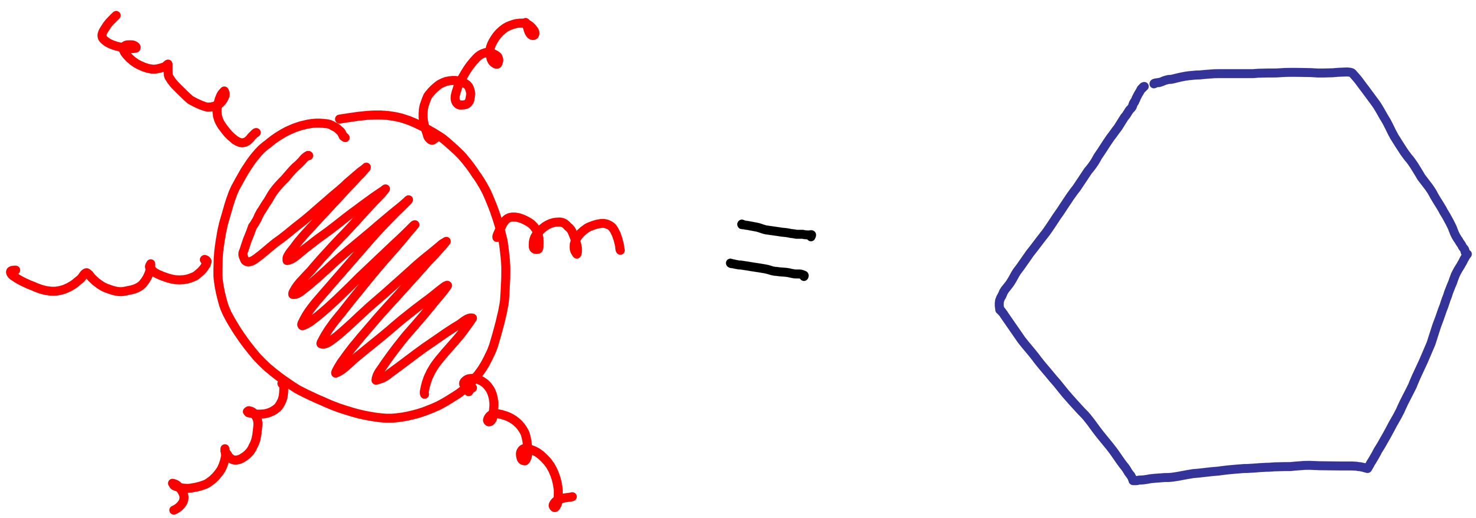
Classical Polylogarithms for Amplitudes and Wilson Loops



A.B. Goncharov ⊗ M. Spradlin ⊗ C. Vergu ⊗ A. Volovich

Introduction

This talk will be concerned with the

Simplest nontrivial multi-loop scattering amplitude in $\mathcal{N}=4$ SYM:

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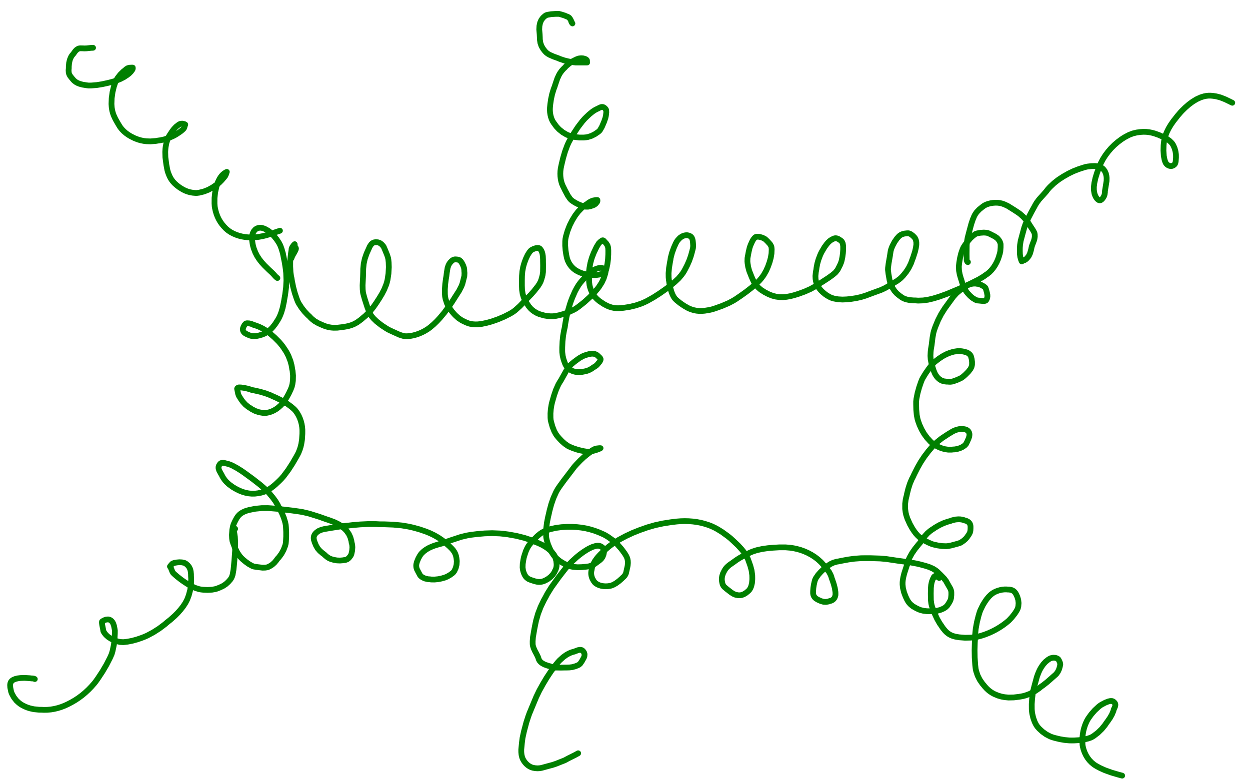
simplest nontrivial multi-loop scattering amplitude in $\mathcal{N}=4$ SYM:

the two-loop six-particle MHV $(++\to++++)$ amplitude



+ many more

The ABDK/BDS Ansatz

The infrared and collinear behavior of amplitudes in massless gauge theories are tightly constrained by general field theory arguments. (Catani, Sterman, Tejeda-Yeomans)

$$
\begin{aligned}
\log A_{\mathrm{MHV}} \sim\ & (\text{known IR divergent terms}) \\
& + (\text{specific finite terms with prescribed collinear behavior}) \\
& + (\text{finite terms with trivial collinear limits}) \\
& + \mathcal{O}(\text{IR regulator}) \text{ terms}
\end{aligned}
$$

Collinear Limits

A function $F_n(k_1, k_2, \ldots, k_n)$ of n 4-vectors k_i has "trivial collinear limits" if

$$F_n(k_1, \ldots, k_i, k_{i+1}, \ldots, k_n) \longrightarrow F_{n-1}(k_1, \ldots, k_i + k_{i+1}, \ldots, k_n)$$

when two cyclically adjacent k_i become parallel.

