Information Geometry and the ASEP

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Random Graphs and Extreme Value Statistics 22-24 Nov

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Apologies

- This talk has nothing to do with random graphs ...
- This talk has nothing to do with extreme value statistics ...

Plan of Talk

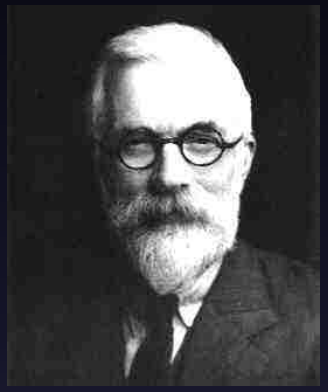
- A bit of Information Geometry
- A bit of ASEP
- A bit of both

A Bit of Information Geometry:

- Fisher, Rao
- Amari, Barndorff-Nielsen
- Ruppeiner, Weinhold
- Janyszek, Mrugała

Rogues gallery

- Fisher, Rao and Amari



- Barndorff-Nielsen, Ruppeiner and Weinhold



Entropy, metric

- Entropy

$$S = -\sum_{\alpha} p_\alpha \ln\left(p_\alpha\right)$$

- Relative Entropy

$$G(p|r) = -\sum_{\alpha} p_\alpha \ln\left(\frac{p_\alpha}{r_\alpha}\right)$$

- Induced metric

$$dl^2 = G\left(p(\theta)|p(\theta+\delta\theta)\right)$$

Entropy, metric II

- Metric

$$dl^2 = \frac{\partial^2 G}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j$$

- Gibbs distribution

$$p_\alpha = \frac{1}{Z}\exp(-\theta_i \Phi_i), \rightarrow S = \langle \theta_i \Phi_i \rangle + \ln Z$$

- (Fisher-Rao) Metric for this

$$dl^2 = \frac{\partial^2 \ln Z}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j$$

(Fisher) Information

- One parameter

$$I(\theta) = \left\langle -\frac{\partial^2}{\partial\theta^2} \log Z \right\rangle$$

- "Classical" uncertainty relation: Cramer-Rao

$$I(\theta) Var(\hat{\theta}) \geq 1$$

- $\hat{\theta}$ any unbiased estimator of θ

Statistical mechanics

- Does the Fisher-Rao (or a similar) metric tell you anything about statistical mechanical systems?
- Fisher-Rao:

$$dl^2 = \frac{\partial^2 F}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j$$

- Ruppeiner:

$$dl^2 = \frac{\partial^2 S}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j$$

- Weinhold:

$$dl^2 = \frac{\partial^2 U}{\partial\theta_i \partial\theta_j} d\theta_i d\theta_j$$

What do we measure

- Geometric formulation

  Co-ordinate free quantities?

- Curvature?

Some examples: Ideal Gas

- Ideal Gas

$$Z(\alpha,\beta) = \frac{1}{N!h^{3N}} \int_0^\infty \left(\int_{\mathbf{\Gamma}} \exp\left(-\beta H\right) \mathrm{dqdp} \right) \exp\left(-\alpha V\right) \mathrm{d}V.$$

With Hamiltonian $H = \sum_{i=1}^{N} \frac{\mathrm{p}_i^2}{2m}$

- With $\beta = 1/k_B T$, $\alpha = P/k_B T$
- Evaluate integrals

$$Z(\alpha,\beta) = \left(\frac{2\pi m}{h^2 \beta}\right)^{3N/2} \alpha^{-(N+1)}$$

Some examples: Ideal Gas II

- Calculate a free energy

$$f(\alpha,\beta) = \lim_{N\to\infty}\left[N^{-1}\ln Z(\alpha,\beta)\right] = \frac{3}{2}\ln\frac{2\pi m}{h^2\beta} - \ln\alpha$$

  And a (Fisher-Rao) metric

$$G_{ij} = \begin{pmatrix} \alpha^{-2} & 0 \\ 0 & \frac{3}{2}\beta^{-2} \end{pmatrix}.$$

