The infinite component for supercritical long-range percolation on the hierarchical lattice is almost surely unique.

Use

Theorem (Gandolfi, Keane and Newman (1992))

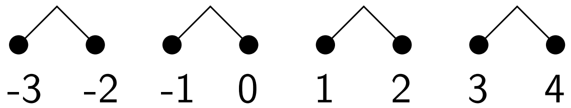
If a supercritical long-range percolation measure on $\mathbb{Z}^d$ is translation invariant and satisfies a finite energy condition, then the infinite component is almost surely unique.

The finite energy condition is that the configuration of edges on $\Omega_N \times \Omega_N \setminus e$, does almost surely not determine whether edge e is present or absent.

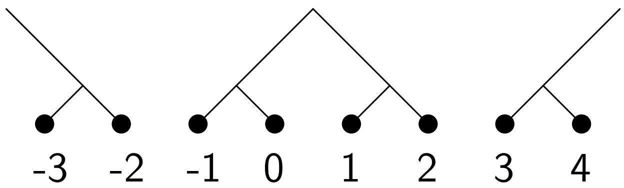
Problem: We do not consider percolation on $\mathbb{Z}^d$.

Idea of solution: Construct random projection of "distance generating tree" in $\mathbb{Z}$, which is translation invariant (even ergodic)

Construction for $N = 2$:
Step 1: ($n \in \mathbb{Z}$) Flip a fair coin: if heads, then $2n$ has distance 1 to $2n+1$, if tails, then $2n$ has distance 1 to $2n-1$.



Step k Flip a fair coin: if heads, then $2^k n$ has distance k to $2^{k-1}(2n+1)$, if tails, then $2^k n$ has distance k to $2^{k-1}(2n-1)$.



Finite energy condition is satisfied and by construction the percolation measure is translation invariant

Continuity of $\theta(\alpha, \beta)$

Theorem

The percolation probability $\theta(\alpha, \beta)$ is continuous for $\alpha, \beta > 0$.

Proof of continuity from the right (resp. left) in α (resp. β) is standard.

Proof of continuity from the left (resp. right) in α (resp. β) is involved. The ideas of the proof are similar to ideas used by Noam Berger (2002).
We need an intermediate lemma.

Lemma

The fraction of vertices in the largest component of long-range percolation graph restricted to $\mathcal{B}_k$ is for large k close to θ, with high probability.

Idea of proof:

1. For every constant $K > 0$ the indicator function of the event that both $|\mathcal{C}(0)| = \infty$ and $|\mathcal{C}_n(0)| < K(\beta/N)^n$ converges a.s. to 0 as $n \to \infty$. (straightforward computation)
2. The fraction of the vertices in $\mathcal{B}_n(0)$ which are in a cluster of size at least $K(\beta/N)^n$, converges a.s. to θ as $n \to \infty$.(ergodicity)
3. Combine the previous two steps: The large clusters at level n, are with high probability all in the same cluster at level $n+1$

Theorem

$\theta(\alpha_c(\beta), \beta) = 0$ *for* $N < \beta < N^2$.

Idea of the proof:

- Assume $\theta := \theta(\alpha, \beta) > 0$: The density of the largest component of random graph restricted to large sub-ball is close to θ, with high probability
- Since subballs are finite, the density of largest cluster in large subball using $\alpha-$ and $\beta+$ is also close to θ
- Rescaled process at level K has parameters $\beta+$ and $\alpha \approx (\alpha-)\theta^2(N^2/\beta)^K$, which can be taken arbitrary large, by choosing K large enough
- So, there is also percolation for $\alpha-$ and $\beta+$, with density arbitrary close to θ

