# Weak coupling tests of the mirror TBA

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IGST10 Stockholm, 28 June -2 July

Based on:
J. Balog, Á. Hegedűs: arXiv:1002.4142
arXiv:1003.4303

# Outline

- Motivation
- Recall of the TBA approach
- Multiparticle Lüscher corrections (4- 5-loops)
- Linearized TBA at 5-loops
- 5-loop Konishi from TBA
- Conclusions

# Motivation

- When computing anomalous dimensions in N=4 SYM theory from 2-pt functions

$$\langle O(x)\, O(y)\rangle = \frac{\text{const}}{|x-y|^{2\Delta}}$$

Two classes of Feynmann graphs arise:

- 1st class: creates interactions among few neighbouring fields of the operators
- 2nd class: creates inteactions among all the fields of the operators (wrapping graphs)

- 1st class is contained in the Asymptotic Bethe Ansatz (BS '05)
- 2nd class: „wrapping interactions" start to appear at $g^{2L}$ and are not contained in the Asymptotic Bethe Ansatz

# Motivation

- The contribution of all „wrapping graphs" is thought to be contained in the mirror TBA system of the light-cone string σ-model in $AdS_5 \times S^5$

Ambjorn, Janik, Kristjansen '05
Arutyunov, Frolov '07

- TBA is based on several conjectures: tests against direct field theory or string-theory computations are neccessary

- Using the mirror TBA equations we calculate the 5-loop anomalous dimensions of the twist-2 operators of the sl(2) sector.

- Twist-2 operators: $\mathrm{Tr}(Z\, D^M\, Z + \ldots)$

$$\Delta(M) = M + 2 + \gamma_{ABA}(M) + \gamma_{wrapp}(M)$$

- Konishi: M=2

# What is interesting at 5-loops?

- Twist-2: wrapping starts at 4-loops i.e $g^8$
- 4-loop wrapping is built into the form of TBA eqs.
- 4-loop wrapping can be expressed by the asymptotic (large L) solution of the TBA
- 5-loop: asymptotic solution is not enough, a linear perturbation of the asymptotic form of the TBA must be treated

- Direct field theory computation at 5-loops? Not available, yet.
- Compare with results of the multiparticle Lüscher approach

# Thermodynamic Bethe Ansatz (TBA)

- Consider the Asymptotic Bethe Ansatz description of the mirror theory

Arutyunov, Frolov '07 '09

- Termodynamics ⟶ extremizing „Witten index density"

TBA integral equations ⟶ $\mathcal{F}[T]$ ⟶ $E_0(L)$

$$\mathcal{F}[T] = \frac{E_0(L)}{L} \qquad T = 1/L$$

Bombardelli, Fioravanti, Tateo '09
Arutyunov, Frolov, Suzuki '09
Gromov, Kazakov, Vieira '09

- Variables: Y-functions

# Thermodynamic Bethe Ansatz

- TBA eqs: $\ln Y_A = s_A + K_{AB} \star \ln(1+Y_B)$
- Energy: $E_0(L) = -\sum_Q \int \frac{du}{2\pi} \frac{d\tilde{p}^Q}{du} \ln(1+Y_Q(u))$

  $\tilde{p}^Q$ = mirror momentum $u$ = rapidity
- SUSY: $E_0(L) = 0$
- Excited states: analytical continuation

  Dorey, Tateo '96
  BLZ '96
- Ground state solutions satisfy Y-system functional eqs.

  Gromov, Kazakov, Vieira '09
  Bombardelli, Fioravanti, Tateo '09
  Arutyunov, Frolov, Suzuki '09

# Thermodynamic Bethe Ansatz

- Y-system is assumed to be universal for all excited states of the model

- Y-system $\longrightarrow$ asymptotic (large L) solution (ABA)