- $\tilde{\alpha} = \ln\alpha\,,\ \tilde{\beta} = \sqrt{(3/2)}\,\ln\beta$
- $\mathcal{R} = 0$
- No interaction $\to$ no curvature

Some examples: 1D Ising

- 1D Ising model in field
- Partition function

$$Z_N = \sum_{\{\sigma\}} \exp\left[\beta \sum_{j=1}^{N} \sigma_j \sigma_{j+1} + h \sum_{j=1}^{N} \sigma_j\right]$$

- Solvable via transfer matrix

Some examples: 1D Ising II

- $Z_N(\beta, h)$ can be conveniently expressed in terms of the transfer matrix

$$V = \begin{pmatrix} V_{++} & V_{+-} \\ V_{-+} & V_{--} \end{pmatrix} \quad = \quad \begin{pmatrix} e^{\beta+h} & e^{-\beta} \\ e^{-\beta} & e^{\beta-h} \end{pmatrix}$$

  Giving $Z_N = Tr(V^N)$
- Diagonalising V gives the eigenvalues

$$\lambda_\pm = e^\beta \left\{ \cosh h \pm \sqrt{\sinh^2 h + e^{-4\beta}} \right\}$$

- And free energy

$$f = (1/N) \ln Z_N = \beta + \ln \left(\cosh h + \sqrt{\sinh^2 h + e^{-4\beta}} \right)$$

Some examples: 1D Ising III

- Use free energy in metric: $f = f(\beta, h)$
- $$\mathcal{R} = \frac{1}{2G^2} \begin{vmatrix} \partial_\beta^2 f & \partial_\beta \partial_h f & \partial_h^2 f \\ \partial_\beta^3 f & \partial_\beta^2 \partial_h f & \partial_\beta \partial_h^2 f \\ \partial_\beta^2 \partial_h f & \partial_\beta \partial_h^2 f & \partial_h^3 f \end{vmatrix},$$

  where $G = \det(G_{ij})$
- $$\mathcal{R}_{\text{Ising}} = 1 + \frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4\beta}}}.$$

Expectations

- Scaling

$$\mathcal{R} \sim \xi^d$$

  Hyperscaling

$$\nu d = 2 - \alpha$$

- Giving...

$$\mathcal{R} \sim |\xi|^{(2-\alpha)/\nu}.$$

Expectations in Ising 1D case

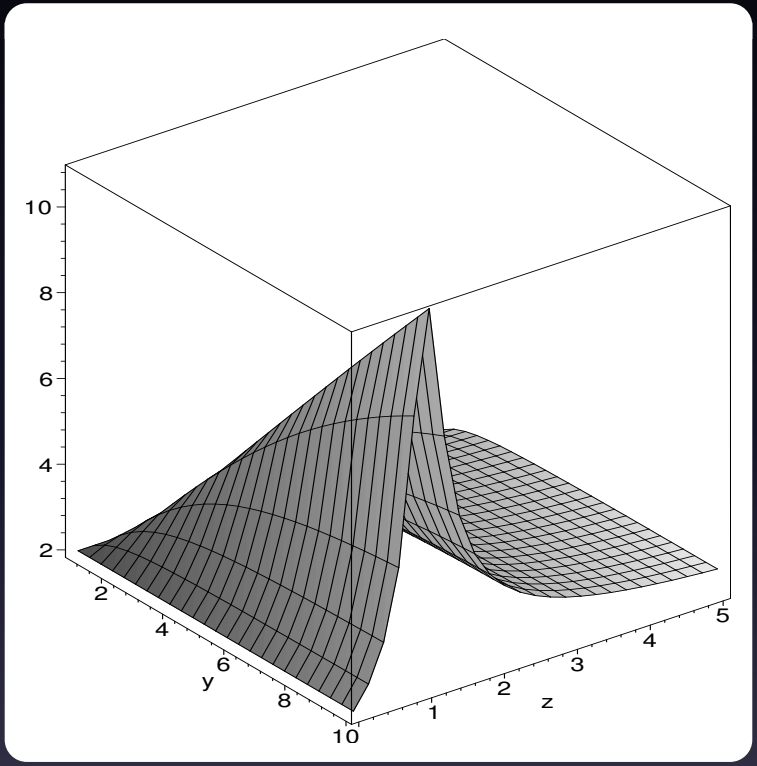
- In Ising case:
- $$\mathcal{R}_{\text{Ising}} = 1 + \frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4\beta}}}.$$
- Correlation length
$$\xi^{-1} = -\ln\left(\tanh(\beta)\right)$$
- $$\xi \sim \exp(2\beta)$$
- Also $\alpha = 1, \nu = 1$
- So expect (and find): $\mathcal{R} \sim |\xi|$
- Transition $\rightarrow$ divergence in $\mathcal{R}$

