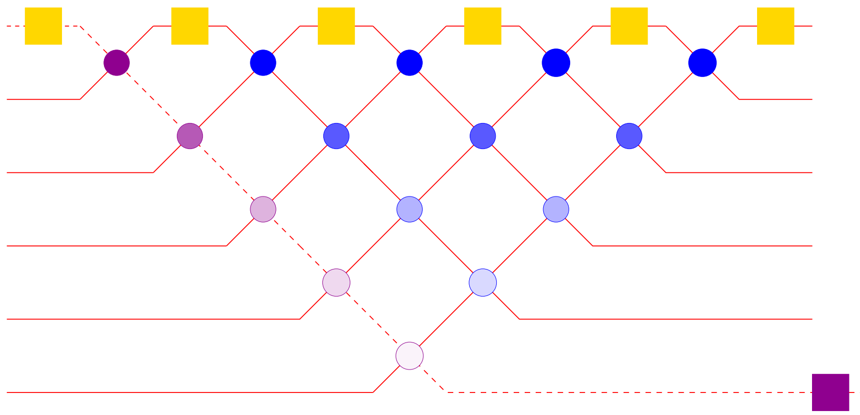
What has quantum mechanics to do with factoring?

Things I wish they had told me about Shor's algorithm



Stockholm, 23 April, 2009

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Answer:

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Answer:

Nothing!

But quantum mechanics is good at diagnosing periodicity, which (for purely arithmetic reasons) helps in factoring.

FACTORING AND PERIOD FINDING

You can factor $N = pq$, with p, q huge (e.g. 300 digit) primes,
if, for integers a having no factors in common with N,
you can find the smallest r with $a^r = 1 \pmod N$

$$b = c \pmod N \Leftrightarrow b \textit{ and } c \textit{ differ by a multiple of } N$$

$a^x \pmod N$ is periodic with period r.

Example:

$5^x \pmod 7$: $5^1 = 5,\ 5^2 = 4,\ 5^3 = 6,$
$5^4 = 2,\ 5^5 = 3,\ 5^6 = 1,\ 5^7 = 5.$

Pick random a. Use *quantum computer* to find r.
Pray for two pieces of good luck!

Quantum computer gives *smallest* r with $a^r - 1$ divisible by $N = pq$

First piece of luck: r even.
Then $(a^{r/2} - 1)(a^{r/2} + 1)$ is divisible by N, but $a^{r/2} - 1$ is *not*,

Second piece of luck: $a^{r/2} + 1$ is *also* not divisible by N.
Then product of $a^{r/2} - 1$ and $a^{r/2} + 1$ is divisible by both p and q although neither factor is divisible by both.

Since p, q primes, one factor divisible by p and other divisible by q.
So one factor is greatest common divisor of N and $a^{r/2} - 1$;
other factor is greatest common divisor of N and $a^{r/2} + 1$.

FINISHED!

Finished, because:

1. Can find greatest common divisor of two integers using method known to ancient Greeks: Euclidean algorithm.

2. If a is picked at random, an hour's argument* shows that the probability is at least 50% that both pieces of luck will hold.

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* N. D. Mermin, *Quantum Computer Science* (2007), Appendix M

Amazing! (but wrong):

> [After the computation] the solutions — the factors of the number being analyzed — will all be in superposition.
>
> — George Johnson, *A Shortcut Through Time.*

> [The computer will] try out all the possible factors simultaneously, in superposition, then collapse to reveal the answer.
>
> — *Ibid.*

Unexciting but correct!

A quantum computer is efficient at factoring because it is efficient at period-finding.

Next question: What's so hard about period finding?

Given graph of $\sin(kx)$ it's easy to find the period $2\pi/k$. Since no value repeats inside a period, $a^x \pmod N$ is even simpler.

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What makes it hard:

Within a period, unlike the smooth, continuous $\sin(kx)$, the function $a^x \pmod N$ looks like *random noise.*

Nothing in a list of r consecutive values gives a *hint* that the next one will be the same as the first.

PERIOD FINDING WITH A QUANTUM COMPUTER

Represent n bit number

$$x = x_0 + 2x_1 + 4x_2 + \cdots + 2^{n-1}x_{n-1} \qquad \text{(each } x_j \text{ 0 or 1)}$$

by product of states $|0\rangle$ and $|1\rangle$ of n 2-state systems (*Qbits*):

$$|x\rangle = |x_{n-1}\rangle \cdots |x_1\rangle|x_0\rangle$$

Classical or *Computational* basis.

Computer acts on states with unitary transformations **U** that can be built from 1-Qbit and 2-Qbit unitary *gates* acting on single Qbits or on pairs of Qbits.