(or, more generally, any number may become parallel
$\Rightarrow$ "multi-collinear" limits)

The ABDK/BDS Ansatz

Bern, Dixon and Smirnov (BDS) made a specific proposal for how to write the terms in terms of one-loop amplitudes

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The BDS ansatz was supported by laborious calculations at $n=4,5$ through 3 loops. (Anastasiou, Bern, Cachazo, Dixon, Kosower, Roiban, Smirnov, MS, Volovich, Wen)

But explicit calculation revealed that the BDS ansatz fails beginning at two-loops for $n=6$ particles.

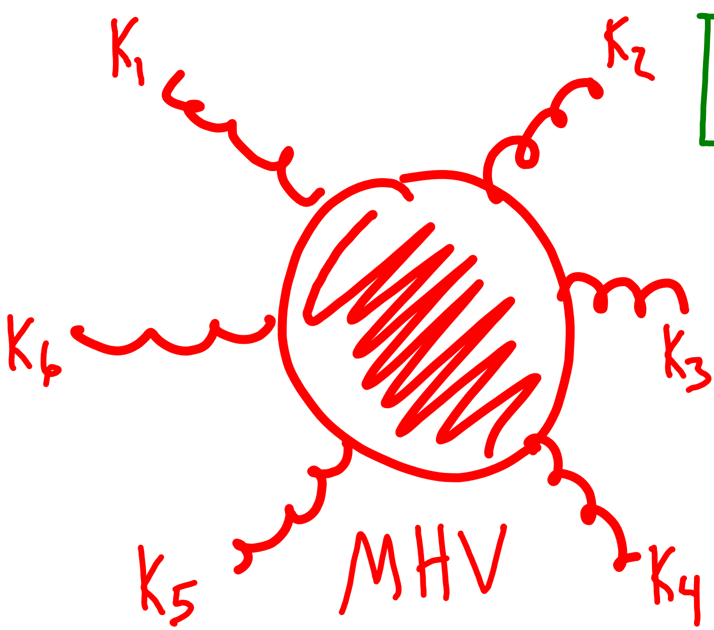
$\Rightarrow$ The BDS "remainder function" is nonzero
(though, as emphasized, it must have trivial collinear limits.)

(Bern, Dixon, Kosower, Roiban, MS, Vergu, Volovich)

(BDKRSVV)

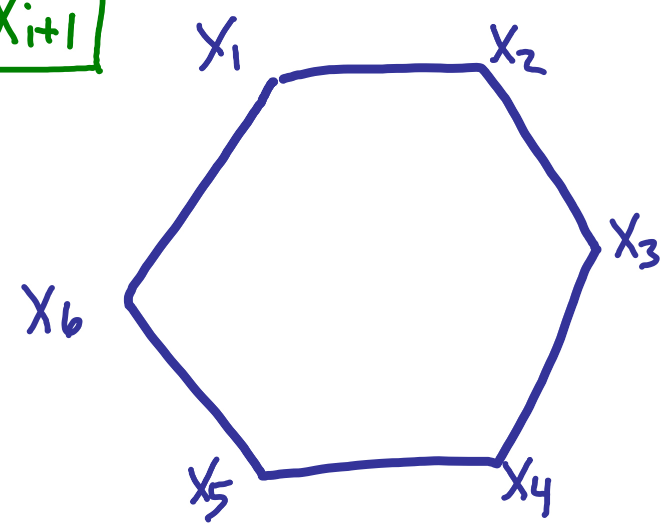
Amplitudes = Wilson Loops

In parallel developments, inspired by the work of Alday and Maldacena at strong coupling, it was experimentally observed that, apparently order by order in perturbation theory,



$$=$$

$$\boxed{k_i = x_i - x_{i+1}}$$



(Drummond, Korchemsky, Sokatchev; Brandhuber, Heslop, Travaglini; BDKRSVV; " + Katsaroumpas, Nguyen, Spence)

Dual Conformal Invariance

Once you accept $A = W$, then the conformal Ward identity which the Wilson loop satisfies (in x space) implies a highly nonobvious relation on A which we call the

dual conformal Ward identity.

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The most general solution to the DCWI is

$$\log A = (\text{BDS ansatz}) + (\text{any finite dual conformaly invariant})$$

ie any function of cross-ratios $\frac{x_{ij}^2 x_{k\ell}^2}{x_{i\ell}^2 x_{jk}^2}$

Modern Understanding

Given the presumption of dual conformal invariance, the BDS ansatz $\log A \sim A^{1\text{-loop}}$ is trivially true for $n=4,5$ to all loops, because there are no possible cross-ratios!

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What <u>is</u> highly nontrivial is the apparent fact that the amplitude remainder function

$$\text{amplitude remainder function} = \text{Wilson loop remainder function}$$

(Bern, Dixon, Kosower, Roiban, MS, Vergu, Volovich; Drummond, Henn, Korchemsky, Sokatchev)

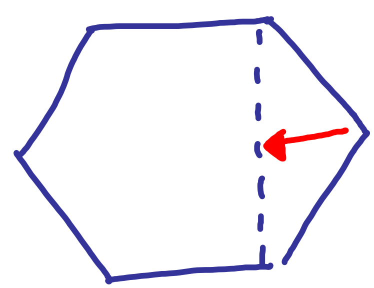
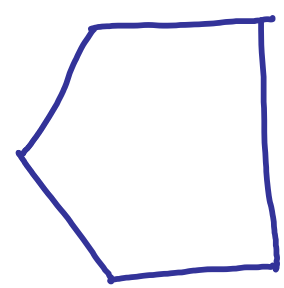
The Simplest Non-Trivial Amplitude/Wilson Loop in SYM

The rest of this talk is about the 2-loop 6-particle remainder function $R_6^{(2)}$. What do we know?