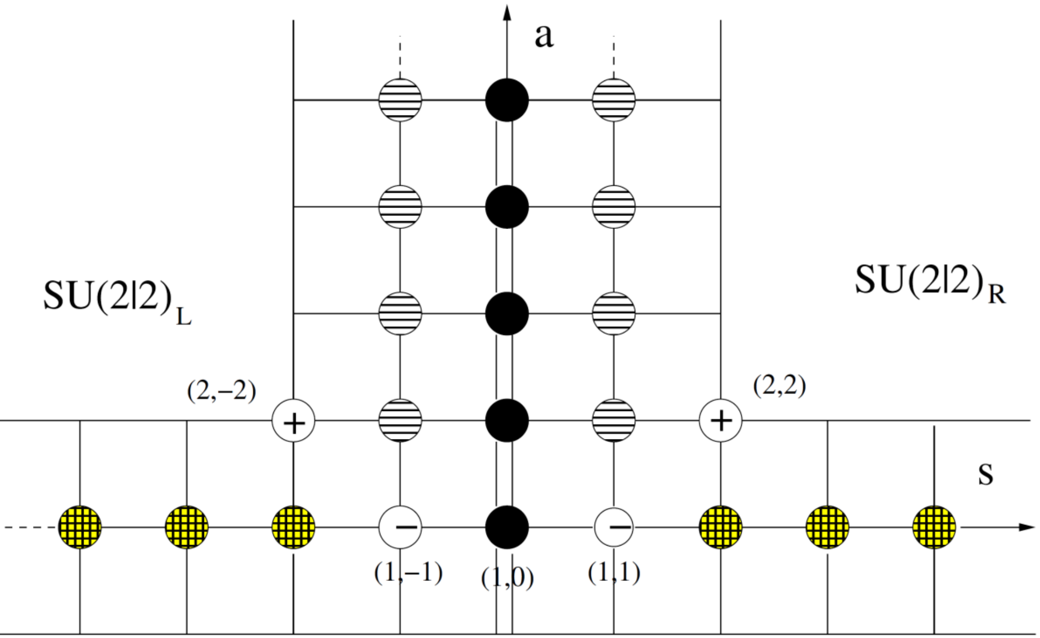
  Gromov, Kazakov, Vieira '09

- TBA for excited states: analytical continuation of the ground state eqs. respecting the asymptotic solution. (Assuming that in a sense the large L solution is close to the exact result)

Gromov, Kazakov, Vieira '09
Arutyunov, Frolov, Suzuki '09

# Asymptotic solution

- **Y-system:** $$Y^+_{a,s}\,Y^-_{a,s}=\frac{(1+Y_{a,s+1})\,(1+Y_{a,s-1})}{(1+\frac{1}{Y_{a+1,s}})(1+\frac{1}{Y_{a+1,s}})}$$ $$f^{\pm a}=f(u\pm\frac{i\,a}{g})$$
- **T-system:** $$T^+_{a,s}\,T^-_{a,s}=T_{a,s-1}\,T_{a,s+1}+T_{a+1,s}\,T_{a-1,s}$$




$$Y_{a,s}=\frac{T_{a,s-1}\,T_{a,s+1}}{T_{a+1,s}\,T_{a-1,s}}$$

**From TBA:**

$$Y_{a,s\neq 0}(L\to\infty)\sim 1$$

$$Y_{a,0}(u)\sim\left(\frac{x^{+a}}{x^{-a}}\right)^L\sim e^{-\tilde{\mathcal{E}}_a\,L}\to 0$$

$$x(u)=\frac{1}{2}(u-i\sqrt{4-u^2})\qquad x+\frac{1}{x}=u$$

- Asymptotic (large L) solution: two SU(2|2) Y-systems (T-systems) decouple

  - ABA √
  - Lüsher F-term √

# Tests of the TBA

- Strong coupling limit: $g \sim L \to \infty$ TBA agrees with 1-loop quasi-classical string solutions

  Gromov '09: $sl(2)$

  Gromov, Kazakov, Sakai '10: $\mathrm{PSU}(2,2|4)$

- It reproduces the Asymptotic Bethe Ansatz of Beisert, Staudacher '05
- Weak coupling limit:

4-loop wrapping → Konishi →
- QFT Velizhanin '08, Fiamberti, Santambrogio,Sieg, Zanon '08
- Lüscher Bajnok, Janik '08