Represent function f taking n-bit to m-bit integers by a linear, norm-preserving (unitary) transformation U_f acting on n-Qbit *input register* and m-Qbit *output register*:

$$\mathsf{U}_f \overset{\substack{\text{input register}\\\downarrow}}{|x\rangle} \underset{\substack{\uparrow\\\text{output register}}}{|0\rangle} = \overset{\downarrow}{|x\rangle} \underset{\uparrow}{|f(x)\rangle}.$$

QUANTUM PARALLELISM

$$\mathbf{U}_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$$

Put input register into *superposition of all possible inputs*:

$$|\phi\rangle = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le x<2^n} |x\rangle$$

$$= \tfrac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big) \cdots \tfrac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big).$$

Applying linear $\mathbf{U}_f$ gives

$$\mathbf{U}_f\big(|\phi\rangle|0\rangle\big) = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le x<2^n} |x\rangle|f(x)\rangle.$$

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Question:

Has *one* invocation of $\mathbf{U}_f$ computed $f(x)$ for *all* x?

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No. Given a single system in an unknown state, *there is no way to learn what that state is.*

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Question:

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Answer:

No. Given a single system in an unknown state, *there is no way to learn what that state is.*

Information is acquired *only* through measurement.

Direct measurement of input register gives random x_0;
Direct measurement of output register then gives $f(x_0)$.

$$\mathbf{U}_f\big(|\phi\rangle|0\rangle\big) = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le x<2^n} |x\rangle|f(x)\rangle.$$

Special form when $f(x) = a^x \pmod N$:

$$\sum_{0\le x<2^n} |x\rangle|a^x\rangle = \sum_{0\le x<r} \Big(|x\rangle + |x+r\rangle + |x+2r\rangle + \cdots\Big)|a^x\rangle$$

Measuring output register leaves input register in state

$$|x\rangle + |x+r\rangle + |x+2r\rangle + \cdots$$

for random $x < r$.

Given n Qbits in the state $|x\rangle + |x+r\rangle + |x+2r\rangle + \cdots$

If you could learn what the state was you would know r.

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> If you could learn what the state was you would know r.
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>
> If you could make exact copies of an unknown state
> you could learn several random multiples of r.
> *But there is no way to duplicate an unknown state.*

Question: How can one learn *anything* about r?

Given n Qbits in the state $|x\rangle + |x + r\rangle + |x + 2r\rangle + \cdots$

If you could learn what the state was you would know r.
But there is no way to learn what the state is.

If you could make exact copies of an unknown state you could learn several random multiples of r.
But there is no way to duplicate an unknown state.

Question: How can one learn *anything* about r?

Answer: Through quantum Fourier analysis!

THE QUANTUM FOURIER TRANSFORM

$$\mathbf{V}_{FT}|x\rangle = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le y<2^n} e^{2\pi i x y/2^n}|y\rangle$$

Acting on superpositions, $\mathbf{V}_{FT}$ Fourier-transforms amplitudes:

$$\mathbf{V}_{FT}\sum \alpha(x)|x\rangle = \sum \beta(x)|x\rangle$$

$$\beta(x) = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le z<2^n} e^{2\pi i x z/2^n}\alpha(z)$$

If α has period r as in $|x\rangle + |x+r\rangle + |x+2r\rangle + \cdots$
then β is sharply peaked at integral multiples of $2^n/r$.

Question: Is *that* all there is to it?

$\mathbf{V}_{FT}$ is *boring*:

1. Just familiar transformation from position to momentum representation.

2. Everybody knows Fourier transform sharply peaked at multiples of inverse period.

But **$\mathbf{V}_{FT}$** is *not* boring because:

1. x has nothing to do with position, real or conceptual.

x is arithmetically useful but physically meaningless:

$$x = x_0 + 2x_1 + 4x_2 + 8x_3 + 16x_4 + \cdots,$$

where $|x_j\rangle = |0\rangle$ or $|1\rangle$ is state of j-th 2-state system.

2. *Sharp* means sharp compared with resolution of apparatus.
But the period r is *hundreds* of digits long.

Need to know r *exactly* — every single digit.
Error in r of 1 in 10^{10} *messes up almost every digit.*

Under $\mathbf{V}_{FT}$ *shifts become phase factors:*

$$\mathbf{V}_{FT}\Big(|x\rangle + |x+r\rangle + |x+2r\rangle + \cdots\Big) =$$

$$= \big(\tfrac{1}{\sqrt{2}}\big)^n \sum_{0\le y<2^n} \Big(1+\alpha+\alpha^2+\alpha^3+\cdots\Big)e^{2\pi i x y/2^n}|y\rangle,$$

$$\alpha = \exp\Big(2\pi i y/(2^n/r)\Big).$$

Sum of powers of α sharply peaked at values of y as close as possible to (i.e. within $\frac{1}{2}$ of) integral multiples of $2^n/r$.

Question: How sharply peaked?

Answer: Probability of measuring such a $y > 40\%$!

So we have a significant ($> 40\%$) chance of learning an integer y within $\frac{1}{2}$ of a (more or less) random integral multiple of $2^n/r$.

Then $y/2^n$ is within $1/2^{n+1}$ of j/r.

Question: Does this pin down a unique rational number j/r?

We have a significant ($> 40\%$) chance of learning an integer y within $\frac{1}{2}$ of $j(2^n/r)$ for some (more or less) random integer j.

Then $y/2^n$ is within $1/2^{n+1}$ of j/r.