- A function of three dual conformal cross-ratios

$$u_1 = \frac{s_{12}\, s_{45}}{s_{123}\, s_{345}} \qquad u_2 = \frac{s_{23}\, s_{56}}{s_{234}\, s_{123}} \qquad u_3 = \frac{s_{35}\, s_{61}}{s_{345}\, s_{234}} \qquad s_{ij\ldots} = (k_i + k_j + \cdots)^2$$

- Symmetric under any permutation of the u's.
- Vanishes in any collinear limit

$R(0, u, 1-u) =$  $=$  $= 0$

In a heroic effort, Del Duca, Duhr and Smirnov found a manageable way to evaluate the appropriate Wilson loop diagrams, and obtained

an analytic formula for $R(u_1, u_2, u_3)$

in terms of Goncharov polylogarithms

$$G(a_k, a_{k-1}, \ldots, a_1; z) = \int_0^z G(a_{k-1}, \ldots, a_1; t) \frac{dt}{t - a_k} \qquad G(z) = 1$$

and the slightly more familiar special cases, the harmonic polylogarithms, with all $a_i = 0$ or 1.

Motivation

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$

$$\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$

$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$

$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+$$

$$G\left(0,\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1,0;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)+$$

$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1},0;1\right)+$$

$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},1;1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$$

$$\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,0,\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2},0;1\right)+$$

$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

$$G\left(\frac{1}{u_3},0,0,\frac{1}{u_2};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_1+u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,0,\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{24}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)+\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)+$$
$$\frac{1}{8}\pi^2\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_2},v_{231};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{312};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,\frac{1}{1-u_3},v_{321};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{123},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{132},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{132},\frac{1}{1-u_1};1\right)+\mathcal{G}\left(0,0,v_{213},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{213},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,0,v_{312},\frac{1}{1-u_3};1\right)+\mathcal{G}\left(0,0,v_{321},0;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,0,v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},0,v_{132};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{213};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},0,v_{231};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{312};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},0,v_{321};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\frac{1}{2}\mathcal{G}\left(0,\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},0,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1},0;1\right)-$$

$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{123},1,\frac{1}{1-u_1};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},0;1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},0,\frac{1}{1-u_1};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},0,\frac{1}{1-u_2};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},0,\frac{1}{1-u_3};1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},0;1\right)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{321},0,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},0;1\right)-\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{123};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,0,v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},0,\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{123};1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},0,v_{132};1\right)-\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)-$$
$$\frac{3}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,1;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},1;1\right)+$$

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3},0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0,\frac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1,\frac{1}{1-u_1};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0,\frac{1}{1-u_1};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1,\frac{1}{1-u_1};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{213};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,0,v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},0,\frac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213},\frac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231},\frac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{213};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},0,v_{231};1\right)-\tfrac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0,\frac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0,\frac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1,\frac{1}{1-u_2};1\right)+
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0,\frac{1}{1-u_2};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1,\frac{1}{1-u_2};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{312};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,0,v_{321};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{312};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},0,\frac{1}{1-u_3},v_{321};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312},\frac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321},\frac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{312};1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},0,v_{321};1\right)-\tfrac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)-\\
&\tfrac{3}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},1;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0,\frac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1,0;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2},0;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,0;1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0,\frac{1}{1-u_3};1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1,\frac{1}{1-u_3};1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},0;1\right)-\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},1;1\right)+\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,0;1\right)-\\
&\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,1;1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0,\frac{1}{1-u_3};1\right)-\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,0;1\right)-\\
&\tfrac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1,\frac{1}{1-u_3};1\right)+\tfrac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},0;1\right)-
\end{aligned}
$$

$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_1},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_1};1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},1,1,\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},0,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,1,\frac{1}{1-u_1};1\right)-\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right)-\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},0,\frac{1}{1-u_2},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},1,0,\frac{1}{1-u_2};1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},1,1,\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},\frac{1}{1-u_2};1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},\frac{1}{1-u_2},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},0,\frac{1}{1-u_3},1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},1,0,\frac{1}{1-u_3};1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},1,1,\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},0;1\right)-$$
$$\frac{5}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},0,1;1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,0;1\right)-\frac{5}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,1;1\right)+\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},\frac{1}{1-u_3},1;1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,1,\frac{1}{1-u_3};1\right)-\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3},1;1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1,1;1\right)-\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)-$$
$$\frac{3}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}G\left(\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}G\left(\frac{1}{1-u_2},1,\frac{1}{u_1};1\right)H\left(0;u_1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(0,v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},1;1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{u_1};1\right)H\left(0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H\left(0;u_1\right)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H\left(0;u_1\right)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H\left(0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H\left(0;u_1\right)-$$

$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},0;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_1)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_1)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_1)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_1)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_1)-$$
$$\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_1)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_1)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_1)+\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_1)-\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_1)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{3}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0;1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)-$$
$$\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)H(0;u_2)-\frac{3}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_2)+$$
$$\frac{1}{2}G\left(\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},1,\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1;1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_2};1\right)H(0;u_2)-$$
$$\frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_2)+$$