1D Lee-Yang edge

- Also pick out Lee-Yang edge
- $$\mathcal{R}_{\mathrm{Ising}} = 1 + \frac{\cosh h}{\sqrt{\sinh^2 h + e^{-4\beta}}}.$$

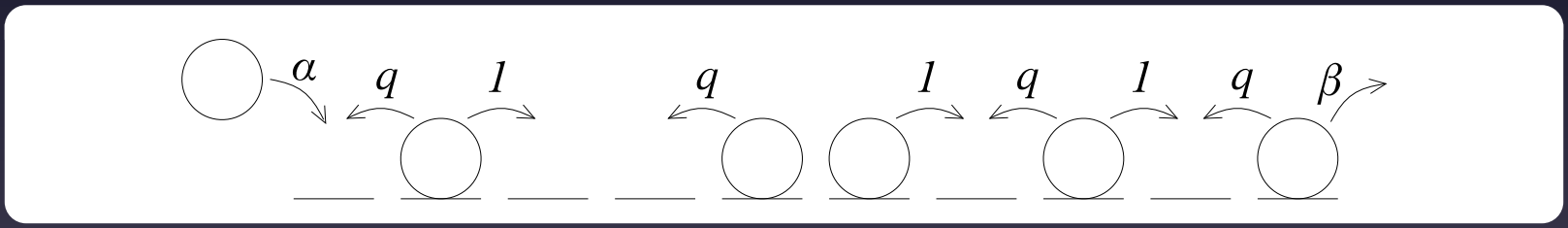
  Edge is given by
  $$\sinh^2 h + e^{-4\beta} = 0$$
- Can be satsified for imaginary h
- $$\sigma = -1/2$$

Gratuitous Picture of $\mathcal{R}$



$$y = \exp(2\beta) \quad z = \exp(h)$$

- A Bit of ASEP:
- One dimensional lattice, N sites
- "Hard" particles
- Forcing



Rogues gallery II

- Derrida and Evans

- TASEP $q = 0$: Derrida, Evans, Hakim, Pasquier
- PASEP $q \neq 0$: Blythe, Evans, Colaiori, Essler
- Schütz, Essler, deGier, Ferrari,...

Setup for solution

- Configuration

$$\mathcal{C}$$

- Statistical weights for configuration

$$f(\mathcal{C})$$

- Normalized probability

$$P(\mathcal{C}) = f(\mathcal{C})/Z$$

- Normalization

$$Z = \sum_{\mathcal{C}} f(\mathcal{C})$$

- Master Equation

$$\frac{\partial P(\mathcal{C},t)}{\partial t} = \sum_{\mathcal{C}' \neq \mathcal{C}} \left[P(\mathcal{C}',t) W(\mathcal{C}' \to \mathcal{C}) - P(\mathcal{C},t) W(\mathcal{C} \to \mathcal{C}') \right]$$

Setup for (Matrix Product) solution

- Represent ball with
$$X_i = D$$
- Represent space with
$$X_i = E$$
- Represent $P(\mathcal{C})$ as
$$P(\mathcal{C}) = \frac{\langle W|X_1 X_2 \ldots X_N|V\rangle}{Z_N}$$
- Make sure behaviour of D, E is compatible with dynamics:
$$\begin{aligned} DE &= D + E \\ \alpha\langle W|E &= \langle W| \\ \beta D|V\rangle &= |V\rangle \end{aligned}$$

Now what?