Question: Does this pin down a unique rational number j/r?

Answer: It depends. Suppose $j'/r' \neq j/r$. Then

$$|j'/r' - j/r| \geq 1/rr' \;\geq 1/N^2$$

Answer is yes, if $1/N^2 > 1/2^n$: $2^n > N^2$

Input register must be large enough to represent N^2.

Then have 40% chance of learning a *divisor* r_0 of r.

(r_0 is r divided by factors it shares with (random) j)

(j and r given from continued-fraction expansion of $y/2^n$)

A comment:

When $N = pq$, easy to show period r necessarily $< N/2$. So

$$\left|\frac{j'}{r'} - \frac{j}{r}\right| > \frac{4}{N^2}$$

and therefore don't need y *as close as possible* to integral multiple of $2^n/r$.

Second, third, or fourth closest do just as well.

Raises probability of learning divisor of r from 40% to 90%.

Have 90% chance of learning a *divisor* r_0 of r.

If j happens to share *no* factors with r, then $r_0 = r$.
Can try it out: Calculate $a^{r_0} \pmod{N}$. Is it 1?

If not, repeat the calculation. Get a *new* (probable) divisor r_0'.
Try for r the *least common multiple* of r_0 and r_0'
(with help from ancient Greeks.)

With several runs of the quantum computation, and some detective work (on a classical computer), one finds r and therefore (unless unlucky) factors N.

Another comment:

Should the period r be 2^m, then $2^n/r$ is itself an integer, and probability of y being multiple of that integer is easily shown to be 1, even if input register contains just a single period.

A pathologically easy case.

Question: When must all periods r be powers of 2?
Answer: When p and q are both primes of form $2^j + 1$.
(Periods are divisors of $(p-1)(q-1)$.)

Therefore factoring $15 = (2+1) \times (4+1)$
— i.e. finding periods modulo 15 —
is not a serious demonstration of Shor's algorithm.

Some neat things about the quantum Fourier transform

$$\mathbf{V}_{FT}|x\rangle = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le y<2^n} e^{2\pi i x y/2^n}|y\rangle$$

