

5-loop test of the AdS/CFT spectral equations

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IGST 2010, Stockholm

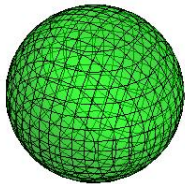
02.07.2010

with Adam Rej and Vitaly Velizhanin

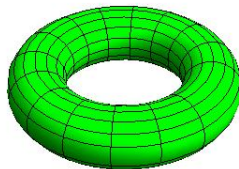
0912.1624, Nuclear Physics B

- Introduction and motivation
- AdS/CFT spectral equations
- Twist operators – test of the spectral equations
- Five-loop result
- Conclusions and perspectives

N^2

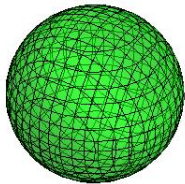


+ 1



+ ...

N^2



Amplitudes

more...

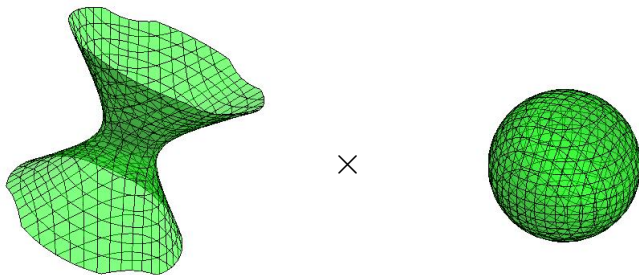
Planar AdS/CFT

Wilson loops

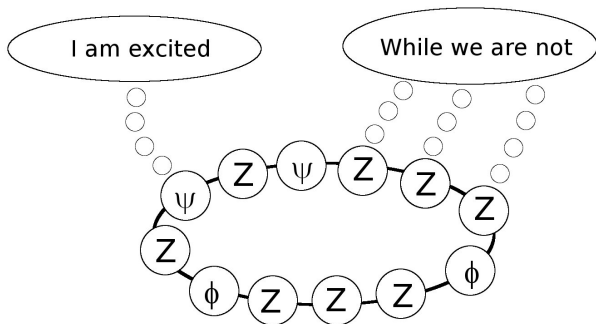
Spectral problem

Planar AdS/CFT

Spectral problem



Superstrings on $AdS_5 \times S^5$



$\mathcal{N} = 4$ super Yang-Mills theory

single trace operators $\longleftrightarrow$ non-interacting strings

Objects and quantities

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- At loop order higher than length wrapping interactions start to play a role

Wrapping corrections

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Leading Lüscher correction

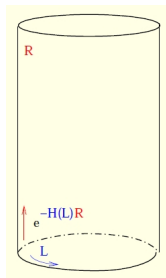
$$E(L) = \sum_k \epsilon(p_k) - \sum_{j,k} \epsilon'(p_k) \left(\frac{\delta B Y_k}{\delta p_j} \right)^{-1} \delta \Phi_j \\ - \int_{-\infty}^{\infty} \frac{d\tilde{p}}{2\pi} \text{Str} [S(\tilde{p}, p_1) S(\tilde{p}, p_2) \dots S(\tilde{p}, p_N)] e^{-\tilde{\epsilon}(\tilde{p})L}$$

TBA technique

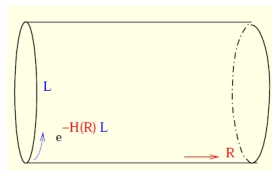
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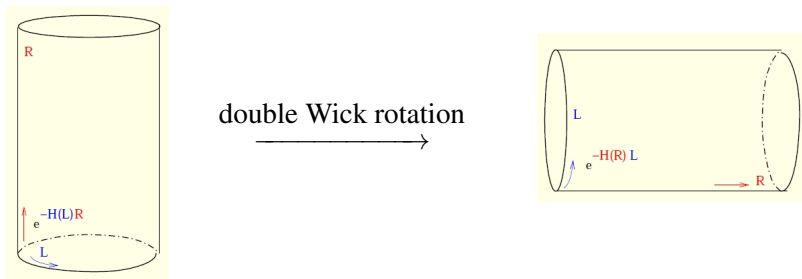


double Wick rotation



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- The mirror model for the planar AdS/CFT has been extensively studied and the infinite volume solution (string hypothesis) has been formulated

[G. Arutyunov, S. Frolov, 2007],[G. Arutyunov, S. Frolov, 2009]

Spectral equations

- The Y-system and TBA equations for the ground state have proposed by different groups.

[D.Bombardielli, D. Fioravanti, R. Tateo '09; N.Gromov, V.Kazakov, A.Kozak, P.Vieira '09; G.Arutyunov, S.Frolov '09]

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$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}$$

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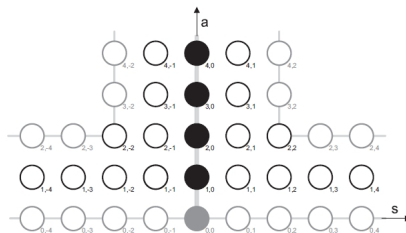
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- infinitely many functions living on the T-hook



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TBA equations should allow us to compute the scaling dimension of any local single-trace operator of the planar $\mathcal{N} = 4$ gauge theory!

... but they are still a conjecture

Twist operators

- A suitable testing ground at weak coupling.