- Various ways to proceed
- Purely algebraic - "normal order"
- Get a representation of D, E - not unique (and infinite)
- Evaluate $Z_N = \langle W|(D+E)^N|V\rangle = \langle W|C^N|V\rangle$ having done this

The solution

- Calculating

$$Z_N = \langle W|(D+E)^N|V\rangle$$

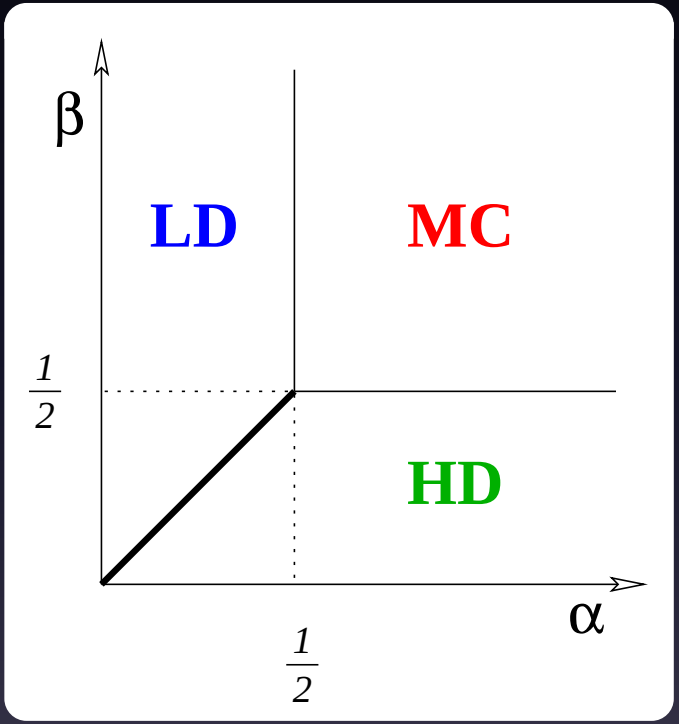
- Gives, for random sequential dynamics

$$Z_N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

- Current is then given by

$$J_N = \frac{Z_{N-1}}{Z_N}$$

The Phase Diagram ($q = 0$)



Behaviour of Z_N

- Can deduce this from

$$\mathcal{Z}(t) = \sum Z_N t^N$$

- $$\mathcal{Z}(t) = \frac{\alpha\beta}{(x(t)-\alpha)(x(t)-\beta)}$$

$$x(t) = \frac{1}{2}\left(1-\sqrt{1-4t}\right)$$

- In maximal current phase with $\alpha < \beta$

$$Z_N \sim \frac{4^N}{\pi^{1/2}N^{3/2}}\left[\frac{1}{(2\alpha-1)^2} - \frac{1}{(2\beta-1)^2}\right]$$

- In HD phase

$$Z_N \sim \frac{\alpha(1-2\beta)}{(\alpha-\beta)(1-\beta)}\frac{1}{(\beta(1-\beta))^N}$$

- A Bit of Both:
-
- Does the Fisher-Rao (or a similar) metric tell you anything about *non-equilibrium* statistical mechanical systems?
-
- Non-equilibrium steady states (NESS)
- ASEP
-

$$Z_N = \sum_{p=1}^{N} \frac{p(2N-1-p)!}{N!(N-p)!} \frac{(1/\beta)^{p+1} - (1/\alpha)^{p+1}}{(1/\beta) - (1/\alpha)}$$