1. Constructed entirely out of 1-Qbit and 2-Qbit gates.

2. Number of gates and therefore time grows only as n^2.

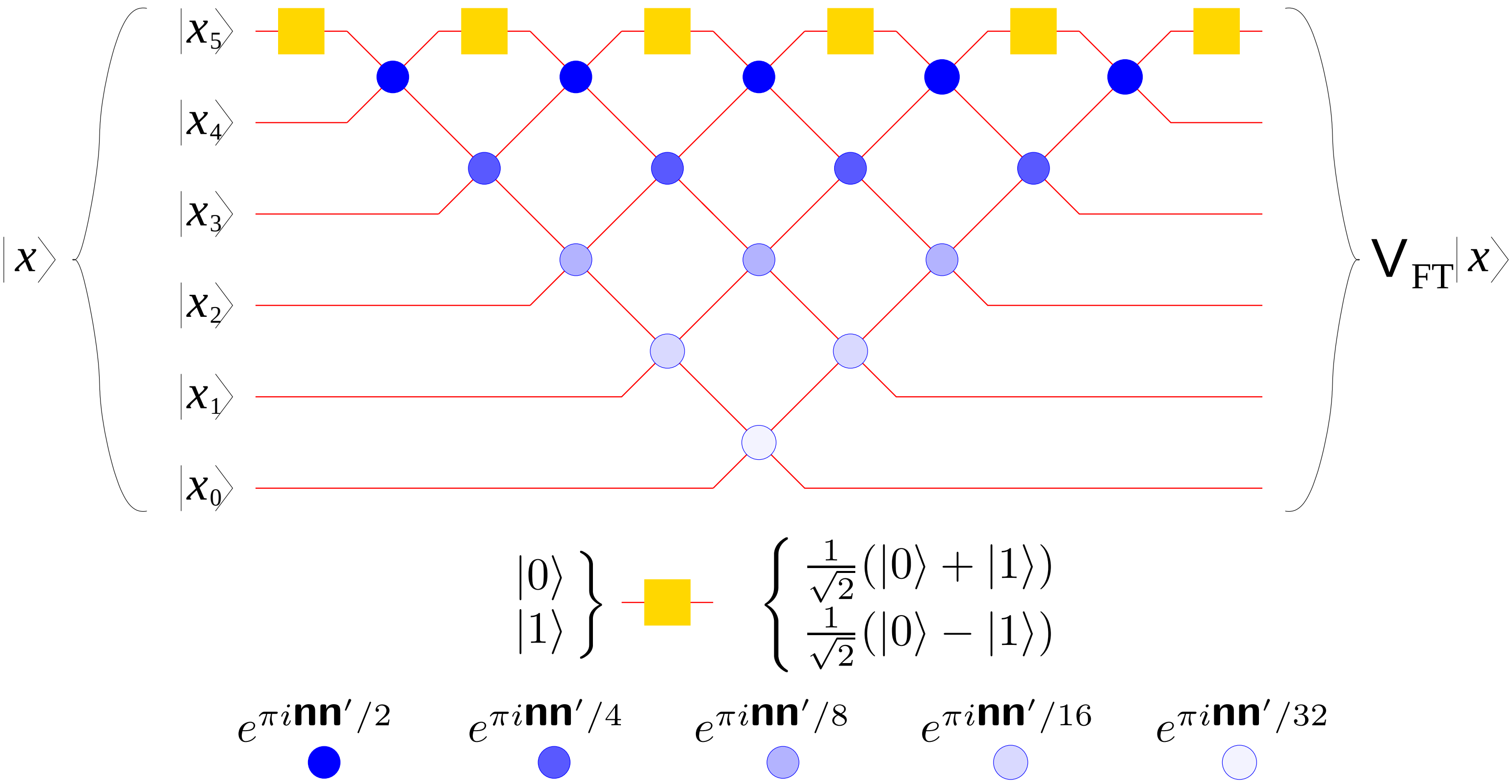
3. With just *one* application,

$$\sum \alpha(x)|x\rangle \longrightarrow \sum \beta(x)|x\rangle,$$

$$\beta(x) = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0\le z<2^n} e^{2\pi i x z/2^n}\alpha(z)$$

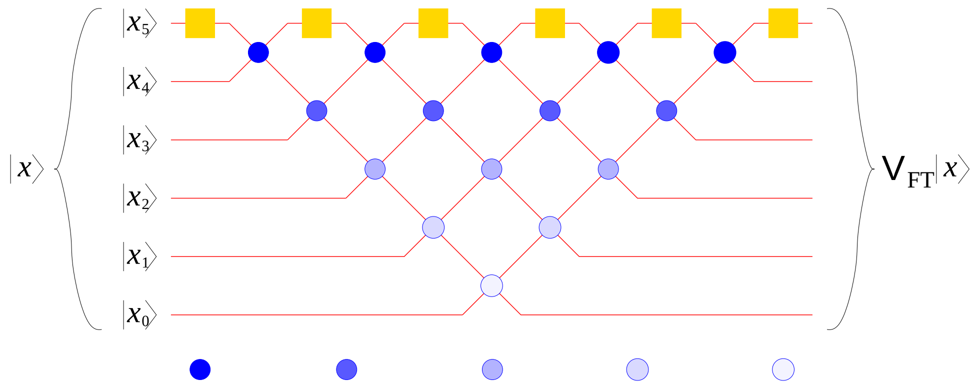
In *classical* "Fast Fourier Transform" time grows as $n2^n$.

But classical FFT gives *all* the $\beta(x)$, while QFT gives *only* $\sum \beta(x)|x\rangle$.



$$|0\rangle|0\rangle,\ |0\rangle|1\rangle,\ |1\rangle|0\rangle \text{ invariant; } \quad |1\rangle|1\rangle \longrightarrow e^{\pi i/2^j}|1\rangle|1\rangle$$

A PROBLEM?



Number n of Qbits: $2^n > N^2$, N hundreds of digits.

Phase gates $e^{\pi i \mathbf{nn}'/2^m}$ *impossible to make* for most m,

since can't control strength or time of interactions to better than parts in $10^{10} = 2^{30}$.

But need to learn period r to parts in 10^{300} or more!

Question:

So is it all based on a silly mistake?

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Answer:

No, all is well.

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No, all is well.

Question:

How can that be?

Answer:

Because of the quantum-computational interplay between analog and digital.

Quantum Computation is Digital

Information is acquired *only* by measuring Qbits.
The reading of each 1-Qbit measurement gate is only 0 or 1.

The 10^3 bits of the output y of Shor's algorithm are given by the readings (0 or 1) of 10^3 1-Qbit measurement gates.

There is no imprecision in those 10^3 readings.
The output is a definite 300-digit number.

But is it the number you wanted to learn?

Quantum Computation is Analog

Before a measurement the Qbits
are acted on by unitary gates with
continuously variable parameters.

These variations affect the amplitudes
of the states prior to measurement
and therefore they affect the *probabilities*
of the readings of the measurement gates.