Twist-two operators

$$\mathcal{O} = \text{Tr} \left(\mathcal{D}^M \mathcal{Z}^2 \right) + \dots$$

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$$\left(\frac{x_k^+}{x_k^-} \right)^2 = \prod_{\substack{j=1 \\ j \neq k}}^M \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp(2i\theta(u_k, u_j)).$$

$$\gamma^{\text{ABA}}(g) = 2g^2 \sum_{k=1}^M \left(\frac{i}{x_k^+} - \frac{i}{x_k^-} \right).$$

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- Finite size corrections given by Lüscher formula

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 - Maximal transcendentality principle
 - Reciprocity symmetry
- Tests of the final result
 - Analytic continuation to $M = -1 + \omega$ vs BFKL equation
 - Analytic continuation to $M = -2 + \omega$ vs double-logarithmic constraints
 - Large M limit vs cusp anomalous dimension

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Nested harmonic sums

$$S_a(M) = \sum_{i=1}^M \frac{(\text{sgn}(a))^i}{i^{|a|}}, \quad S_{a_1, \dots, a_n}(M) = \sum_{i=1}^M \frac{(\text{sgn}(a_1))^i}{i^{|a_1|}} S_{a_2, \dots, a_n}(i).$$

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- Maximal transcendentality principle fixes a **finite** basis of transcendental functions for every loop order

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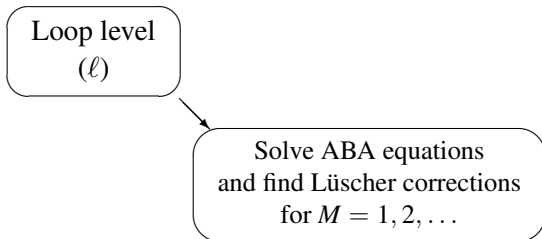
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- Anomalous dimension cannot be expressed in terms of the binomial sums

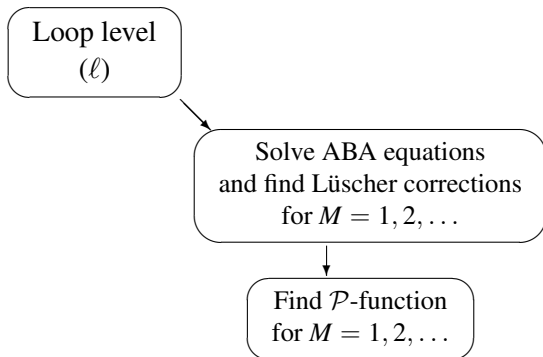
How does it work?

Loop level
(ℓ)

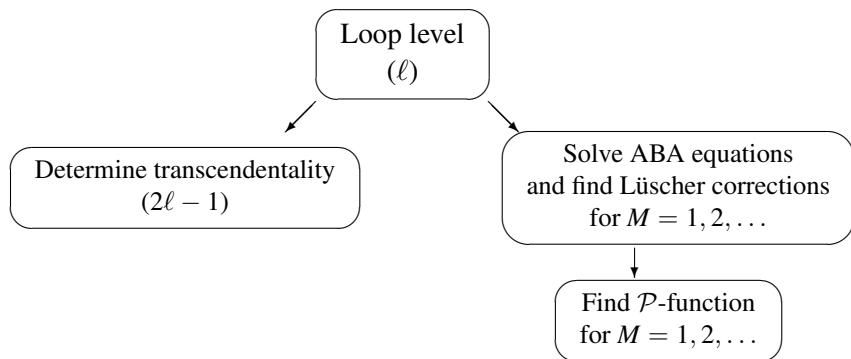
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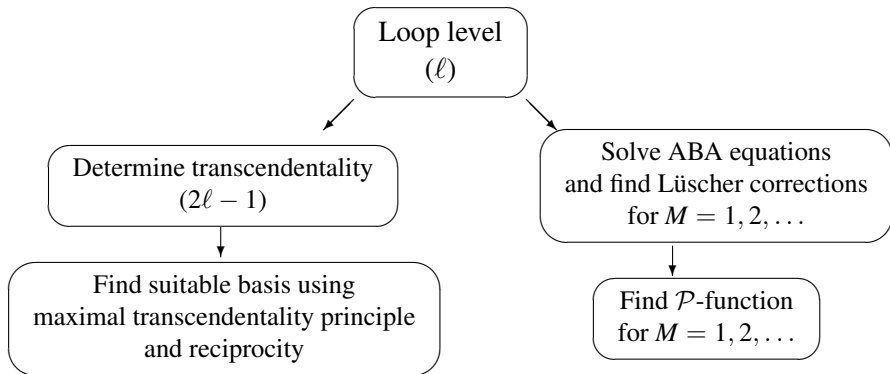
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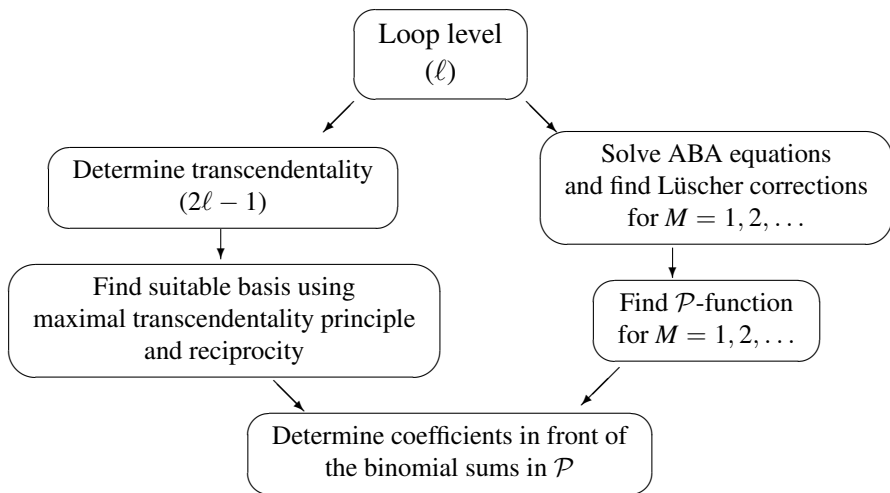
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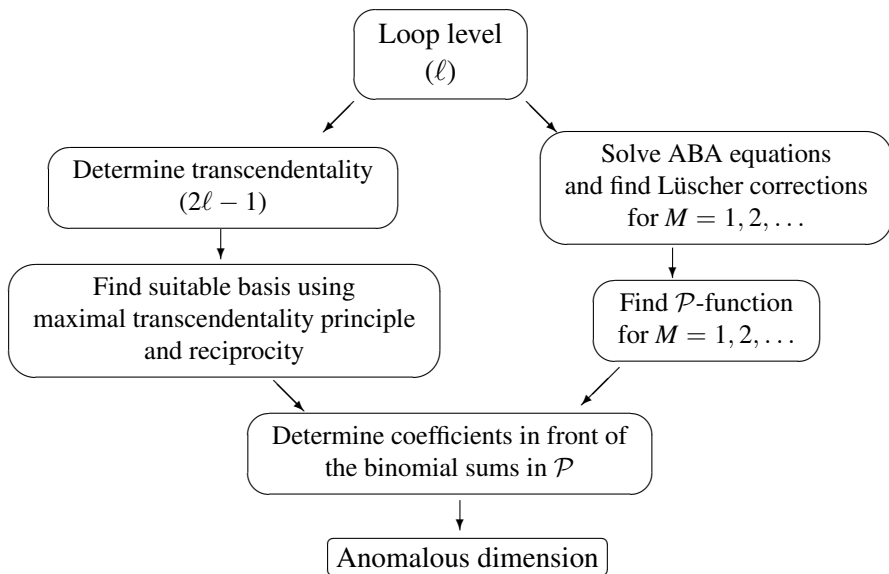
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- Cusp anomalous dimension up to five-loop order