$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(0,u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(0,v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0;1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_2)-$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_2)+$$
$$\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},1;1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312},\frac{1}{1-u_3};1\right)H(0;u_2)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},0;1\right)H(0;u_2)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},0;1\right)H(0;u_2)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_2)+$$
$$\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_2)+\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_2)-$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H(0;u_2)-\\
&\frac{3}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H(0;u_2)-\frac{3}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H(0;u_2)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H(0;u_1)H(0;u_2)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H(0;u_1)H(0;u_2)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H(0;u_1)H(0;u_2)+\frac{5}{24}\pi^2H(0;u_1)H(0;u_2)-\\
&\frac{1}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)-\\
&\frac{3}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},1;1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_1},0,\frac{1}{u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0;1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_2},0,\frac{1}{u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H(0;u_3)-\frac{3}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(0;u_3)+\\
&\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(0;u_3)+\frac{1}{2}G\left(\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)+
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(0,u_{123},\frac{1}{u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(0,u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\frac{1}{2}\mathcal{G}\left(0,v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(0,v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(0,v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(0,v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{123};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},0,v_{132};1\right)H(0;u_3)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{123};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},\frac{1}{1-u_1},v_{132};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{u_3};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},0;1\right)H(0;u_3)-\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{123},\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},0;1\right)H(0;u_3)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},v_{132},\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{213};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},0,v_{231};1\right)H(0;u_3)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{213};1\right)H(0;u_3)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},\frac{1}{1-u_2},v_{231};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},0;1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231},\frac{u_3-1}{u_2+u_3-1};1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},0;1\right)H(0;u_3)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{213},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},0;1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2},v_{231},\frac{1}{1-u_2};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{312};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{321};1\right)H(0;u_3)+\\
&\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{312};1\right)H(0;u_3)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{321};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312},\frac{1}{1-u_3};1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},1;1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321},\frac{1}{1-u_3};1\right)H(0;u_3)-\\
&\frac{3}{4}\mathcal{G}\left(v_{123},1,\frac{1}{1-u_1};1\right)H(0;u_3)-\frac{3}{4}\mathcal{G}\left(v_{123},\frac{1}{1-u_1},1;1\right)H(0;u_3)+\\
&\frac{1}{4}\mathcal{G}\left(v_{132},1,\frac{1}{1-u_1};1\right)H(0;u_3)+\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1;1\right)H(0;u_3)-\\
&\frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2};1\right)H(0;u_3)-\frac{1}{4}\mathcal{G}\left(v_{213},\frac{1}{1-u_2},1;1\right)H(0;u_3)+
\end{aligned}
$$

$$\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2};1\right)H\left(0;u_3\right)+\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0;u_1\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0;u_1\right)H\left(0;u_3\right)+$$
$$\frac{5}{24}\pi^2H\left(0;u_1\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0;u_2\right)H\left(0;u_3\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0;u_2\right)H\left(0;u_3\right)+\frac{5}{24}\pi^2H\left(0;u_2\right)H\left(0;u_3\right)+$$
$$3H\left(0;u_2\right)H\left(0,0;u_1\right)H\left(0;u_3\right)+3H\left(0;u_1\right)H\left(0,0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,1;\left(u_1+u_3\right)\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}H\left(0;u_1\right)H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(0;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,1;\left(u_2+u_3\right)\right)H\left(0;u_3\right)+$$
$$\frac{3}{4}H\left(0;u_2\right)H\left(1,0;u_1\right)H\left(0;u_3\right)+\frac{3}{4}H\left(0;u_1\right)H\left(1,0;u_2\right)H\left(0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_1\right)-\frac{23}{24}\pi^2H\left(0,0;u_1\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_2\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{312};1\right)H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{321};1\right)H\left(0,0;u_2\right)-$$
$$\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_2\right)-\frac{23}{24}\pi^2H\left(0,0;u_2\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{123};1\right)H\left(0,0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},v_{132};1\right)H\left(0,0;u_3\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{213};1\right)H\left(0,0;u_3\right)+$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},v_{231};1\right)H\left(0,0;u_3\right)+3H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,0;u_3\right)-$$
$$\frac{25}{4}H\left(0,0;u_1\right)H\left(0,0;u_3\right)-\frac{25}{4}H\left(0,0;u_2\right)H\left(0,0;u_3\right)-\frac{23}{24}\pi^2H\left(0,0;u_3\right)+\frac{1}{12}\pi^2H\left(0,1;u_1\right)+$$
$$\frac{1}{12}\pi^2H\left(0,1;u_2\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\left(u_1+u_2\right)\right)+$$

$$\frac{1}{12}\pi^2H\left(0,1;\left(u_1+u_2\right)\right)+\frac{1}{12}\pi^2H\left(0,1;u_3\right)+\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{24}\pi^2H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\frac{1}{12}\pi^2H\left(0,1;\left(u_1+u_3\right)\right)-\frac{1}{24}\pi^2H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{12}\pi^2H\left(0,1;\left(u_2+u_3\right)\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(1,0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_1\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},u_{312};1\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_1\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_1\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)H\left(1,0;u_1\right)-\frac{1}{3}\pi^2H\left(1,0;u_1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_2\right)-$$
$$\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123};1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_2\right)-\frac{3}{4}H\left(0,0;u_3\right)H\left(1,0;u_2\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_1+u_2-1}{u_2-1}\right)H\left(1,0;u_2\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_2\right)-\frac{1}{3}\pi^2H\left(1,0;u_2\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{3}\pi^2H\left(1,0;u_3\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)H\left(1,0;u_3\right)-\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2},u_{231};1\right)H\left(1,0;u_3\right)+$$
$$\frac{3}{4}H\left(0;u_1\right)H\left(0;u_2\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_1\right)H\left(1,0;u_3\right)-\frac{3}{4}H\left(0,0;u_2\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{4}H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_1\right)H\left(1,0;u_3\right)-\frac{1}{4}H\left(1,0;u_2\right)H\left(1,0;u_3\right)+$$
$$\frac{1}{24}\pi^2H\left(1,1;u_1\right)+\frac{1}{24}\pi^2H\left(1,1;u_2\right)+\frac{1}{24}\pi^2H\left(1,1;u_3\right)+\frac{1}{2}H\left(0;u_2\right)H\left(0,0,0;u_1\right)+$$
$$\frac{1}{2}H\left(0;u_3\right)H\left(0,0,0;u_2\right)+\frac{1}{2}H\left(0;u_1\right)H\left(0,0,0;u_3\right)-\frac{1}{2}H\left(0;u_2\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-H\left(0;u_1\right)H\left(0,0,1;\left(u_1+u_2\right)\right)-$$
$$H\left(0;u_2\right)H\left(0,0,1;\left(u_1+u_2\right)\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$