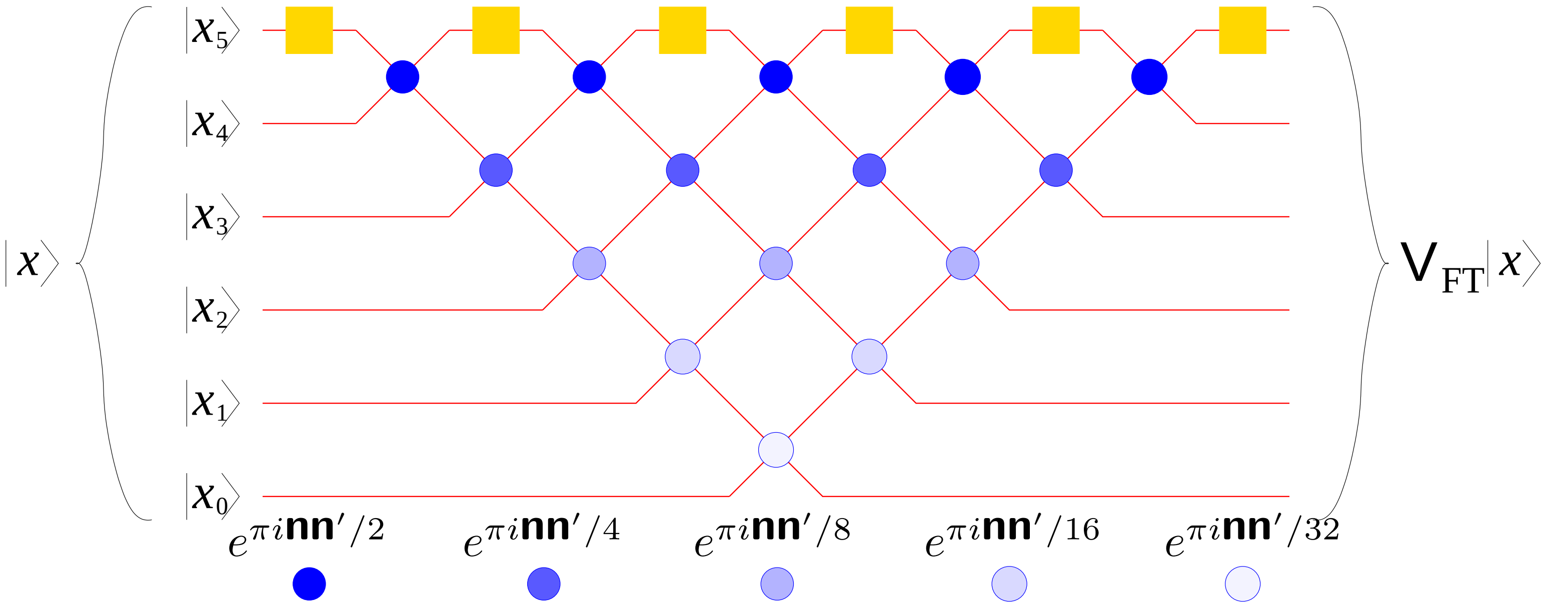
So all is indeed well

"Huge" errors (parts in 10^4) in the phase gates
may result in comparable errors in the *probability* that
the 300 digit number given *precisely* by the measurement gates
is *the right* 300 digit number.

So the probability of getting a useful number
may not be 90% but only 89.99%.

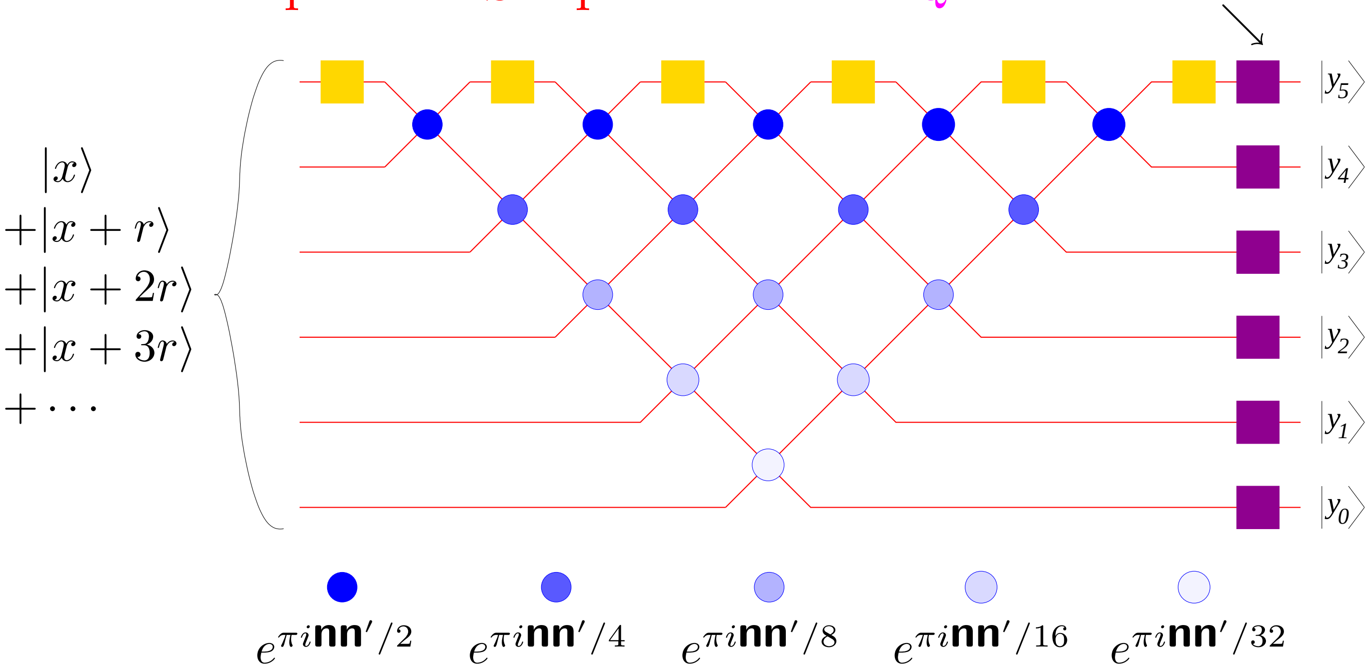
Since "90%" is actually "about 90%"
this makes no difference.