$$\begin{aligned} 2\gamma_{\text{cusp}}(g) = & 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16 \left(\frac{73}{630}\pi^6 + 4\zeta(3)^2 \right) g^8 \\ & + 32 \left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2 \zeta(3)^2 + 40\zeta(3)\zeta(5) \right) g^{10} + \dots \end{aligned}$$

- The harmonic sums can be analytically continued in M in the whole complex space, e.g.

[A. Kotikov, V. Velizhanin, 2005]

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- Double-logarithmic constraints for $M = -2 + \omega$

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Four-loop result has to be supplemented with the wrapping corrections!

Wrapping corrections at four loops

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The formula (1) may also be derived from the spectral equations!

[N. Gromov, V. Kazakov, P. Vieira, 2009]

Is four-loop order enough to feel satisfied ?

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Not exactly!

Do we need a 5-loop test?

The structure of the TBA equations

- Quantization condition

$$Y_1(u_k) = -1$$

- Energy formula

$$E = \sum_k \epsilon_1(u_k) + \sum_j \int_{-\infty}^{\infty} \frac{du}{2\pi i} \partial_u \epsilon_j \log(1 + Y_j)$$

- TBA equations

$$\log Y_k = \sum_j K_{kj} \star \log(1 + Y_j) \quad \forall_k$$

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- ABA is implemented in TBA equations from the beginning
→ only wrapping is interesting
- For four-loop wrapping the ABA quantization is sufficient
→ the full quantization condition does not need to be solved
- We **do not need to iterate** the TBA equations

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- For four-loop wrapping the ABA quantization is sufficient
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- The really interesting testing opportunity would thus furnish the five-loop result...

[T. Ł., A. Rej, V. Velizhanin, 2009]

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- There are 256 different binomial sums with degree 9.
- Here comes the result:

$$\begin{aligned}
 & (20480S_{-5} - 8192S_{-3}S_{-2} + 2048S_5 - 20480S_{-4,1} - 16384S_{-3,2} - \frac{28672}{3}S_{-2,3} \\
 & + \frac{32768}{3}S_{-3,1,1} + \frac{16384}{3}S_{-2,1,2} + \frac{16384}{3}S_{-2,2,1})S_1^4 + (20480S_{-3}^2 + 4096S_3^2 + 81920S_{-6} \\
 & + S_{-2}(30720S_{-4} + 8192S_4) + 30720S_6 - 98304S_{-5,1} - 12288S_{-4,-2} - 102400S_{-4,2} \\
 & - 8192S_{-3,-3} - 90112S_{-3,3} + S_3(24576S_{-3} - 16384S_{-2,1}) - 57344S_{-2,4} + 4096S_{4,2} \\
 & + 16384S_{5,1} + 122880S_{-4,1,1} - 16384S_{-3,-2,1} + 106496S_{-3,1,2} + 106496S_{-3,2,1} \\
 & - 16384S_{-2,-3,1} - 8192S_{-2,-2,2} + S_2(-8192S_{-2}^2 + 49152S_{-4} + 8192S_4 - \frac{131072}{3}S_{-3,1} \\
 & - \frac{81920}{3}S_{-2,2} + \frac{65536}{3}S_{-2,1,1}) + 65536S_{-2,1,3} + 65536S_{-2,2,2} + 65536S_{-2,3,1} \\
 & - 98304S_{-3,1,1,1} - 49152S_{-2,1,1,2} - 49152S_{-2,1,2,1} - 49152S_{-2,2,1,1})S_1^3 + ((12288S_{-3} \\
 & + 9216S_3)S_{-2}^2 + (53248S_{-5} + 24576S_5 - 61440S_{-4,1} - 40960S_{-3,2} - 20480S_{-2,3} \\
 & + 32768S_{-3,1,1} + 16384S_{-2,1,2} + 16384S_{-2,2,1})S_{-2} + 113664S_{-7} + 3072S_7 - 163840S_{-6,1} \\
 & - 172032S_{-5,2} - 174080S_{-4,3} - 163840S_{-3,4} + S_2^2(36864S_{-3} + 12288S_3 - 24576S_{-2,1}) \\
 & + (-12288S_{-4} - 36864S_4)S_{-2,1} - 118784S_{-2,5} + 8192S_{4,3} + 8192S_{5,2} - 40960S_{6,1} \\
 & + 253952S_{-5,1,1} + 24576S_{-4,-2,1} + 24576S_{-4,1,-2} + 266240S_{-4,1,2} + 266240S_{-4,2,1} \\
 & + 16384S_{-3,-3,1} - 8192S_{-3,-2,2} + 16384S_{-3,1,-3} + 249856S_{-3,1,3} + 8192S_{-3,2,-2} \\
 & + 258048S_{-3,2,2} + 249856S_{-3,3,1} - 16384S_{-2,-3,2} - 16384S_{-2,-2,3} + S_{-3}(14336S_{-4} \\
 & + 43008S_4 - 49152S_{-3,1} - 24576S_{-2,2} + 32768S_{-2,1,1}) + S_3(52224S_{-4} + 12288S_4 \\
 & - 57344S_{-3,1} - 40960S_{-2,2} + 49152S_{-2,1,1}) + 172032S_{-2,1,4} + 180224S_{-2,2,3}
 \end{aligned}$$