$$\frac{1}{2}H\left(0;u_2\right)H\left(0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-H\left(0;u_1\right)H\left(0,0,1;\left(u_1+u_3\right)\right)-$$
$$H\left(0;u_3\right)H\left(0,0,1;\left(u_1+u_3\right)\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-H\left(0;u_2\right)H\left(0,0,1;\left(u_2+u_3\right)\right)-$$
$$H\left(0;u_3\right)H\left(0,0,1;\left(u_2+u_3\right)\right)-\frac{1}{2}H\left(0;u_2\right)H\left(0,1,0;u_1\right)-\frac{1}{2}H\left(0;u_3\right)H\left(0,1,0;u_2\right)-$$
$$\frac{1}{2}H\left(0;u_1\right)H\left(0,1,0;u_3\right)+\frac{1}{4}H\left(0;u_2\right)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(0;u_1\right)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(0,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+$$
$$\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H\left(0;u_2\right)H\left(1,0,0;u_1\right)-\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_1\right)-$$
$$\frac{1}{2}H\left(0;u_1\right)H\left(1,0,0;u_2\right)+\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_2\right)+\frac{1}{2}H\left(0;u_1\right)H\left(1,0,0;u_3\right)-$$
$$\frac{1}{2}H\left(0;u_2\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)H\left(1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-$$
$$7H\left(0,0,0,0;u_1\right)-7H\left(0,0,0,0;u_2\right)-7H\left(0,0,0,0;u_3\right)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+$$
$$3H\left(0,0,0,1;\left(u_1+u_2\right)\right)+\frac{3}{2}H\left(0,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+3H\left(0,0,0,1;\left(u_1+u_3\right)\right)+$$
$$\frac{3}{2}H\left(0,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+3H\left(0,0,0,1;\left(u_2+u_3\right)\right)+\frac{9}{4}H\left(0,0,1,0;u_1\right)+$$
$$\frac{9}{4}H\left(0,0,1,0;u_2\right)+\frac{9}{4}H\left(0,0,1,0;u_3\right)-\frac{1}{2}H\left(0,1,0,0;u_1\right)-\frac{1}{2}H\left(0,1,0,0;u_2\right)-$$
$$\frac{1}{2}H\left(0,1,0,0;u_3\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{2}H\left(0,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{2}H\left(0,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(0,1,1,0;u_1\right)+H\left(0,1,1,0;u_2\right)+H\left(0,1,1,0;u_3\right)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_2-1}{u_2-1}\right)-\frac{1}{4}H\left(0,1,1,1;\frac{u_1+u_3-1}{u_1-1}\right)-$$
$$\frac{1}{4}H\left(0,1,1,1;\frac{u_2+u_3-1}{u_3-1}\right)+H\left(1,0,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+H\left(1,0,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$H\left(1,0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+2H\left(1,0,1,0;u_1\right)+2H\left(1,0,1,0;u_2\right)+2H\left(1,0,1,0;u_3\right)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_2-1}{u_2-1}\right)+\frac{1}{4}H\left(1,1,0,1;\frac{u_1+u_3-1}{u_1-1}\right)+$$
$$\frac{1}{4}H\left(1,1,0,1;\frac{u_2+u_3-1}{u_3-1}\right)+\frac{1}{2}H\left(1,1,1,0;u_1\right)+\frac{1}{2}H\left(1,1,1,0;u_2\right)+\frac{1}{2}H\left(1,1,1,0;u_3\right)-$$
$$\frac{1}{24}\pi^2H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{u_{123}}\right)-\frac{1}{24}\pi^2H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{u_{231}}\right)-\frac{1}{24}\pi^2H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{8}\pi^2H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)-\frac{1}{8}\pi^2H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{123}}\right)+\frac{1}{24}\pi^2H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-$$

$$\frac{1}{24}\pi^2H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{132}}\right)-\frac{1}{24}\pi^2H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)+\frac{1}{24}\pi^2H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{8}\pi^2H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H\left(0;u_3\right)\mathcal{H}\left(1;\frac{1}{v_{231}}\right)+\frac{1}{8}\pi^2H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{8}\pi^2H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{312}}\right)+\frac{1}{24}\pi^2H\left(0;u_1\right)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-\frac{1}{24}\pi^2H\left(0;u_2\right)\mathcal{H}\left(1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)-\frac{1}{4}H\left(1,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-\frac{1}{4}H\left(1,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-\frac{1}{4}H\left(1,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)+\frac{1}{24}\pi^2\mathcal{H}\left(0,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H\left(0,0;u_3\right)\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{231}}\right)-$$
$$\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H\left(0,0;u_1\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H\left(0,0;u_2\right)\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)+\frac{1}{6}\pi^2\mathcal{H}\left(0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}H\left(0;u_2\right)H\left(0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{1}{2}H\left(0,0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+$$
$$\frac{1}{2}H\left(0,0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{123}}\right)-\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{132}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{213}}\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{1}{2}H\left(0,0;u_1\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{2}H\left(0,0;u_3\right)\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{231}}\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+$$
$$\frac{1}{2}H\left(0,0;u_1\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{1}{2}H\left(0,0;u_2\right)\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)+\frac{11}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{312}}\right)-$$
$$\frac{1}{24}\pi^2\mathcal{H}\left(1,1;\frac{1}{v_{321}}\right)+\frac{1}{2}H\left(0;u_2\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+\frac{1}{2}H\left(0;u_3\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{123}}\right)+$$
$$\frac{1}{2}H\left(0;u_1\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H\left(0;u_3\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{231}}\right)+\frac{1}{2}H\left(0;u_1\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{2}H\left(0;u_2\right)\mathcal{H}\left(0,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H\left(0;u_3\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H\left(0;u_1\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{231}}\right)+$$

$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(0,1,1;\frac{1}{v_{321}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{123}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{u_{312}}\right)+\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{123}}\right)-$$
$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{132}}\right)+\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-$$
$$\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{213}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+\frac{1}{4}H(0;u_3)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{231}}\right)+$$
$$\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{312}}\right)-\frac{1}{4}H(0;u_1)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+$$
$$\frac{1}{4}H(0;u_2)\,\mathcal{H}\left(1,0,1;\frac{1}{v_{321}}\right)+H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{123}}\right)-$$
$$H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_3)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{231}}\right)+H(0;u_1)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-$$
$$H(0;u_2)\,\mathcal{H}\left(1,1,1;\frac{1}{v_{312}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{123}}\right)-\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{231}}\right)-$$
$$\frac{3}{2}\mathcal{H}\left(0,0,0,1;\frac{1}{u_{312}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{132}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{213}}\right)-3\mathcal{H}\left(0,0,0,1;\frac{1}{v_{321}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,0,1,1;\frac{1}{u_{312}}\right)-$$
$$\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(0,1,0,1;\frac{1}{u_{312}}\right)+$$
$$\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{123}}\right)+\frac{1}{4}\mathcal{H}\left(0,1,1,1;\frac{1}{v_{132}}\right)+\zeta_3H(0;u_1)+\zeta_3H(0;u_2)+\zeta_3H(0;u_3)+$$
$$\frac{5}{2}\zeta_3H(1;u_1)+\frac{5}{2}\zeta_3H(1;u_2)+\frac{5}{2}\zeta_3H(1;u_3)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{123}}\right)+\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{231}}\right)+$$
$$\frac{1}{2}\zeta_3\mathcal{H}\left(1;\frac{1}{u_{312}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{123}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{231}}\right)-\frac{1}{2}\mathcal{H}\left(1,0,0,1;\frac{1}{u_{312}}\right)+$$
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There has to be a better way...