4-loop wrapping → Twist-2 – Lüscher Bajnok, Janik, Lukowski '09

5-loops → Konishi - Lüscher Bajnok, Hegedűs, Janik, Lukowski '09

Arutyunov, Frolov, Suzuki '10

Balog, Hegedűs '10

5-loops → Twist-2 – Lüscher Lukowski ,Rej, Velizhanin '09

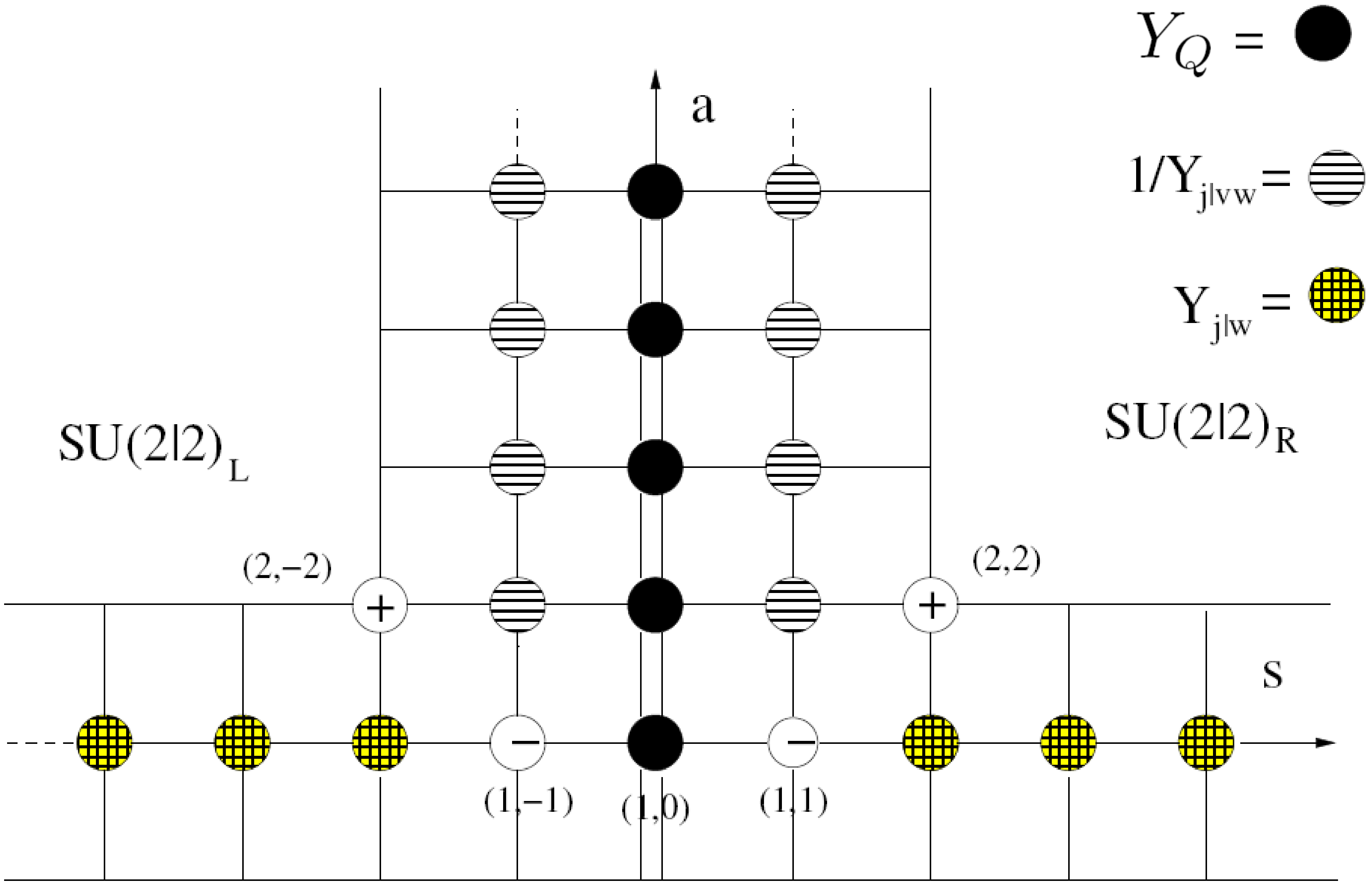
Balog, Hegedűs '10

# Conventions for the TBA variables

- $Y_Q$ functions belong to Q-bound states: $Y_Q(u) = Y_{Q,0}(u)$
- $Y_{j|vw}^{(\pm)}$ functions belong to the j-vw-string: $Y_{j|vw}^{(\pm)} = \frac{1}{Y_{j+1,\pm 1}}$
- $Y_{j|w}^{(\pm)}$ functions belong to the j-w-string: $Y_{j|w}^{(\pm)} = Y_{1,\pm(j+1)}$
- $Y_{\pm}^{(\pm)}$ functions belong to y-particles $\begin{cases} Y_{+}^{(\pm)} = -Y_{2,\pm 2} \\ Y_{-}^{(\pm)} = -\frac{1}{Y_{1,\pm 1}} \end{cases}$

Conventions used by Arutyunov, Frolov, Suzuki '09

# Conventions for the TBA variables



# What is new at 5-loops?

- Wrapping corrections to ABA start to play role

- Asymptotic solution is not enough. Linearization of the TBA eqs. must be „solved”.

- *What to compare with?*
- With 5-loop multiparticle Lüscher computations { BHJL '09, LRV '09

- They satisfied so many non-trivial constraints, that there is little doubt that they give the correctly the 5-loop anomalous dimensions of the twist-2 operators.

- Advantage: it is enough to get the starting formulae of the Lüscher approach

# About the Lüscher approach

- The Lüscher approach expresses the exponentially small corrections to the volume dependence of the energy in terms of the ∞ volume scattering data.

- 1-particle states ⟶ QFT techniques
  - Lüscher '86 (relat.)
  - Klassen, Melzer '91 (relat.)
  - Janik,Lukowski '08 (non-relat.)

- The multiparticle Lüscher formula is a conjecture based on:

  Bajnok, Janik '08

- The 1-particle results
- Explicit TBA computations in diagonally scattering relat. and (hypothetical) non-relat. models
- TBA and NLIE computations in certain non-diagonally scattering relativistic theories

# Lüscher approach

The Bajnok, Janik formula was supported by explicit calculations:

- In non-diagonally scattering relativistic models, like:

O(4) NLS model — Gromov,Kazakov, Vieira '08

O(2), O(3) NLS model
SU(N) PCM
} Balog, Hegedűs '10 (unpublished)

- Non-relativistic support: 4-loop Konishi (QFT)

and the 4- and 5-loop twist-2 (consistency)
and 5- and 6-loop twist-3 (*simplest case: QFT* & consistency)

Bajnok,Janik '08, Fiamberti, Santambrogio,Sieg, Zanon '08, Velizhanin '08
Bajnok,Janik,Lukowski '08, BHJL '09, Lukowski,Rej,Velizhanin '09