$$\begin{aligned}
 &+180224S_{-2,3,2} + 172032S_{-2,4,1} - 8192S_{4,1,2} - 8192S_{4,2,1} - 32768S_{5,1,1} \\
 &-368640S_{-4,1,1,1} + 32768S_{-3,-2,1,1} - 344064S_{-3,1,1,2} - 344064S_{-3,1,2,1} - 344064S_{-3,2,1,1} \\
 &+32768S_{-2,-3,1,1} + 16384S_{-2,-2,1,2} + 16384S_{-2,-2,2,1} + S_2(92160S_{-5} + S_{-2}(49152S_{-3} \\
 &+24576S_3) + 30720S_5 - 122880S_{-4,1} - 12288S_{-3,-2} - 122880S_{-3,2} - 86016S_{-2,3} \\
 &+12288S_{4,1} + 172032S_{-3,1,1} - 24576S_{-2,-2,1} + 122880S_{-2,1,2} + 122880S_{-2,2,1} \\
 &-147456S_{-2,1,1,1}) - 221184S_{-2,1,1,3} - 221184S_{-2,1,2,2} - 221184S_{-2,1,3,1} \\
 &-221184S_{-2,2,1,2} - 221184S_{-2,2,2,1} - 221184S_{-2,3,1,1} + 393216S_{-3,1,1,1,1} \\
 &+196608S_{-2,1,1,1,2} + 196608S_{-2,1,1,2,1} + 196608S_{-2,1,2,1,1} + 196608S_{-2,2,1,1,1})S_1^2 \\
 &+(2048S_2^4 + 8192S_{-2}S_2^3 + (9216S_{-2}^2 + 24576S_{-4} + 9216S_4 - 36864S_{-3,1} - 30720S_{-2,2} \\
 &+49152S_{-2,1,1})S_2^2 + (4096S_{-2}^3 + (32768S_{-4} + 24576S_4 - 49152S_{-3,1} - 24576S_{-2,2} \\
 &+32768S_{-2,1,1})S_{-2} + 6144S_3^2 + 53248S_{-6} + 6144S_6 - 90112S_{-5,1} - 94208S_{-4,2} \\
 &-94208S_{-3,3} + S_3(32768S_{-3} - 32768S_{-2,1}) - 16384S_{-3}S_{-2,1} - 77824S_{-2,4} + 8192S_{4,2} \\
 &-16384S_{5,1} + 163840S_{-4,1,1} + 16384S_{-3,-2,1} + 16384S_{-3,1,-2} + 172032S_{-3,1,2} \\
 &+172032S_{-3,2,1} - 16384S_{-2,-2,2} + 139264S_{-2,1,3} + 147456S_{-2,2,2} + 139264S_{-2,3,1}
 \end{aligned}$$

$$\begin{aligned}
 & -16384S_{4,1,1} - 294912S_{-3,1,1,1} + 32768S_{-2,-2,1,1} - 245760S_{-2,1,1,2} - 245760S_{-2,1,2,1} \\
 & - 245760S_{-2,2,1,1} + 393216S_{-2,1,1,1,1})S_2 + 13824S_4^2 + 4608S_4^2 + 16384S_{-3,1}^2 \\
 & + 14336S_{-2,2}^2 + 57344S_{-8} + S_{-2}^2(3072S_{-4} + 12288S_4) + 64512S_8 - 98304S_{-7,1} \\
 & - 30720S_{-6,-2} - 98304S_{-6,2} - 16384S_{-5,-3} - 102400S_{-5,3} - 3072S_{-4,-4} - 98304S_{-4,4} \\
 & - 98304S_{-3,5} - 92160S_{-2,6} - 15360S_{4,4} - 12288S_{5,3} + 26624S_{6,2} + 36864S_{7,1} \\
 & + 163840S_{-6,1,1} - 24576S_{-5,-2,1} + 180224S_{-5,1,2} + 180224S_{-5,2,1} - 24576S_{-4,-3,1} \\
 & - 6144S_{-4,-2,-2} - 18432S_{-4,-2,2} + 184320S_{-4,1,3} + 196608S_{-4,2,2} + 184320S_{-4,3,1} \\
 & - 8192S_{-3,-4,1} - 4096S_{-3,-3,-2} - 28672S_{-3,-3,2} - 4096S_{-3,-2,-3} + 12288S_{-3,-2,3} \\
 & + 180224S_{-3,1,4} + 192512S_{-3,2,3} + 192512S_{-3,3,2} + 176128S_{-3,4,1} + 8192S_{-2,-5,1} \\
 & - 22528S_{-2,-4,2} + 4096S_{-2,-3,3} + 30720S_{-2,-2,4} + S_{-3,1}(36864S_{-2,2} - 16384S_{-2,1,1}) \\
 & - 8192S_{-2,2}S_{-2,1,1} + S_{-4}(-14336S_{-3,1} - 10240S_{-2,2} + 36864S_{-2,1,1}) \\
 & + S_4(30720S_{-4} - 51200S_{-3,1} - 43008S_{-2,2} + 69632S_{-2,1,1}) + 139264S_{-2,1,5} \\
 & + S_{-2,1}(-4096S_{-5} - 20480S_5 + 24576S_{-4,1} + 36864S_{-3,2} + 28672S_{-2,3} - 16384S_{-3,1,1} \\
 & - 8192S_{-2,1,2} - 8192S_{-2,2,1}) + 145408S_{-2,2,4} + 147456S_{-2,3,3} + 143360S_{-2,4,2} \\
 & + 131072S_{-2,5,1} - 8192S_{4,1,3} - 8192S_{4,2,2} - 8192S_{4,3,1} - 16384S_{5,1,2} - 16384S_{5,2,1} \\
 & - 294912S_{-5,1,1,1} - 319488S_{-4,1,1,2} - 319488S_{-4,1,2,1} - 319488S_{-4,2,1,1} + 49152S_{-3,-3,1,1} \\
 & + 8192S_{-3,-2,-2,1} + 16384S_{-3,-2,1,2} + 16384S_{-3,-2,2,1} - 16384S_{-3,1,1,-3} - 311296S_{-3,1,1,3} \\
 & - 327680S_{-3,1,2,2} - 311296S_{-3,1,3,1} - 16384S_{-3,2,-2,1} - 327680S_{-3,2,1,2} - 327680S_{-3,2,2,1} \\
 & - 311296S_{-3,3,1,1} + 73728S_{-2,-4,1,1} + 8192S_{-2,-3,-2,1} + 40960S_{-2,-3,1,2} + 40960S_{-2,-3,2,1}
 \end{aligned}$$