ASEP as a test case

- Z_N "like" a partition function
- Define something "like" a free energy (density)

$$f_N = \frac{1}{N} \ln Z_N$$

- Thermodynamic limit

$$f = \lim_{N \to \infty} \left(\frac{1}{N} \ln Z_N \right)$$

- Use this in metric definition

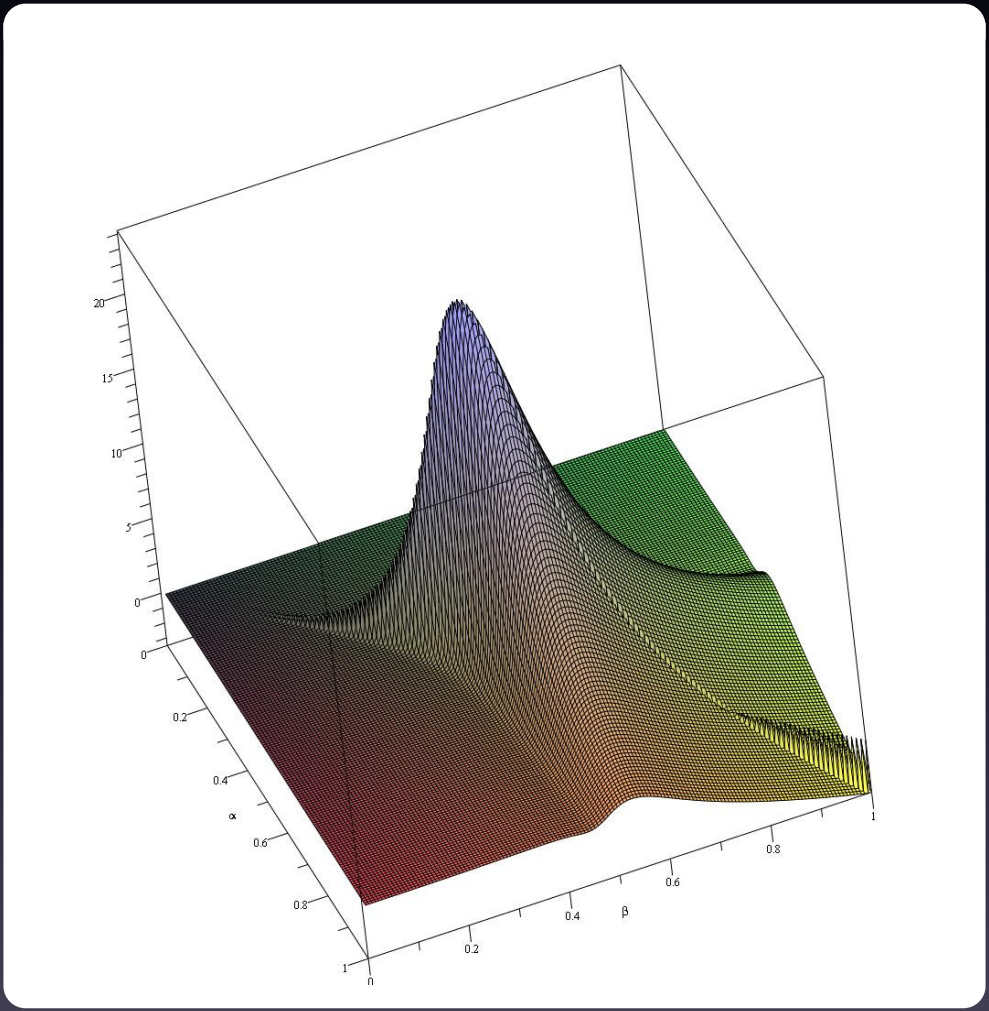
$$dl^2 = \frac{\partial^2 f}{\partial \alpha \, \partial \beta} \, d\alpha \, d\beta$$