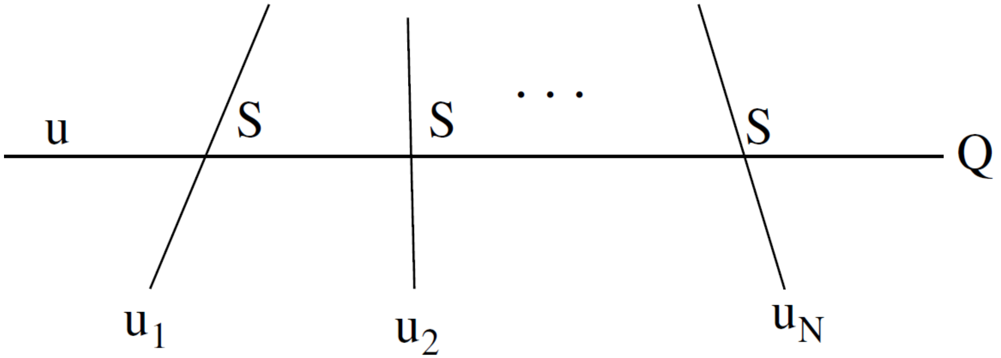
Beccaria, Forini, Lukowski, Zieme '09, Fiamberti, Santambrogio, Sieg '09
Velizhanin '10

# The multiparticle Lüscher formulae

$$E=\underbrace{\sum_j \mathcal{E}(p_j)}_{\text{ABA}}+\underbrace{\sum_j \mathcal{E}'(p_j)\,\delta p_j}_{\text{ABA modification}}-\underbrace{\sum_Q \int \frac{du}{2\pi}\,\frac{d\tilde{p}^Q}{du}\,Y_Q(u)}_{\text{F-term}}$$

- F-term:

$$Y_Q \sim \Lambda_Q\, e^{-\tilde{\mathcal{E}}_Q\, L}$$




$$\Lambda_Q(u,\{u_j\})=\mathrm{Str}(S_{Q1}(u,u_1)\,S_{Q2}(u,u_2)\ldots S_{QN}(u,u_N))$$

- Twist-2: F-term starts at 4-loops

# The modification of the ABA

$$\left.\frac{\partial \text{ABA}}{\partial p_k}\right|_0 \delta p_k + i\Phi_k = 0$$

$$\Phi_k \sim \sum_Q \int \frac{du}{2\pi} \text{Str}(S_{Q1}(u,u_1)\ldots\partial_u S_{Qk}(u,u_k)\ldots S_{QN}(u,u_N))\, e^{-\tilde{\mathcal{E}}_Q(u)\,L}$$

$$\mathcal{E}(p) = \sqrt{1+4g^2\sin^2\frac{p}{2}}$$

- Twist-2: $\mathcal{E}'(p_j)\,\delta p_j \sim g^2\, g^8$

Wrapping correction to the ABA starts at 5-loops

- Luckily: $\Phi_k = \sum_Q \int \frac{du}{2\pi}\, \partial_{u_k} Y_Q(u,\{u_j\})\Big|_{\underline{u}=\underline{u}^o} + \mathcal{O}(g^{10})$

# Magnitudes

- Magnitude of integrals: $$\left.\begin{array}{r}\text{F-term}\\ \Phi_k\end{array}\right\} \sim e^{-\tilde{\mathcal{E}}_Q L} \sim g^{2L}$$
- Twist-2: $\sim g^8$
- The ABA modification term appears at one order higher: $\mathcal{E}'(p_j)\,\delta p_j \sim g^2\, g^8 \sim g^{10}$
- 5-loop twist-2 computations have been done recently
  BHJL '09 (Konishi), Lukowski, Rej, Velizhanin '09
- Twist-2 results seem to be correct as they satisfy lots of non-trivial consistency checks

# Contraints satisfied by twist-2 results

- At 4-loops: $\gamma_8(M) = \gamma_8^{\mathrm{ABA}}(M) + \gamma_8^{wrapp}(M)$
- Wrapping part has no piece $\sim \log M$ ( $\Delta_{cusp}$ unmodified)
- Maximal transcendentality principle of Kotikov, Lipatov.
  $\gamma_8(M)$ has transcendentality degree 7.
- $\gamma(M)$ analytically continued to M=-1+ω has the prescribed pole structure from BFKL and NLO BFKL eqs.

$$\gamma_8^{wrapp}(M) \sim -640\, S_1^2\, \zeta(5) - 512\, S_1^2\, S_{-2}\, \zeta(3) +$$

$$+256\, S_1^2 \left(-S_5 + S_{-5} + 2\, S_{4,1} - 2\, S_{3,-2} + 2S_{-2,-3} - 4\, S_{-2,-2,1}\right)$$

where $S_k \equiv S_k(M) = \sum_{n=1}^{M} 1/n^k$ etc. Bajnok,Janik,Lukowski '08

# Contraints satisfied by twist-2 results

- At 5-loops:
- Transcendentality √
- BFKL and NLO BFKL √ Lukowski, Rej, Velizhanin '09
- The success of the Lüscher approach makes natural to compare TBA with the multiparticle Lüscher formulae!