$$\begin{aligned}
 &+8192S_{-2,-2,-3,1} + 4096S_{-2,-2,-2,2} + 16384S_{-2,-2,1,3} + 16384S_{-2,-2,2,2} + 16384S_{-2,-2,3,1} \\
 &-24576S_{-2,1,1,-4} + S_3(40960S_{-5} + 16384S_5 - 28672S_{-4,1} - 22528S_{-3,2} - 22528S_{-2,3} \\
 &+4096S_{4,1} + 49152S_{-3,1,1} - 8192S_{-2,-2,1} + 36864S_{-2,1,2} + 36864S_{-2,2,1} \\
 &-49152S_{-2,1,1,1}) + S_3(40960S_{-5} + 8192S_5 - 53248S_{-4,1} - 51200S_{-3,2} - \frac{112640S_{-2,3}}{3} \\
 &+ \frac{212992}{3}S_{-3,1,1} + \frac{143360}{3}S_{-2,1,2} + \frac{143360}{3}S_{-2,2,1} - 49152S_{-2,1,1,1}) - 221184S_{-2,1,1,4} \\
 &-8192S_{-2,1,2,-3} - 237568S_{-2,1,2,3} - 237568S_{-2,1,3,2} - 221184S_{-2,1,4,1} - 16384S_{-2,2,-3,1} \\
 &-8192S_{-2,2,-2,2} - 8192S_{-2,2,1,-3} + S_{-2}(4096S_{-3}^2 + 8192S_3^2 + 56320S_{-6} + 25600S_6 \\
 &-32768S_{-5,1} - 26624S_{-4,2} - 28672S_{-3,3} + S_3(20480S_{-3} - 8192S_{-2,1}) - 24576S_{-2,4} \\
 &+2048S_{4,2} + 8192S_{5,1} + 36864S_{-4,1,1} - 8192S_{-3,-2,1} + 36864S_{-3,1,2} + 36864S_{-3,2,1} \\
 &-8192S_{-2,-3,1} - 4096S_{-2,-2,2} + 24576S_{-2,1,3} + 24576S_{-2,2,2} + 24576S_{-2,3,1} \\
 &-49152S_{-3,1,1,1} - 24576S_{-2,1,1,2} - 24576S_{-2,1,2,1} - 24576S_{-2,2,1,1}) - 237568S_{-2,2,1,3} \\
 &-245760S_{-2,2,2,2} - 237568S_{-2,2,3,1} - 16384S_{-2,3,-2,1} - 237568S_{-2,3,1,2} - 237568S_{-2,3,2,1} \\
 &-221184S_{-2,4,1,1} + 24576S_{4,1,1,2} + 24576S_{4,1,2,1} + 24576S_{4,2,1,1} + 98304S_{5,1,1,1}
 \end{aligned}$$

$$\begin{aligned}
 &+491520S_{-4,1,1,1,1} - 98304S_{-3,-2,1,1,1} - 32768S_{-3,1,-2,1,1} + 491520S_{-3,1,1,1,2} \\
 &+491520S_{-3,1,1,2,1} + 491520S_{-3,1,2,1,1} + 491520S_{-3,2,1,1,1} - 98304S_{-2,-3,1,1,1} \\
 &-49152S_{-2,-2,1,1,2} - 49152S_{-2,-2,1,2,1} - 49152S_{-2,-2,2,1,1} - 32768S_{-2,1,-3,1,1} \\
 &-16384S_{-2,1,-2,1,2} - 16384S_{-2,1,-2,2,1} + 327680S_{-2,1,1,1,3} + 327680S_{-2,1,1,2,2} \\
 &+327680S_{-2,1,1,3,1} + 327680S_{-2,1,2,1,2} + 327680S_{-2,1,2,2,1} + 327680S_{-2,1,3,1,1} \\
 &-16384S_{-2,2,-2,1,1} + 327680S_{-2,2,1,1,2} + 327680S_{-2,2,1,2,1} + 327680S_{-2,2,2,1,1} \\
 &+327680S_{-2,3,1,1,1} - 655360S_{-3,1,1,1,1,1} - 327680S_{-2,1,1,1,1,2} - 327680S_{-2,1,1,1,2,1} \\
 &-327680S_{-2,1,1,2,1,1} - 327680S_{-2,1,2,1,1,1} - 327680S_{-2,2,1,1,1,1})S_1 + 512S_3^3 - 7168S_{-9} \\
 &+7168S_9 - 18432S_{-8,1} - 2048S_{-2,-7} + S_3^2(3072S_{-3} - 2048S_{-2,1}) + S_2^3(1024S_{-3} \\
 &+1024S_3 - 2048S_{-2,1}) + S_{-2}(3072S_{-3}S_4 - 6144S_{-2,1}S_4 + S_3(3072S_{-4} + 6144S_4 \\
 &-4096S_{-3,1} - 2048S_{-2,2})) - 8192S_{1,-8} + 8192S_{1,8} - 16384S_{2,-7} + 16384S_{2,7} \\
 &-3072S_{3,-6} + 3072S_{3,6} - 13824S_{4,-5} + 4608S_{4,5} - 34816S_{5,-4} - 2048S_{5,4} - 35328S_{6,-3} \\
 &-4608S_{6,3} + 10240S_{7,-2} + 9216S_{7,2} + 16384S_{8,1} + 26624S_{-7,1,1} - 27648S_{-6,-2,1} \\
 &-6144S_{-6,1,-2} + 12288S_{-6,1,2} + 12288S_{-6,2,1} - 18432S_{-5,-3,1} - 2048S_{-5,-2,-2} \\
 &-4096S_{-5,-2,2} - 18432S_{-5,1,-3} - 4096S_{-5,2,-2} + 26624S_{-4,-4,1} + 44032S_{-4,-3,-2} \\
 &+51200S_{-4,-3,2} + 70656S_{-4,-2,-3} + 12288S_{-4,-2,3} + 13312S_{-4,1,-4} + 17408S_{-4,1,4} \\
 &+7168S_{-4,2,-3} - 1024S_{-4,3,-2} + 44032S_{-4,4,1} - 10240S_{-3,-5,1} + 45056S_{-3,-4,-2} \\
 &+51200S_{-3,-4,2} + 157696S_{-3,-3,-3} + 33792S_{-3,-3,3} + 73728S_{-3,-2,-4} + 8192S_{-3,-2,4}
 \end{aligned}$$