(otherwise we should all abandon $\mathcal{N}=4$ SYM!)

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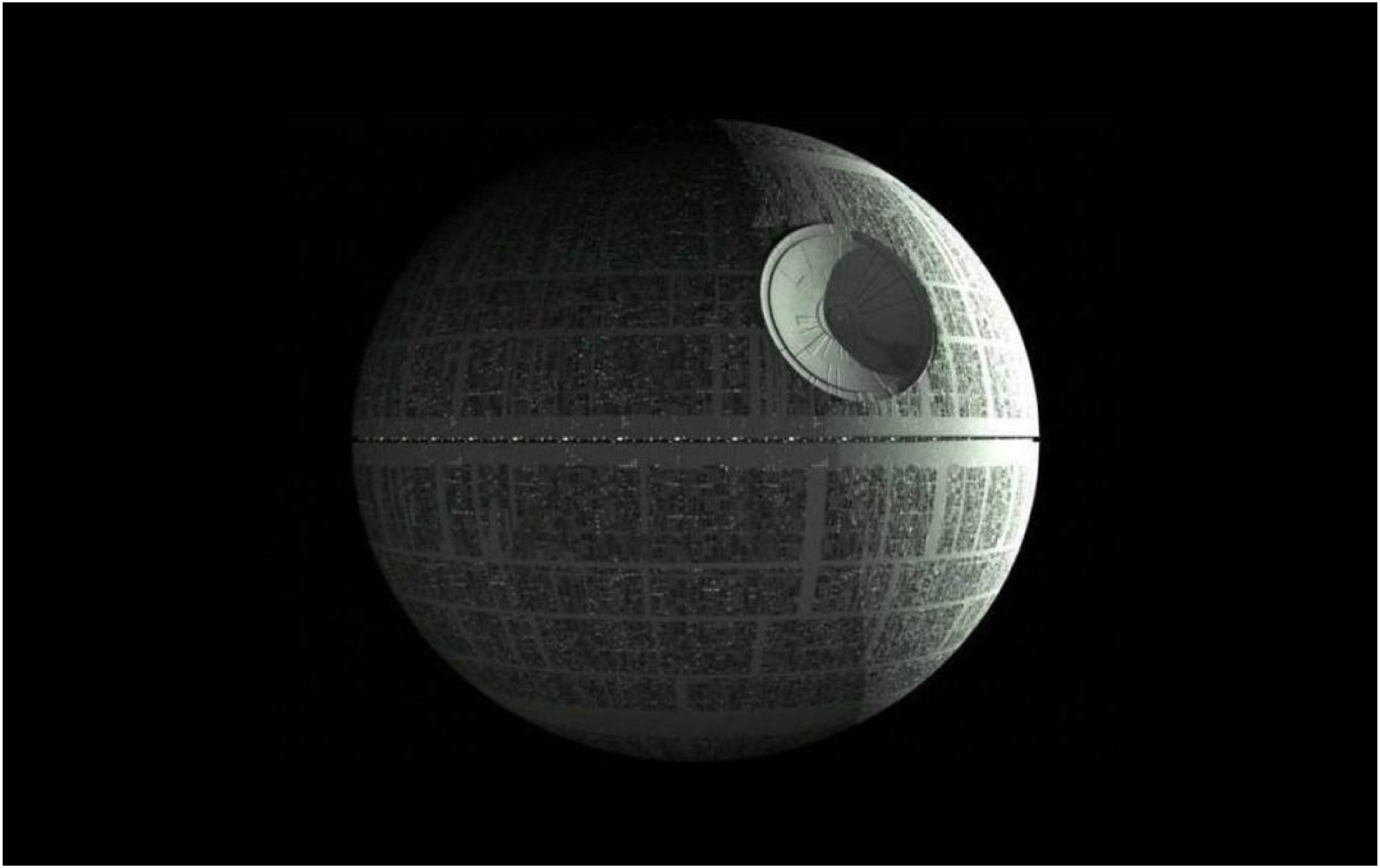
But to do battle against the Goncharov polylogarithms we need a serious weapon –

Some Hi-Tech from the theory of Mixed Motives



Transcendentality

The two-loop remainder function has uniform transcendentality degree 4.

A function of degree k is one which can be written as

$$T_k = \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} dt_k \; \gamma^*(d\log R_1(t_1)) \circ \cdots \gamma^*(d\log R_k(t_k))$$

where
- R_i are rational functions with rational coefficients
- $\gamma(0)=a$, $\gamma(1)=b$, a, b are rational points in $\mathbb{C}^n$
- $\int d\log R_1 \circ \cdots \circ d\log R_k = \int_a^b \left(\int_a^t d\log R_1 \circ \cdots \circ d\log R_{k-1} \right) d\log R_k(t)$
- The integral is taken along a path in $\mathbb{C}^n$ and one needs to check local homotopy invariance.

The Symbol of a Trancendental Function

A very useful quantity associated to a function of uniform degree k is its symbol.

The symbol is an element of the

k-fold tensor product of the multiplicative group of rational functions (modulo constants)

$$\boxed{\text{symbol}(T_k) = R_1 \otimes R_2 \otimes \cdots \otimes R_k}$$

The Symbol made Simple

A function of degree k is one which can be written as a k-fold iterated integral of a rational integrand

$$T_k(x_1,\dots,x_m)=\int^1 dt_1 \int^{t_1} dt_2 \cdots \int^{t_{k-1}} dt_k \, R(x_1,\dots,x_m;t_1,\dots,t_k)$$

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The symbol is a way to express the integrand in a way which preserves information about the order (but not the path) of integration.