# 5-loop twist-2 from TBA

- Energy fomula: $J = 2$ for twist-2 operators

$$E = J + \sum_{i=1}^{N} \mathcal{E}(p_i) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \frac{d\tilde{p}^Q}{du} \log(1 + Y_Q)$$

- 1-particle energy: $\mathcal{E}(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$
- Linear expansion:

$$E = J + \sum_{i=1}^{N} \mathcal{E}(p_i^o) + \sum_{i=1}^{N} \mathcal{E}'(p_i^o)\delta p_i - \underbrace{\sum_{Q=1}^{\infty} \int \frac{du}{2\pi} \frac{d\tilde{p}^Q}{du} Y_Q}_{\text{Lüscher-F-term}} + \mathcal{O}(g^{16})$$

- Only the wrapping correction to the ABA should be checked!

# Wrapping corrected ABA for twist-2

- Leading order wrapping corrections:

$$\pi(2n_k+1) = J\,p_k + i\sum_{j=1}^{N}\log S^{1_*1_*}_{\mathfrak{sl}(2)}(u_j,u_k) + \delta\mathcal{R}_k$$

$$\delta\mathcal{R}_k = \delta\mathcal{R}_k^{(\mathrm{BJ})} + \mathcal{O}(g^{10})$$

$$\delta\mathcal{R}_k^{(\mathrm{BJ})} = \frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{\partial}{\partial u_k}\,Y_Q^o(u)\Big|_{\{u_j\}=\{u_j^o\}}$$

- We will show this formula from TBA in case of the Konishi operator (N=2)

# Exact Bethe equations (Twist-2)

$$\pi i(2n_k+1) = \log Y_{1_*}(u_k) = iL\,p_k - \sum_{j=1}^{N} \log S_{\mathfrak{sl}(2)}^{1_* 1_*}(u_j,u_k)$$

$$+2\sum_{j=1}^{N} \log \mathrm{Res}(S) \star K_{vwx}^{11_*}(u_j^-,u_k) - 2\sum_{j=1}^{N} \log\left(u_j-u_k-\frac{2i}{g}\right)\frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+}$$

$$-2\sum_{j=1}^{n_1}\left(\log S\,\hat{\star}\,K_{y1_*}(r_j^{(1)-},u_k) - \log S(r_j^{(1)}-u_k)\right)$$

$$+\log(1+Y_Q)\star\left(K_{\mathfrak{sl}(2)}^{Q1_*} + 2\,s\star K_{vwx}^{Q-1,1_*}\right) + 2\log\left(1+Y_{1|vw}\right)\star\left(s\,\hat{\star}\,K_{y1_*}+\tilde{s}\right)$$

$$-2\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\,s\star K_{vwx}^{11_*} + \log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\,K_1 + \log\left(1-\frac{1}{Y_-}\right)\left(1-\frac{1}{Y_+}\right)\hat{\star}\,K_{y1_*}$$

$$L = N + J \qquad J = 2$$

- Konishi: $N = 2$

Gromov, Kazakov,Kozak,Vieira ‘09
Arutyunov,Frolov,Suzuki ‘09

# Linearization of mirror TBA

Arutyunov, Frolov, Suzuki '10

- $Y_Q \to Y_Q^o \sim e^{-\tilde{\mathcal{E}}_Q(u)\,L}$ $\qquad \ln(1+Y_Q^o) \to Y_Q^o$

- Expand all other Y-functions around their asymptotic solution:

$$Y = Y^o(1+\mathscr{Y}) \qquad \mathscr{Y}(u) = \sum_Q \int du' \mathcal{K}_y(u,u') \underbrace{Y_Q^o(u')}_{\sim e^{-\tilde{\mathcal{E}}_Q(u)\,L} \sim g^8} + \ldots$$

- Solve the linearized eqs. at $e^{-\tilde{\mathcal{E}}_Q(u)\,L}$ order i.e. linearly in $Y_Q^o$

- Terms $\sim Y_Q^o$ give the Lüscher (wrapping) correction to ABA

# Linearized TBA eqs. (Konishi)

Arutyunov, Frolov, Suzuki '10

- m|w-strings: $m \geq 1\ ,\ \mathcal{Y}_{0|w} = 0$

$$\mathcal{Y}_{m|w} = (A_{m-1|w}\mathcal{Y}_{m-1|w} + A_{m+1|w}\mathcal{Y}_{m+1|w}) \star s + \delta_{m1}\left(\frac{\mathcal{Y}_+}{1-Y^o_+} - \frac{\mathcal{Y}_-}{1-Y^o_-}\right)\hat{\star} s$$