Calculating $\mathcal{R}$

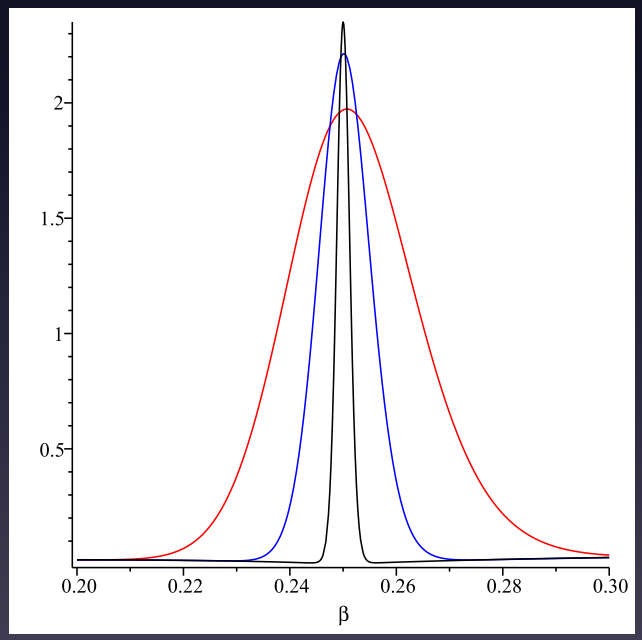
- ASEP: $f = f(\alpha, \beta)$
- $$\mathcal{R} = \frac{1}{2G^2} \begin{vmatrix} \partial_\alpha^2 f & \partial_\alpha \partial_\beta f & \partial_\beta^2 f \\ \partial_\alpha^3 f & \partial_\alpha^2 \partial_\beta f & \partial_\alpha \partial_\beta^2 f \\ \partial_\alpha^2 \partial_\beta f & \partial_\alpha \partial_\beta^2 f & \partial_\beta^3 f \end{vmatrix}$$

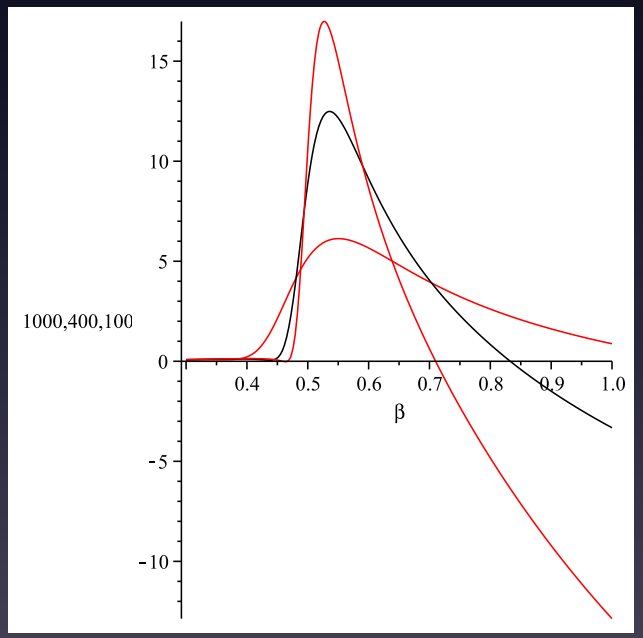
where $G = \det(G_{\alpha,\beta})$

ASEP $\mathcal{R}$



ASEP - slices of $\mathcal{R}$

- Cuts for $N = 100, 400, 1000$
- $\alpha = 1/4$ and $\alpha = 2/3$
- 



ASEP - scaling $\mathcal{R}$

- At peak, $\alpha = \beta = 1/2$, $\mathcal{R} \sim N$
- At corner, $\alpha = \beta = 1$, $\mathcal{R} \sim N$
- Can deduce from