In fact this makes things even better



Since only top 20 layers of phase gates matter when $N > 2^{20} = 10^6$,
time for QFT scales not quadratically but linearly in number of Qbits.

Another Important Simplification



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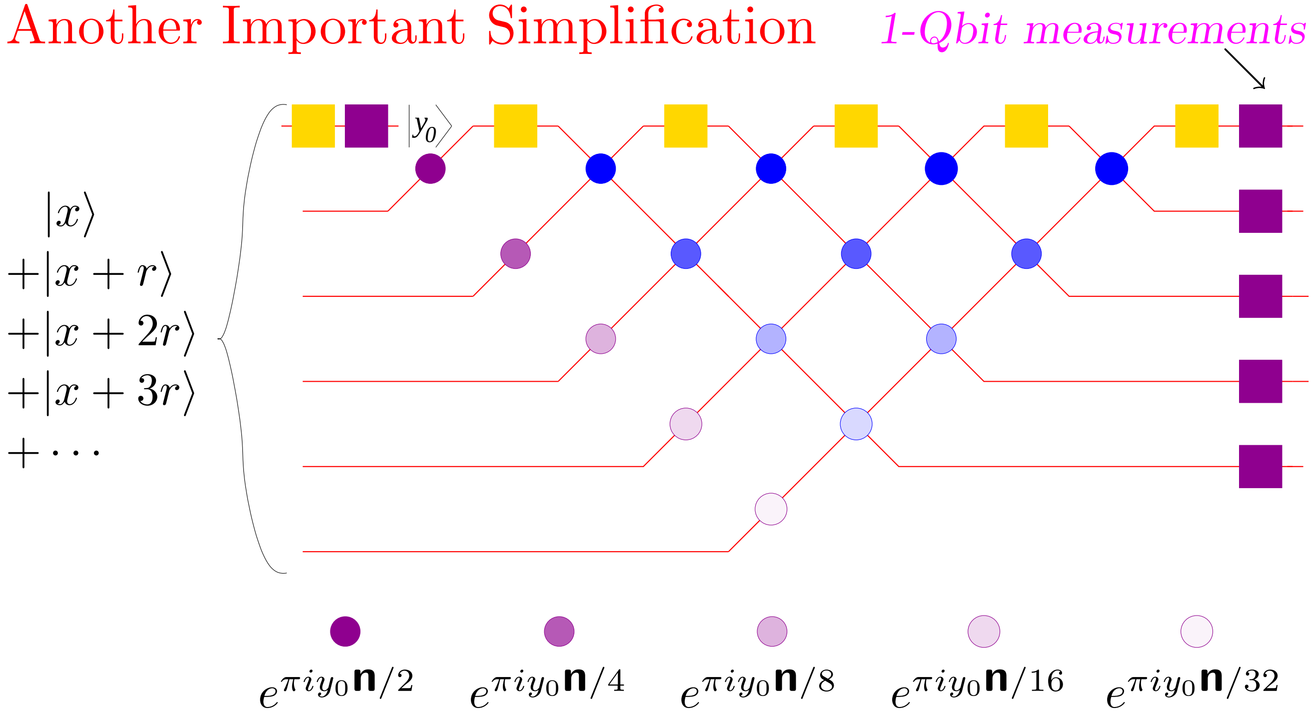


To execute the Quantum Fourier transformation and then measure its output you only need 1-Qbit gates!

References:

Quantum Computer Science: An Introduction

N. David Mermin

Cambridge University Press

Physics Today, April and October, 2007

March, 2008

Quantum Versus Classical Programming Styles

Question: How do you calculate a^x when x is a 300 digit number?
Answer: Not by multiplying a by itself 10^{300} times!

How else, then?

Write x as a binary number: $x = x_{999}x_{998}\cdots x_2x_1x_0$.

Next square a, square the result, square *that* result …, getting the 1,000 numbers a^{2^j}.

Finally, multiply together all the a^{2^j} for which $x_j = 1$.

$$\prod_{j=0}^{999}\left(a^{2^j}\right)^{x_j} = a^{\sum_j x_j 2^j} = a^x$$

Classical: Cbits Cheap; Time Precious

$$a^x = \prod_{j=0}^{999} \left(a^{2^j}\right)^{x_j}$$

Once and for all, make and store a look-up table:

$$a,\ a^2,\ a^4,\ a^8, \ldots ,\ a^{2^{999}}$$

A thousand entries, each of a thousand bits.

For each x multiply together all the a^{2^j} in the table for which $x_j = 1$.