coefficients: $A_{m|w} = \frac{Y^o_{m|w}}{1+Y^o_{m|w}}$

$$\hat{\star} = \int_{-2}^{2} du \ldots$$

- m|vw-strings: $m \geq 1\ ,\ \mathcal{Y}_{0|vw} = 0$

$$\mathcal{Y}_{m|vw} = (A_{m-1|vw}\mathcal{Y}_{m-1|vw} + A_{m+1|vw}\mathcal{Y}_{m+1|vw}) \star s - Y^o_{m+1} \star s$$
$$+ \delta_{m1}\left(\frac{\mathcal{Y}_-}{1-\frac{1}{Y^o_-}} - \frac{\mathcal{Y}_+}{1-\frac{1}{Y^o_+}}\right)\hat{\star} s$$

$$s(x) = \frac{g}{4\cosh(\frac{\pi g x}{2})}$$

coefficients: $A_{m|vw} = \frac{Y^o_{m|vw}}{1+Y^o_{m|vw}}$

# Linearized TBA

- y-particles:

$$\mathscr{Y}_+ - \mathscr{Y}_- = Y_Q^o \star K_{Qy}\,,$$

$$\mathscr{Y}_+ + \mathscr{Y}_- = 2(A_{1|vw}\mathscr{Y}_{1|vw} - A_{1|w}\mathscr{Y}_{1|w}) \star s - Y_Q^o \star s + 2Y_Q^o \star K_{xv}^{Q1} \star s$$

- Perturbative orders:

$$Y_Q^o(u) = O(g^8) \qquad\qquad \mathscr{Y}_+ - \mathscr{Y}_- = O(g^9)$$

- $Y_+^o(u)$ and $Y_-^o(u)$ coincide at leadning order:

$$\frac{Y_+^o(u)}{Y_-^o(u)} = 1 + O(g^2)$$

- Decoupling:

$$\mathscr{Y}_\pm = O(g^8) \qquad \mathcal{Y}_{m|vw} = O(g^8) \qquad \mathcal{Y}_{m|w} = O(g^9)$$

# Decoupling at $g^8$

- The vw-type linear problem decouples: $A_{j|vw}\,\mathscr{Y}_{j|vw} = \delta L_j$

$$D_m^\gamma\,\delta L_m^\gamma - K\star(\delta L_{m+1}^\gamma + \delta L_{m-1}^\gamma) = -K^\gamma\star Y_{m+1}^o, \qquad m = 1, 2, \ldots$$

$$D_m(u) = \frac{1}{A_{m|vw}(u)} \qquad\qquad K(x) = \frac{1}{4\,\cosh(\frac{\pi x}{2})}$$

- Wrapping corrected Bethe equations:

$$\pi(2n_k + 1) = 2\,p_k + i\sum_{j=1}^{2}\log S_{\mathfrak{sl}(2)}^{1_*1_*}(\frac{u_j}{g}, \frac{u_k}{g}) + \delta\mathcal{R}_k$$

$$\delta\mathcal{R}_k = \sum_Q \int du\, \underbrace{\mathcal{K}_Q(u_k, u)}_{\sim 1}\, \underbrace{Y_Q^o(u)}_{\sim g^8} + \int du\, \underbrace{\mathcal{K}_1(u_k, u)}_{\sim 1}\, \underbrace{\mathscr{Y}_{1|vw}(u)}_{\sim g^8} +$$

$$+ \int du\, \underbrace{\mathcal{K}_+(u_k, u)}_{\sim g}\, \underbrace{[\mathscr{Y}_+(u) + \mathscr{Y}_-(u)]}_{\sim g^8} + \int du\, \underbrace{\mathcal{K}_-(u_k, u)}_{\sim 1}\, \underbrace{[\mathscr{Y}_+(u) - \mathscr{Y}_-(u)]}_{\sim g^9}$$

# The $O(g^8)$ wrapping correction for ABA

$$\delta\mathcal{R}_k = \delta\mathcal{R}_k^{(1)} + \delta\mathcal{R}_k^{(2)} + \delta\mathcal{R}_k^{(3)}$$

$$\delta\mathcal{R}_k^{(1)} = \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_m^o(u)\,\frac{u-u_k}{(m+1)^2+(u-u_k)^2}$$

$$\boxed{\delta\mathcal{R}_k^{(2)} = \int_{-\infty}^{\infty}\mathrm{d}u\;\frac{\delta L_1(u)}{2\sinh\frac{\pi}{2}(u-u_k)}}$$

$$\delta\mathcal{R}_k^{(3)} = \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^o(u)\,\left\{\mathcal{F}_m(u-u_k) - \frac{u-u_k}{m^2+(u-u_k)^2}\right\}$$

$$\mathcal{F}_m(u) = \frac{-i}{4}\left\{\psi\left(\frac{m+iu}{4}\right) - \psi\left(\frac{m-iu}{4}\right) - \psi\left(\frac{m+2+iu}{4}\right) + \psi\left(\frac{m+2-iu}{4}\right)\right\}$$

$$\psi(z) = (\ln\Gamma)'(z)$$

# The linear problem in detail

$$D_m^\gamma\,\delta L_m^\gamma - K\star(\delta L_{m+1}^\gamma + \delta L_{m-1}^\gamma) = -K^\gamma \star Y_{m+1}^o, \qquad m = 1, 2, \ldots$$

$$D_m = \frac{y_m}{1+y_m} \qquad\qquad K(x) = \frac{1}{4\cosh(\frac{\pi x}{2})}$$