$$\mathcal{Z}(t) = \sum Z_N t^N$$

- $$\mathcal{Z}(t) = \frac{\alpha\beta}{(x(t)-\alpha)(x(t)-\beta)}$$

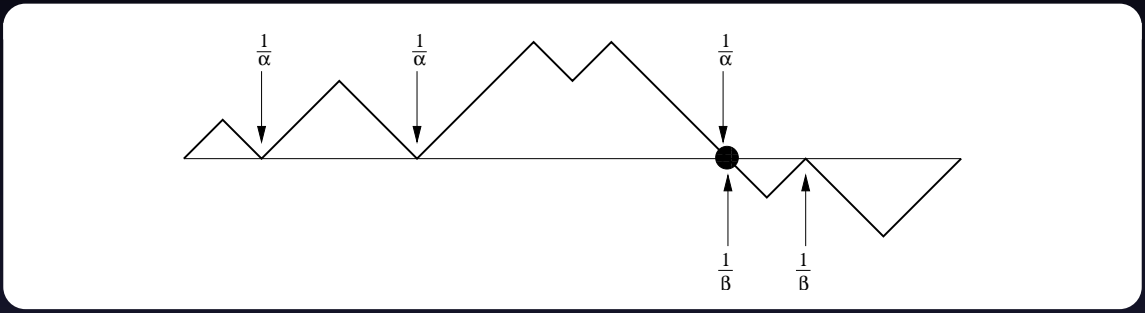
$$x(t) = \frac{1}{2}\left(1 - \sqrt{1-4t}\right)$$

Observations

- NESS transition $\rightarrow$ divergence in $\mathcal{R}$
- (Where $\mathcal{R}$ is calculated using f as a "free energy")

Endnote - Not a huge surprise

- $\mathcal{Z}(t) = \sum Z_N t^N$ is recognizable as the generating function for a lattice path problem.



Get generating function by iterating

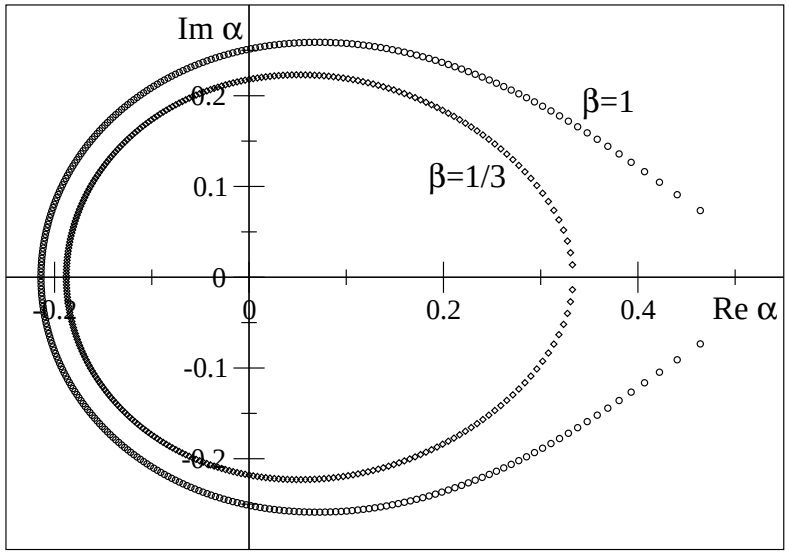
$$G_E(z) = z\left(1 + G_E(z) + [G_E(z)]^2 + \cdots\right) = \frac{z}{1 - G_E(z)}$$

- Then iterate again with contact weights $1/\alpha$

$$\mathcal{Z}(z) = \alpha\beta\, G_D(1/\alpha, z)\, G_D(1/\beta, z)$$

Similar approach - Lee-Yang Zeros

- Similar approach taken to Lee-Yang zeros taken by Blythe and Evans
- Treat f_N "like" a free energy...



Observation, question

- Divergences in $\mathcal{R}$ show phase structure for NESS *in the ASEP*
- Is this a feature of this very particular model (and relations)?

THE END :)