Quantum: Time Cheap; Qbits Precious

Circuit that executes

$$a^x = \prod_{j=0}^{999} \left(a^{2^j}\right)^{x_j}$$

is not applied 2^n times to input register for each $|x\rangle$.
It is applied *just once* to input register in the state

$$|\phi\rangle = \left(\tfrac{1}{\sqrt{2}}\right)^n \sum_{0 \le x < 2^n} |x\rangle.$$

So after each conditional (on $x_j = 1$) multiplication by a^{2^j} can store $\left(a^{2^j}\right)^2 = a^{2^{j+1}}$ using *same* 1000 Qbits that formerly held a^{2^j}.

Some other things I wish they had told me:

Question:

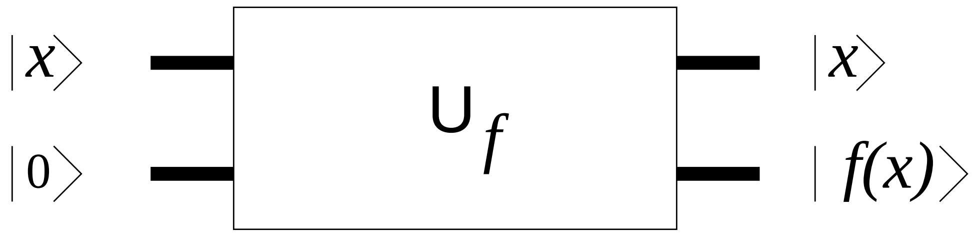
Why must a quantum computation be reversible (except for measurements)?

Superficial answer:

Because linear + norm-preserving $\Rightarrow$ unitary and unitary transformations have inverses.

Real answer:

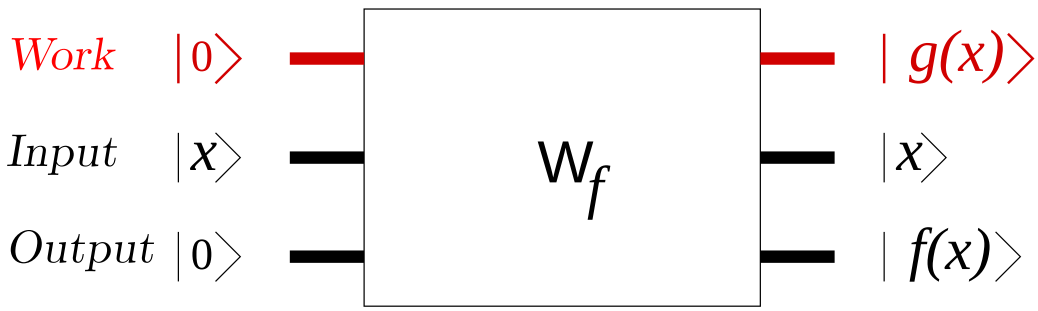
Because standard architecture for evaluating $f(x)$,



oversimplifies the actual architecture:

Need additional work registers for doing calculation:

Registers



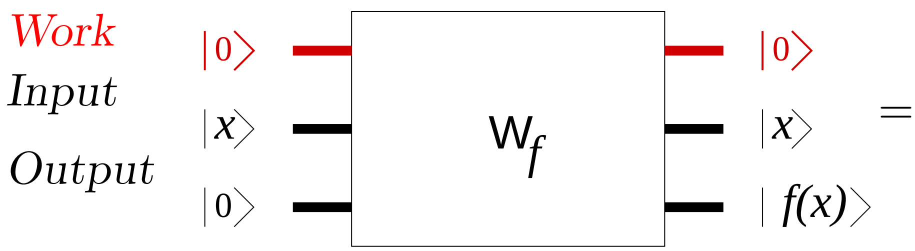
If input register starts in standard state $\sum_x |x\rangle$
then final state of all registers is $\sum_x |g(x)\rangle|x\rangle|f(x)\rangle$.

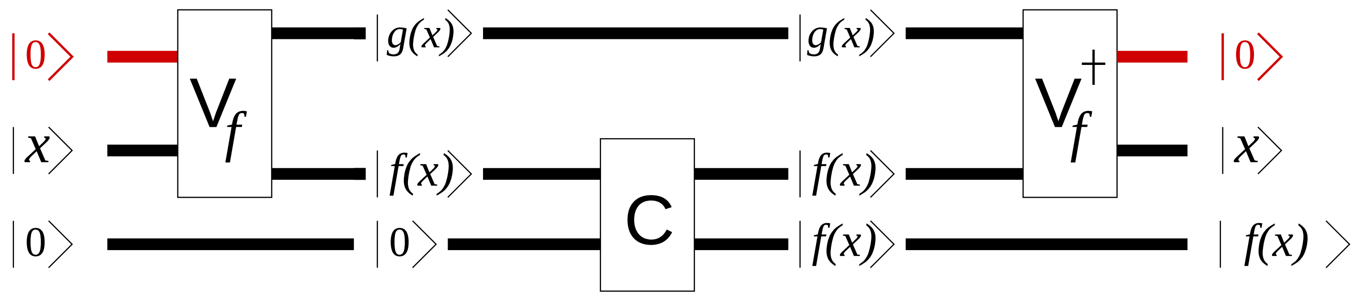
Work register entangled with input and out registers,
Quantum parallelism breaks down.