$$\begin{aligned}
 & -8192S_{-3,1,-5} + 61440S_{-3,1,5} + 14336S_{-3,2,-4} + 20480S_{-3,2,4} - 3072S_{-3,3,-3} \\
 & + 10240S_{-3,4,-2} + 45056S_{-3,4,2} + 90112S_{-3,5,1} - 13312S_{-2,-6,1} + 1024S_{-2,-5,-2} \\
 & - 4096S_{-2,-5,2} + 68608S_{-2,-4,-3} + 12288S_{-2,-4,3} + 70656S_{-2,-3,-4} + 8192S_{-2,-3,4} \\
 & + 15360S_{-2,-2,-5} + 7168S_{-2,-2,5} - 7168S_{-2,1,-6} + 21504S_{-2,1,6} - 10240S_{-7,-2} \\
 & - 13312S_{-7,2} + 16896S_{-6,-3} - 5632S_{-6,3} + 5120S_{-5,-4} + 1024S_{-5,4} + 3584S_{-4,-5} \\
 & - 27136S_{-4,5} + 9216S_{-3,-6} - 23552S_{-3,6} - 4096S_{-2,2,-5} + 28672S_{-2,2,5} \\
 & + 1024S_{-2,3,4} + 8192S_{-2,4,-3} + 11264S_{-2,4,3} + 13312S_{-2,5,-2} + 40960S_{-2,5,2} \\
 & + 35840S_{-2,6,1} + 40960S_{1,-7,1} - 11264S_{1,-6,-2} + 8192S_{1,-6,2} - 32768S_{1,-5,-3} \\
 & + 4096S_{1,-5,3} + 18432S_{1,-4,-4} + 23552S_{1,-4,4} - 10240S_{1,-3,-5} + 71680S_{1,-3,5} \\
 & - 11264S_{1,-2,-6} + 25600S_{1,-2,6} + 32768S_{1,1,-7} - 32768S_{1,1,7} + 8192S_{1,2,-6} - 8192S_{1,2,6} \\
 & + 4096S_{1,3,-5} + 35840S_{1,4,-4} - 6144S_{1,4,4} + 83968S_{1,5,-3} + 18432S_{1,5,3} + 17408S_{1,6,-2} \\
 & + 22528S_{1,6,2} - 32768S_{1,7,1} + 14336S_{2,-6,1} - 20480S_{2,-5,-2} - 8192S_{2,-5,2} \\
 & + 22528S_{2,-4,-3} + 1024S_{2,-4,3} + 32768S_{2,-3,-4} + 30720S_{2,-3,4} - 6144S_{2,-2,-5} \\
 & + 38912S_{2,-2,5} + 8192S_{2,1,-6} - 8192S_{2,1,6} - 4096S_{2,2,-5} + 16384S_{2,2,5} - 1024S_{2,3,-4} \\
 & - 5120S_{2,3,4} + 43008S_{2,4,-3} + 9216S_{2,4,3} + 32768S_{2,5,-2} + 40960S_{2,5,2} + 6144S_{2,6,1} \\
 & + 2048S_{3,-5,1} - 3072S_{3,-4,-2} - 3072S_{3,-4,2} + 12288S_{3,-3,-3} + 1024S_{3,-3,3} + 5120S_{3,-2,-4}
 \end{aligned}$$