- $Y_{m|vw}^0 = y_m$ are the $Q(u) = u$ Y-functions the site-2 XXX model with inhomogeneities:

$$u_1 = -u_2 = \frac{1}{\sqrt{3}} + \mathcal{O}(g^2)$$ being the leading order solutions of the ABA:

$$\pi(2n_k+1) = 2\,p_k + i\sum_{j=1}^{2}\log S_{\mathfrak{sl}(2)}^{1_*1_*}\left(\frac{u_j}{g}, \frac{u_k}{g}\right)$$

# Linear problem in a more general context

- Consider a more general linear problem: $\mathbf{M}\,\xi = j$

$$\mathbf{M} = \begin{pmatrix} D_1^\gamma & -K\star & 0 & \dots \\ -K\star & D_2^\gamma & -K\star & \dots \\ 0 & -K\star & D_3^\gamma & \dots \\ & \vdots & & \end{pmatrix} \qquad \xi = \begin{pmatrix} \delta L_1^\gamma \\ \delta L_2^\gamma \\ \vdots \end{pmatrix}$$

$$j = I = \begin{pmatrix} -K^\gamma \star Y_2^o \\ -K^\gamma \star Y_3^o \\ \vdots \end{pmatrix}$$

# Linear problem in a more general context

- $\mathbf{M}$ is symmetric $\mathbf{M}^T = \mathbf{M}$
- $\mathbf{R} = \mathbf{M}^{-1}$ is also symmetric: $\mathbf{R}^T = \mathbf{R}$
- Formal solution: $\xi_m(x) = \sum_{m'=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}y \, R_{mm'}(x,y) \, j_{m'}(y)$
- Exploiting the symmetry : $R_{mm'}(x,y) = R_{m'm}(y,x)$
- The important quantity: $\delta\mathcal{R}_k^{(2)} = \int_{-\infty}^{\infty} \mathrm{d}u \, \frac{\delta L_1^\gamma(u)}{2\sinh\frac{\pi}{2}(u + i\gamma - u_k)}$

$$= \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v \, \frac{1}{2\sinh\frac{\pi}{2}(u + i\gamma - u_k)} R_{1m}(u,v) I_m(v)$$

# Idea of the „solution"

- The important quantity:

$$\delta\mathcal{R}_k^{(2)} \sim \langle j_k|\mathbf{M}^{-1}\,I\rangle = \langle I|\mathbf{M}^{-1} j_k\rangle$$

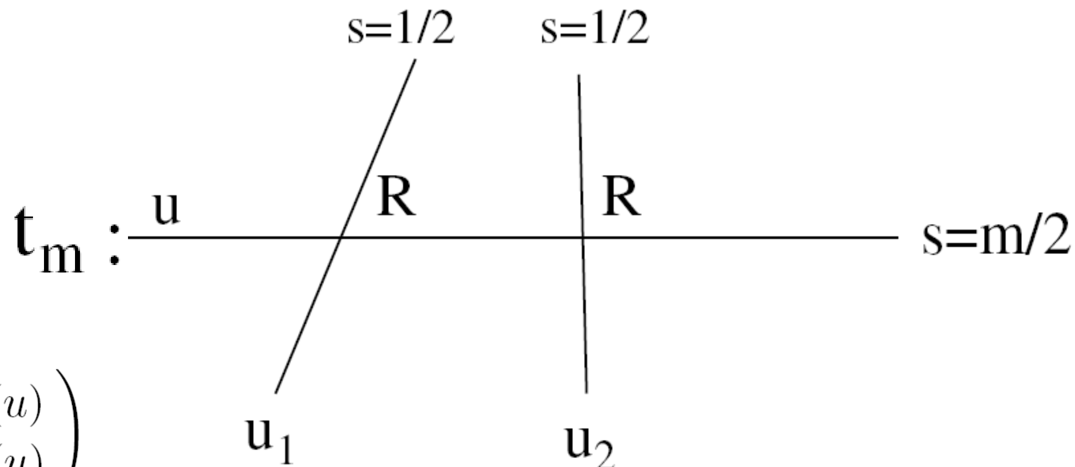
$$(j_k)_m(u) = j_m(u) = -\frac{\pi\,\delta_{m1}}{2\sinh\frac{\pi}{2}(u+i\gamma-u_k)}$$

- We cannot solve $\mathbf{M}\,\xi = I$ in general
- We cannot solve $\mathbf{M}\,\xi = I$ with $I_m = -K \star Y^o_{m+1}$
- If we can solve $\mathbf{M}\,\xi = j_k$ , then we are ready!
- The $\mathbf{M}\,\xi = j_k$ linear problem is related to the Y-system of the XXX-magnet.