The symbol can be computed recursively. If

$$dT_k = \sum_i T^i_{k-1} \, d\log R_i$$

Then $\text{symbol}(T_k) = \sum_i \text{symbol}(T^i_{k-1}) \otimes R_i$

Note that the symbol satisfies

$$R_1 \otimes \cdots \otimes (R_a R_b) \otimes \cdots \otimes R_k$$
$$= R_1 \otimes \cdots \otimes R_a \otimes \cdots \otimes R_k \quad + \quad R_1 \otimes \cdots \otimes R_b \otimes \cdots \otimes R_k$$

$$R_1 \otimes \cdots \otimes (c R_i) \otimes \cdots \otimes R_k$$
$$= R_1 \otimes \cdots \otimes R_i \otimes \cdots \otimes R_k \qquad \text{for any constant } c$$

... properties it inherits from $d\log R$.

Examples

- degree 0 $\quad T_0 = R \qquad \text{symbol}(T_0) = 0$
- degree 1 $\quad T_1 = \log R \qquad dT_1 = d\log R \Rightarrow \text{symbol}(T_1) = R$
- degree 2 $\quad T_2 = \mathrm{Li}_2(R) = \int_0^R -\log(1-t)\, d\log t$

$$dT_2 = -\log(1-R)\, d\log R$$

$$\Rightarrow \quad \text{symbol}(T_2) = -(1-R) \otimes R$$

or consider $\quad T_2 = \log R_1 \log R_2$

$$dT_2 = \log R_1\, d\log R_2 + \log R_2\, d\log R_1$$

$$\Rightarrow \text{symbol}(T_2) = R_1 \otimes R_2 + R_2 \otimes R_1$$

Classical Polylogarithms

The functions Li_k are defined recursively by

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t)\, d\log t \qquad \mathrm{Li}_1(z) = -\log(1-z)$$

$$\Rightarrow \quad \mathrm{symbol}(\mathrm{Li}_k(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{k-1 \text{ times}}$$

Using these definitions it is straightforward to calculate the symbol of all harmonic & Goncharov polylogs, and hence the symbol S of the DDS formula.

But what is it good for?

Uses of the Symbol

The symbol converts polylog functional equations into rational function identities

$$\text{symbol}(\mathrm{Li}_2(z)) = -(1-z)\otimes z$$

$$\Rightarrow \text{symbol}(\mathrm{Li}_2(1-1/x)) = -(1/x)\otimes(1-1/x) = x\otimes\left(\frac{x-1}{x}\right)$$

$$= x\otimes\left(\frac{1-x}{x}\right) = x\otimes(1-x) - x\otimes x$$

$$= \text{symbol}\left(-\mathrm{Li}_2(1-x) - \frac{1}{2}\log(x)^2\right)$$

$$\Rightarrow \boxed{\mathrm{Li}_2(1-1/x) = -\mathrm{Li}_2(1-x) - \frac{1}{2}\log(x)^2}$$

Ambiguities

Here we got lucky, but consider

$$\mathrm{Li}_2(1-x) + \log(x)\log(1-x) = -\mathrm{Li}_2(x) + \frac{\pi^2}{6}$$

The symbol doesn't see this term

$\Rightarrow$ The symbol only fixes the "leading transcendentality" piece, i.e. modulo constants times functions of lower degree

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A related ambiguity is that the symbol obviously has no knowlege of where to put branch cuts

$\log(x)$, $\log(-x)$, $-\log(\frac{1}{x})$, $\frac{1}{2}\log(x^2)$, $\frac{3}{2}\log(-x^{2/3})$ have the same symbol

The Symbol of the DDS Function

The symbol $\mathcal{S}$ is a mess! It involves arguments

u_i,

$1-u_i$,

u_i+u_j,

$1-u_i-u_j$

$$v_{jk\ell}^{\pm} = \frac{u_k - u_\ell \pm \sqrt{(u_k+u_\ell)^2 - 4u_j u_k u_\ell}}{2(1-u_j)u_k}$$

$$u_{jk\ell}^{\pm} = \frac{1-u_j-u_k+u_\ell \pm \sqrt{\Delta}}{2(1-u_j)u_\ell}$$

$$\Delta = (1-u_1-u_2-u_3)^2 - 4u_1u_2u_3$$

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This is a redundant set; they satisfy many algebraic identies.

Eliminate these in terms of the rest (not obvious that this had to be possible!)

Momentum Twistors

We still have a $\sqrt{\Delta}$ we don't want; we need to find variables on a covering space.

The "right" variables are

$$u_1 = \frac{z_{23} z_{56}}{z_{25} z_{36}} \qquad u_2 = \frac{z_{16} z_{34}}{z_{14} z_{36}} \qquad u_3 = \frac{z_{12} z_{45}}{z_{14} z_{25}} \qquad \begin{array}{l} z_{ij} = z_i - z_j \\ z_i \in \mathbb{P}^1 \end{array}$$

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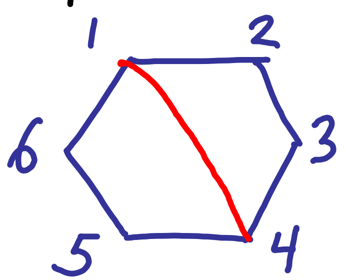
or, if you prefer momentum twistors,

$$u_1 = \frac{\langle 1234 \rangle \langle 4561 \rangle}{\langle 1245 \rangle \langle 3461 \rangle} \quad \text{etc (cyclically)}$$

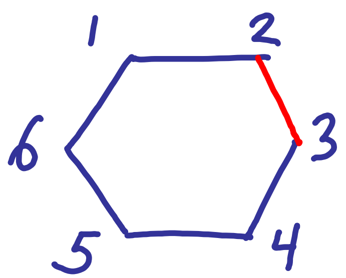
Then Δ becomes a perfect square, so the symbol can be expressed in terms of cross-ratios of z's.