$$\begin{aligned}
 &+7168S_{3,-2,4} + 4096S_{3,1,-5} - 1024S_{3,2,-4} - 5120S_{3,2,4} + 3072S_{3,3,-3} + 9216S_{3,4,-2} \\
 &+9216S_{3,4,2} + 8192S_{3,5,1} + 39936S_{4,-4,1} - 6144S_{4,-3,-2} + 31744S_{4,-3,2} - 6144S_{4,-2,-3} \\
 &+15360S_{4,-2,3} + 32768S_{4,1,-4} - 6144S_{4,1,4} + 36864S_{4,2,-3} + 9216S_{4,2,3} + 8192S_{4,3,-2} \\
 &+9216S_{4,3,2} - 6144S_{4,4,1} + 86016S_{5,-3,1} + 8192S_{5,-2,-2} + 36864S_{5,-2,2} + 81920S_{5,1,-3} \\
 &+18432S_{5,1,3} + 32768S_{5,2,-2} + 40960S_{5,2,2} + 18432S_{5,3,1} + 50176S_{6,-2,1} + 20480S_{6,1,-2} \\
 &+22528S_{6,1,2} + 22528S_{6,2,1} - 18432S_{7,1,1} - 24576S_{-6,1,1,1} + 8192S_{-5,-2,1,1} \\
 &+28672S_{-5,1,-2,1} + 8192S_{-5,1,1,-2} - 102400S_{-4,-3,1,1} - 88064S_{-4,-2,-2,1} \\
 &-53248S_{-4,-2,1,-2} - 59392S_{-4,-2,1,2} - 59392S_{-4,-2,2,1} - 55296S_{-4,1,-3,1} \\
 &-34816S_{-4,1,-2,-2} - 43008S_{-4,1,-2,2} - 14336S_{-4,1,1,-3} - 2048S_{-4,1,2,-2} - 12288S_{-4,2,-2,1} \\
 &-2048S_{-4,2,1,-2} - 102400S_{-3,-4,1,1} - 188416S_{-3,-3,-2,1} - 126976S_{-3,-3,1,-2} \\
 &-155648S_{-3,-3,1,2} - 155648S_{-3,-3,2,1} - 180224S_{-3,-2,-3,1} - 24576S_{-3,-2,-2,-2} \\
 &-90112S_{-3,-2,-2,2} - 155648S_{-3,-2,1,-3} - 36864S_{-3,-2,1,3} - 65536S_{-3,-2,2,-2} \\
 &-81920S_{-3,-2,2,2} - 36864S_{-3,-2,3,1} - 61440S_{-3,1,-4,1} - 102400S_{-3,1,-3,-2} \\
 &-122880S_{-3,1,-3,2} - 159744S_{-3,1,-2,-3} - 30720S_{-3,1,-2,3} - 28672S_{-3,1,1,-4} \\
 &-40960S_{-3,1,1,4} - 12288S_{-3,1,2,-3} + 2048S_{-3,1,3,-2} - 98304S_{-3,1,4,1} - 61440S_{-3,2,-3,1} \\
 &-40960S_{-3,2,-2,-2} - 49152S_{-3,2,-2,2} - 12288S_{-3,2,1,-3} + 4096S_{-3,3,-2,1} + 2048S_{-3,3,1,-2} \\
 &-90112S_{-3,4,1,1} + 8192S_{-2,-5,1,1} - 83968S_{-2,-4,-2,1} - 53248S_{-2,-4,1,-2} - 59392S_{-2,-4,1,2} \\
 &-59392S_{-2,-4,2,1} - 169984S_{-2,-3,-3,1} - 24576S_{-2,-3,-2,-2} - 83968S_{-2,-3,-2,2} \\
 &-151552S_{-2,-3,1,-3} - 36864S_{-2,-3,1,3} - 65536S_{-2,-3,2,-2} - 81920S_{-2,-3,2,2} \\
 &-36864S_{-2,-3,3,1} - 75776S_{-2,-2,-4,1} - 24576S_{-2,-2,-3,-2} - 79872S_{-2,-2,-3,2}
 \end{aligned}$$

$$\begin{aligned}
 & -24576S_{-2,-2,-2,-3} - 22528S_{-2,-2,-2,3} - 69632S_{-2,-2,1,-4} - 8192S_{-2,-2,1,4} \\
 & -73728S_{-2,-2,2,-3} - 18432S_{-2,-2,2,3} - 16384S_{-2,-2,3,-2} - 18432S_{-2,-2,3,2} \\
 & -8192S_{-2,-2,4,1} + 12288S_{-2,1,-5,1} - 38912S_{-2,1,-4,-2} - 43008S_{-2,1,-4,2} \\
 & -157696S_{-2,1,-3,-3} - 30720S_{-2,1,-3,3} - 71680S_{-2,1,-2,-4} - 8192S_{-2,1,-2,4} \\
 & + 8192S_{-2,1,1,-5} + S_{-4}(4608S_{-5} + 1536S_5 - 9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} \\
 & + 18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} - 36864S_{-2,1,1,1}) + S_4(4608S_{-5} + 1536S_5 \\
 & - 9216S_{-4,1} - 9216S_{-3,2} - 9216S_{-2,3} + 18432S_{-3,1,1} + 18432S_{-2,1,2} + 18432S_{-2,2,1} \\
 & - 36864S_{-2,1,1,1}) + S_2^2(3072S_{-5} + 1024S_5 - 6144S_{-4,1} - 6144S_{-3,2} + S_{-2}(2048S_{-3} \\
 & + 4096S_3 - 4096S_{-2,1}) - 6144S_{-2,3} + 12288S_{-3,1,1} + 12288S_{-2,1,2} + 12288S_{-2,2,1} \\
 & - 24576S_{-2,1,1,1}) + S_{-2,2}(-3072S_{-5} - 1024S_5 + 6144S_{-4,1} + 6144S_{-3,2} + 6144S_{-2,3} \\
 & - 12288S_{-3,1,1} - 12288S_{-2,1,2} - 12288S_{-2,2,1} + 24576S_{-2,1,1,1}) + S_{-3,1}(-6144S_{-5} \\
 & - 2048S_5 + 12288S_{-4,1} + 12288S_{-3,2} + 12288S_{-2,3} - 24576S_{-3,1,1} - 24576S_{-2,1,2} \\
 & - 24576S_{-2,2,1} + 49152S_{-2,1,1,1}) - 57344S_{-2,1,1,5} - 8192S_{-2,1,2,-4} - 14336S_{-2,1,2,4} \\
 & + 4096S_{-2,1,3,-3} - 12288S_{-2,1,4,-2} - 43008S_{-2,1,4,2} - 90112S_{-2,1,5,1} - 20480S_{-2,2,-4,1}
 \end{aligned}$$