Quantum parallelism maintained if $|g(x)\rangle = |0\rangle$, for any x.

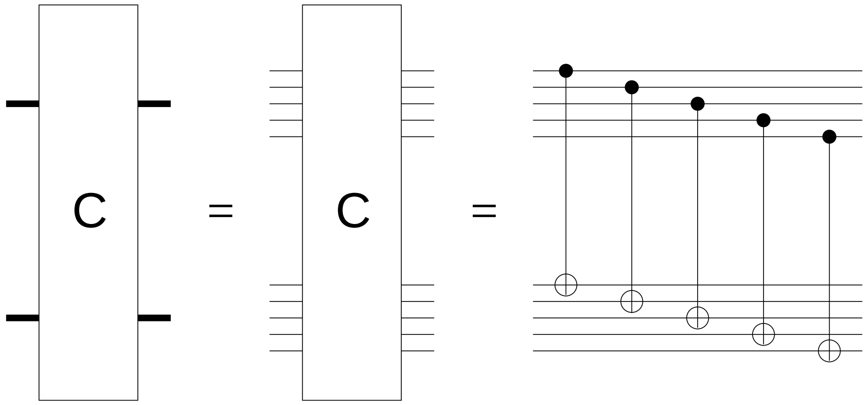
Final state is then $|0\rangle \left(\sum_x |x\rangle|f(x)\rangle\right)$.

How to keep the work register unentangled:

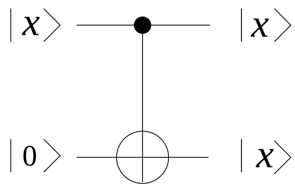




C *is built out of 1-Qbit controlled-NOT gates:*



controlled-NOT:



Question:

How do you do arithmetic on a quantum computer?

Answer:

By copying the (pre-existing) classical theory
of reversible computation.

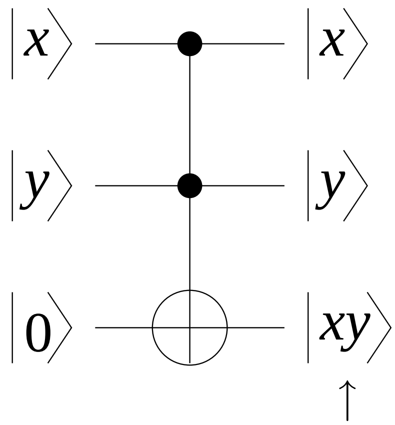
Question (from reversible-classical-computer scientist):

But that theory requires an irreducibly
3-Cbit doubly-controlled-NOT (Toffoli) gate!

Answer:

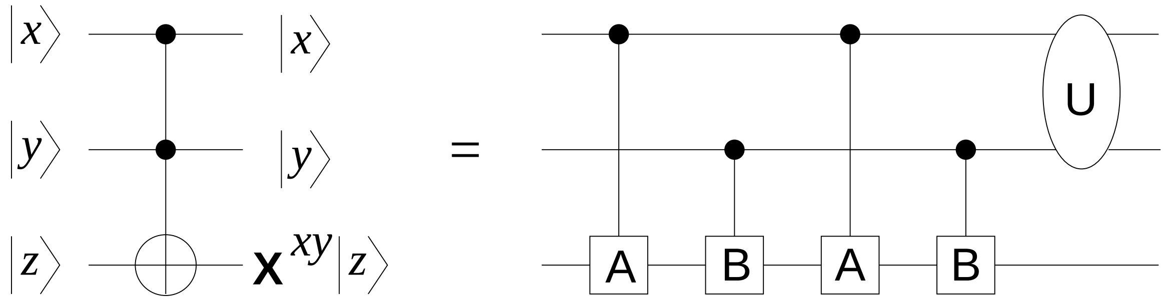
In a quantum computer 3-Qbit Toffoli gate
can be built from a few 2-Qbit gates.

The 3-Cbit Doubly-Controlled-NOT (Toffoli) gate:



logical AND
of x and y

Building 3-Qbit Doubly-Controlled-NOT gate from 2-Qbit gates:



$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \qquad \mathbf{U} = e^{-\pi i \mathbf{n}\mathbf{n}'/2}$$

$$\mathbf{A} = \hat{\mathbf{a}} \cdot \sigma \qquad \mathbf{B} = \hat{\mathbf{b}} \cdot \sigma \qquad \hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{x}} \sin\theta \qquad \mathbf{A}^2 = \mathbf{B}^2 = \mathbf{1}$$

$$\mathbf{AB} = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} + i\hat{\mathbf{a}} \times \hat{\mathbf{b}} \cdot \sigma = \cos\theta + i\sigma_x \sin\theta$$

$$(\mathbf{AB})^2 = \cos 2\theta + i\sigma_x \sin 2\theta$$

If angle θ between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is $\pi/4$ then $(\mathbf{AB})^2 = i\mathbf{X} = e^{\pi i/2}\mathbf{X}$

References:

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