Cross-Ratios

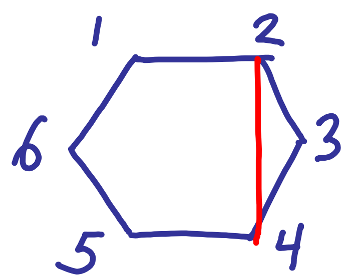
There are 15 different kinds of cross-ratios (of course, only 3 are independent, they satisfy algebraic relations)



$$\frac{z_{23} z_{56}}{z_{25} z_{36}} = u_1$$ "diagonal" u_1, u_2, u_3



$$\frac{z_{14} z_{56}}{z_{16} z_{45}} \equiv -x_1^+$$ "edge" $x_1^\pm, x_2^\pm, x_3^\pm$



$$\frac{z_{13} z_{56}}{z_{16} z_{35}}$$ "chord" 6 of these

$$x_i^\pm = u_i \, x^\pm, \qquad x^\pm = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2 u_1 u_2 u_3} \qquad x_i^+ x_i^- = \frac{u_i^2}{u_1 u_2 u_3}$$

Miracle #1

In z coordinates, the symbol could have been a linear combination of terms like

$$p_1(r_i) \otimes p_2(r_i) \otimes p_3(r_i) \otimes p_4(r_i)$$

for arbitrary polynomials of the cross-ratios r_i

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We found that in fact it takes the form

$$S = \sum_{i,j,k,l=1}^{15} c_{ijkl}\, r_i \otimes r_j \otimes r_k \otimes r_l$$ for rational c_{ijkl}

(There is considerable ambiguity in writing it this way due to identities like $\frac{z_{14} z_{56}}{z_{16} z_{45}} \cdot \frac{z_{45} z_{13}}{z_{14} z_{35}} = \frac{z_{13} z_{56}}{z_{16} z_{35}}$)

Miracle #2

We find that S can be expressed in terms of the diagonal and edge cross-ratios only; chords drop out.

(This should provide a possible hint about the structure of the n-point remainder function for $n>6$)

Let us decompose the symbol S into four pieces depending on the symmetry properties under exchange of the first two entries, and last two entries

$$
S \rightarrow
\begin{cases}
S \otimes S & \text{symmetric} \otimes \text{symmetric} \\
A \otimes S & \text{antisymmetric} \otimes \text{symmetric} \\
S \otimes A & \text{symmetric} \otimes \text{antisymmetric} \\
A \otimes A & \text{antisymmetric} \otimes \text{antisymmetric}
\end{cases}
$$

i.e. $(A \otimes A)_{ijkl} = \frac{1}{4}(C_{ijkl} - C_{jikl} - C_{ijlk} + C_{jilk})$ etc.

Miracle #3

We find that $A \wedge A = 0$

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Motivic hi-tech: any function of transcendentality degree 4 whose symbol satisfies $A \wedge A = 0$ can be expressed in terms of the classical (poly)logs $\mathrm{Li}_4(x)$, $\mathrm{Li}_3(x)$, $\mathrm{Li}_2(x)$, $\mathrm{Li}_1(x)$ & $\log(x)$ only!

A comment: there is no constructive algorithm for determining what arguments can appear inside the polylogs, they could be in principle arbitrary rational functions of your original variables.

e.g.
$$\begin{aligned}(1+x+x^2)\otimes x &= \left(\frac{1-x^3}{1-x}\right)\otimes x \\ &= (1-x^3)\otimes x - (1-x)\otimes x \\ &= \frac{1}{3}(1-x^3)\otimes x^3 - (1-x)\otimes x\end{aligned}$$

$$\Rightarrow -\frac{1}{3}\mathrm{Li}_2(x^3) + \mathrm{Li}_2(x)$$

In practice, guessing, experimentation & luck required!

A Divide & Conquer Algorithm

We want to find a function with given symbol S in terms of

	$A \otimes A$	$S \otimes A$	$A \otimes S$	$S \otimes S$	
$\mathrm{Li}_4(x)$	✗	✗	✓	✓	← do third
$\mathrm{Li}_3(x)\log(y)$	✗	✗	✓	✓	← do third
$\mathrm{Li}_2(x)\mathrm{Li}_2(y)$	✓	✓	✓	✓	← do first
$\mathrm{Li}_2(x)\log(y)\log(z)$	✗	✓	✓	✓	← do second
$\log(x)\log(y)\log(z)\log(w)$	✗	✗	✗	✓	← do last

(Note that $\mathrm{Li}_2(x)\mathrm{Li}_2(y)$ satisfies $A \wedge A = 0$!)

Analyticity

This algorithm gives a prototype, $\tilde{R}$, which has the same symbol as that of the DDS formula for $R_6^{(2)}$.

The most annoying part was finding an an expression which put all branch cuts in the right place.

Physical input specifies that we expect the remainder function to be smooth (and real-valued) everywhere in the Euclidean regime where $u_i > 0$.

This frequently involved "unsimplifying"

i.e. $$\frac{1}{2}\log(x^+/x^-) \longrightarrow \sum_{i=1}^{3} \ell_1(x_i^+) - \ell_1(x_i^-)$$

where $$\ell_n(x) = \frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}(-1)^n \mathrm{Li}_n(1/x)$$

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In terms of the function

$$L_4(x^+, x^-) = \ell_4(x^+) - \frac{1}{2}\log(x^+x^-)\,\ell_3(x^+) + \frac{1}{8}\log(x^+x^-)^2\,\ell_2(x^+) - \frac{1}{48}\log(x^+x^-)^3\,\ell_1(x) + \frac{1}{384}\log(x^+x^-)^4 + (x^+ \longleftrightarrow x^-)$$

we find...

Final (?) Result

$$R = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathrm{Li}_4(1 - 1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1 - 1/u_i)\right)^2 + \frac{J^4}{12}$$

$$+ \frac{\pi^2}{12}\left(J^2 + \frac{\pi^2}{6}\right) \times \underbrace{\begin{cases} -2 & \text{if } \Delta > 0,\ 1 < u_1 + u_2 + u_3 \\ +1 & \text{otherwise} \end{cases}}_{\text{This piece is required to cancel discontinuities across various branch cuts.}}$$

The full function is completely smooth, manifestly symmetric in u_1, u_2, u_3, and valid for all $u_i > 0$.

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our formula provides hope, where before there was none, to the idea that we might be able to really unlock the secrets of multi-loop SYM amplitudes, and hopefully connect to strong coupling (Alday, Gaiotto, Maldacena, Sever, Vieira)

End of the Coefficient × Integral Paradigm?

Despite all the recent progress in amplitudes, the paradigm for computing loop amplitudes remains

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But perhaps there is no piper!