# The auxiliary linear problem (XXX)

- Consider an inhomogeneous site-2 s=1/2 XXX model
- Let $t_m(u)$ the transfer matrices with s=m/2 auxiliary spaces
- Consider eigenvalues:

$$Q(u) = u - \frac{u_1 + u_2}{2}$$




$$B(\frac{u_1 + u_2}{2})|\uparrow\uparrow\rangle \qquad T_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$t_m(u) = (m+1)\left\{(u-u_1)(u-u_2) + m(m+2)\right\}$$

- They satisfy the T-sytem relations: $m = 0, 1, 2, \ldots$

$$t_m(u+i)\, t_m(u-i) = t_{m+1}(u)\, t_{m-1}(u) + t_0(u+(m+1)i)\, t_0(u-(m+1)i)$$

# Y-system of the XXX-model

- Y-system elements:

$$y_m(u) = \frac{t_{m+1}(u)\, t_{m-1}(u)}{t_0(u+(m+1)i)\, t_0(u-(m+1)i)}$$

- Y-system equations:

$$y_m(u+i)\, y_m(u-i) = [1+y_{m+1}(u)]\,[1+y_{m-1}(u)]$$

- Observation:

$$D_m(u) = \frac{1}{A_{m|vw}(u)} = \frac{1+y_m(u)}{y_m(u)}$$

# TBA Lemma

- Let f,G satisfy: $$f(x+i)\,f(x-i)=G(x)$$
- Let f analytic and free of poles in the strip $\mathrm{Im}(x)\leq 1$ and has zeroes at $r_k$ i.e $f(r_k)=0$
- Let G analytic non-zero and regular along the real line
- Then they satisfy the integral equation:

$$f(x)=\prod_k t(x-r_k)\,\exp(K\star\ln G)(x)$$

$$t(x+i)\,t(x-i)=1 \qquad t(x)=\tanh(\frac{\pi}{4}x)$$

$$(f\star g)(x)=\int dy\,f(x-y)\,g(y) \qquad \ln t'(x)=\frac{1}{2\pi}\frac{1}{4\,\sinh\frac{\pi x}{2}}$$

# Linearization of the XXX Y-system

- XXX TBA equations:

$$y_m(u) = \{t(u-u_1)\,t(u-u_2)\}^{\delta_{m1}} \exp\{K \star (L_{m+1} + L_{m-1})(u)\}$$

$$t(u) = \tanh\frac{\pi}{4}u \qquad\qquad L_m(u) = \ln(1 + y_m(u))$$

- Taking the derivative w.r.t $u_k$ :

$$D_m\,\partial_k L_m(u) - (K \star \partial_k L_{m+1})\,(u) - (K \star \partial_k L_{m-1})\,(u) = -\frac{\pi\,\delta_{m1}}{2\sinh\frac{\pi}{2}(u-u_k)}$$

- Analogy: $\boxed{\mathbf{M}\,\xi = j}$

$$\xi_m(u) = \partial_k L_m^\gamma(u) \qquad\qquad j_m(u) = -\frac{\pi\,\delta_{m1}}{2\sinh\frac{\pi}{2}(u+i\gamma-u_k)}$$

$$\boxed{\partial_k L_m^\gamma(u) = -\frac{\pi}{2}\int_{-\infty}^{\infty} \mathrm{d}v\,\frac{R_{m1}(u,v)}{\sinh\frac{\pi}{2}(v+i\gamma-u_k)}}$$

# Wrapping correction to ABA

- Formally: $\mathbf{M}^{-1} j_k \sim \partial_k L$

$$\delta\mathcal{R}_k^{(2)} \sim \langle j_k | \mathbf{M}^{-1}\, I \rangle = \langle I | \mathbf{M}^{-1} j_k \rangle = \langle I | \partial_k L \rangle$$

- The relevant term:

$$\delta\mathcal{R}_k^{(2)} = -\frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v\, I_m(v)\, \partial_k L_m^{\gamma}(v)$$

$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}v \int_{-\infty}^{\infty} \mathrm{d}u\, K(v + i\gamma - u)\, Y_{m+1}^{o}(u)\, \partial_k L_m^{\gamma}(v)$$

$$= \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u\, Y_{m+1}^{o}(u)\, \left(K \star \partial_k L_m\right)(u).$$

# The final result

- Putting everything together:

$$\delta\mathcal{R}_k = \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_m^o(u)\, \partial_k \log j_m(u)$$

$$j_m(u) = \frac{64 g^8}{(u^2+m^2)^4}\,\frac{t_{m-1}^2(u)}{t_0(u-im-i)\,t_0(u-im+i)\,t_0(u+im-i)\,t_0(u+im+i)}$$

- Identity:

$$Y_m^o(u) = C_1\, j_m(u)$$

- Finally:

$$\boxed{\delta\mathcal{R}_k = \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, \partial_k\, Y_m^o(u)}$$

# Conclusions

- „TBA" and „Lüscher" agrees at 5-loops for twist -2
- 5-loop test required the study of the linearization of the TBA equations
- „Deficiency" of the 5-loop test: decoupling

(only the $Y_{j|vw}^{(\pm)}$ nodes contribute)

- Twist-2 at higher loops?

Numerically: can be done without difficulty

Analytically: the improvement of the XXX-trick is not straightforward

- Twist-3? Velizhanin '10