Continuity of $\alpha_c(\beta)$

Theorem

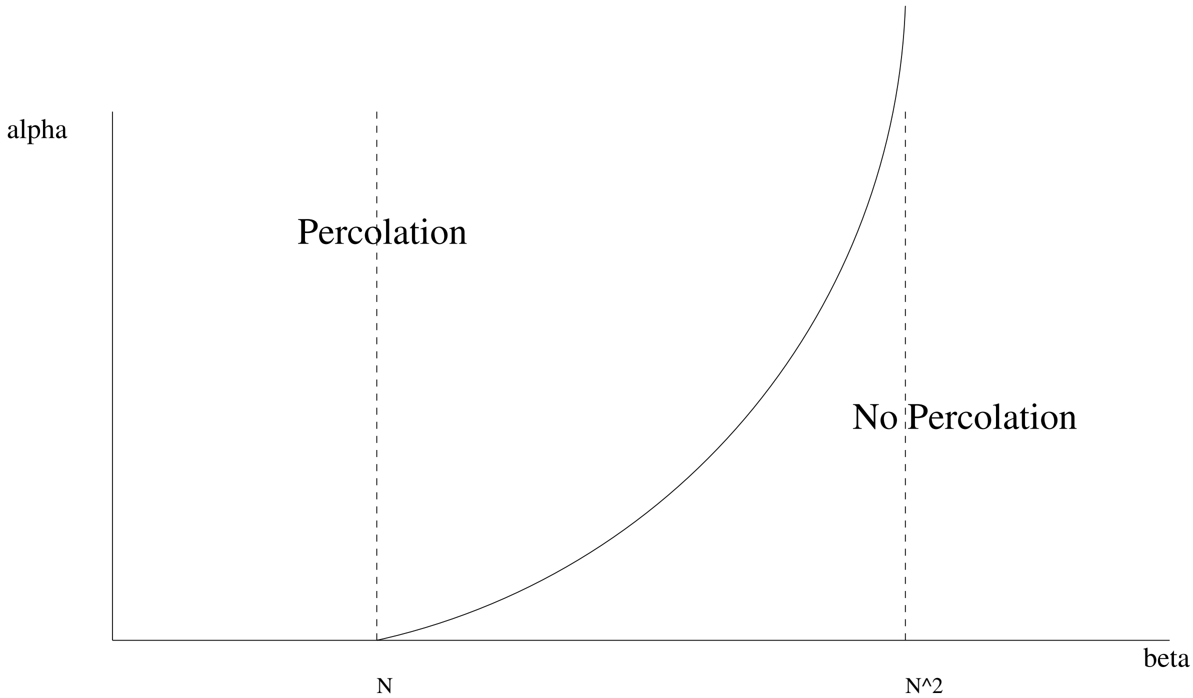
$\alpha_c(\beta)$ is continuous on $\beta \in (0, N^2)$ and strictly increasing on $\beta \in [N, N^2)$. Furthermore, $\alpha_c(\beta) \nearrow \infty$ for $\beta \nearrow N^2$.

The proof relies on the result by Aizenman and Barsky (1987) that for independent long-range percolation on $\mathbb{Z}^d$:

$$\inf\{\alpha : \theta(\alpha, \beta) > 0\} = \sup\{\alpha : \mathbb{E}_{\alpha,\beta}(|\mathcal{C}(0)|) < \infty\}$$

Close inspection of their proof shows that this result also holds for independent long-range percolation on the hierarchical lattice.

- By continuity of θ: If $\theta(\alpha, \beta) > 0$, then $\theta(\alpha-, \beta+) > 0$. This implies continuity from the right of $\alpha_c(\beta)$
- If $\mathbb{E}_{\alpha,\beta}(|\mathcal{C}(0)|) < \infty$, then (after some work:) $\mathbb{E}_{\alpha+,\beta-}(|\mathcal{C}(0)|) < \infty$
  This implies continuity from the left of $\alpha_c(\beta)$



M. Aizenman and D.J. Barsky.
Sharpness of the phase transition in percolation models.
Comm. Math. Phys., 108(3):489–526, 1987.

N. Berger.
Transience, recurrence and critical behavior for long-range percolation.
Comm. Math. Phys., 226(3):531–558, 2002.

A. Gandolfi, M.S. Keane, and C.M. Newman.
Uniqueness of the infinite component in a random graph with applications to percolation and spin glasses.
Probab. Theory Related Fields, 92(4):511–527, 1992.