$$\begin{aligned}
 & -43008S_{-2,2,-3,-2} - 49152S_{-2,2,-3,2} - 79872S_{-2,2,-2,-3} - 12288S_{-2,2,-2,3} \\
 & -8192S_{-2,2,1,-4} + S_{-3}(7680S_{-6} + 2560S_6 - 12288S_{-5,1} - 12288S_{-4,2} - 12288S_{-3,3} \\
 & -9216S_{-2,4} + 18432S_{-4,1,1} + 18432S_{-3,1,2} + 18432S_{-3,2,1} + 12288S_{-2,1,3} + 12288S_{-2,2,2} \\
 & + 12288S_{-2,3,1} - 24576S_{-3,1,1,1} - 12288S_{-2,1,1,2} - 12288S_{-2,1,2,1} - 12288S_{-2,2,1,1}) \\
 & + S_3(2560S_{-3}^2 - 6144S_{-2,1}S_{-3} + 2048S_{-2,1}^2 + 7680S_{-6} + 2560S_6 - 12288S_{-5,1} \\
 & - 12288S_{-4,2} - 12288S_{-3,3} - 9216S_{-2,4} + 18432S_{-4,1,1} + 18432S_{-3,1,2} + 18432S_{-3,2,1} \\
 & + 12288S_{-2,1,3} + 12288S_{-2,2,2} + 12288S_{-2,3,1} - 24576S_{-3,1,1,1} - 12288S_{-2,1,1,2} \\
 & - 12288S_{-2,1,2,1} - 12288S_{-2,2,1,1}) + S_{-2,1}(-15360S_{-6} - 5120S_6 + 24576S_{-5,1} \\
 & + 24576S_{-4,2} + 24576S_{-3,3} + 18432S_{-2,4} - 36864S_{-4,1,1} - 36864S_{-3,1,2} \\
 & - 36864S_{-3,2,1} - 24576S_{-2,1,3} - 24576S_{-2,2,2} - 24576S_{-2,3,1} + 49152S_{-3,1,1,1} \\
 & + 24576S_{-2,1,1,2} + 24576S_{-2,1,2,1} + 24576S_{-2,2,1,1}) - 14336S_{-2,2,1,4} \\
 & - 51200S_{-2,2,4,1} + 2048S_{-2,3,-3,1} - 2048S_{-2,3,-2,-2} - 2048S_{-2,3,-2,2} + 4096S_{-2,3,1,-3} \\
 & - 4096S_{-2,4,-2,1} - 12288S_{-2,4,1,-2} - 38912S_{-2,4,1,2} - 38912S_{-2,4,2,1} \\
 & - 81920S_{-2,5,1,1} - 16384S_{1,-6,1,1} + 40960S_{1,-5,-2,1} + 24576S_{1,-5,1,-2} - 83968S_{1,-4,-3,1} \\
 & - 51200S_{1,-4,-2,-2} - 59392S_{1,-4,-2,2} - 28672S_{1,-4,1,-3} + 2048S_{1,-4,1,3} - 4096S_{1,-4,2,-2} \\
 & + 2048S_{1,-4,3,1} - 96256S_{1,-3,-4,1} - 129024S_{1,-3,-3,-2} - 155648S_{1,-3,-3,2} \\
 & - 165888S_{1,-3,-2,-3} - 36864S_{1,-3,-2,3} - 51200S_{1,-3,1,-4} - 59392S_{1,-3,1,4} \\
 & - 40960S_{1,-3,2,-3} + 8192S_{1,-3,3,-2} - 96256S_{1,-3,4,1} + 8192S_{1,-2,-5,1} - 51200S_{1,-2,-4,-2}
 \end{aligned}$$

$$\begin{aligned}
 & -73728S_{2,-3,1,-3} + 4096S_{2,-3,1,3} - 16384S_{2,-3,2,-2} + 4096S_{2,-3,3,1} - 55296S_{2,-2,-4,1} \\
 & -69632S_{2,-2,-3,-2} - 81920S_{2,-2,-3,2} - 86016S_{2,-2,-2,-3} - 18432S_{2,-2,-2,3} \\
 & -30720S_{2,-2,1,-4} - 32768S_{2,-2,1,4} - 28672S_{2,-2,2,-3} + 6144S_{2,-2,3,-2} - 49152S_{2,-2,4,1} \\
 & + 16384S_{2,1,-5,1} - 2048S_{2,1,-4,-2} + 4096S_{2,1,-4,2} - 110592S_{2,1,-3,-3} + 4096S_{2,1,-3,3} \\
 & -34816S_{2,1,-2,-4} - 28672S_{2,1,-2,4} + 8192S_{2,1,1,-5} - 32768S_{2,1,1,5} - 36864S_{2,1,4,-2} \\
 & -40960S_{2,1,4,2} - 65536S_{2,1,5,1} - 16384S_{2,2,-3,-2} - 8192S_{2,2,-3,2} - 65536S_{2,2,-2,-3} \\
 & -49152S_{2,2,4,1} + 8192S_{2,3,-3,1} + 10240S_{2,3,-2,-2} + 8192S_{2,3,-2,2} - 49152S_{2,4,-2,1} \\
 & -36864S_{2,4,1,-2} - 40960S_{2,4,1,2} - 40960S_{2,4,2,1} - 81920S_{2,5,1,1} + 6144S_{3,-4,1,1} \\
 & -22528S_{3,-3,-2,1} - 2048S_{3,-3,1,-2} - 4096S_{3,-3,1,2} - 4096S_{3,-3,2,1} - 26624S_{3,-2,-3,1} \\
 & -18432S_{3,-2,-2,-2} - 18432S_{3,-2,-2,2} - 10240S_{3,-2,1,-3} + 2048S_{3,-2,1,3} - 2048S_{3,-2,2,-2} \\
 & + 2048S_{3,-2,3,1} - 2048S_{3,1,-4,1} + 10240S_{3,1,-3,-2} + 4096S_{3,1,-3,2} - 14336S_{3,1,-2,-3} \\
 & -4096S_{3,1,-2,3} + 2048S_{3,1,1,-4} + 10240S_{3,1,1,4} - 14336S_{3,1,4,1} + 8192S_{3,2,-3,1} \\
 & + 10240S_{3,2,-2,-2} + 8192S_{3,2,-2,2} - 6144S_{3,3,-2,1} - 18432S_{3,4,1,1} - 63488S_{4,-3,1,1} \\
 & + 8192S_{4,-2,-2,1} + 4096S_{4,-2,1,-2} - 38912S_{4,-2,1,2} - 38912S_{4,-2,2,1} - 65536S_{4,1,-3,1} \\
 & + 8192S_{4,1,-2,-2} - 24576S_{4,1,-2,2} - 73728S_{4,1,1,-3} - 18432S_{4,1,1,3} - 32768S_{4,1,2,-2} \\
 & -40960S_{4,1,2,2} - 18432S_{4,1,3,1} - 40960S_{4,2,-2,1} - 32768S_{4,2,1,-2} - 40960S_{4,2,1,2} \\
 & -40960S_{4,2,2,1} - 18432S_{4,3,1,1} - 73728S_{5,-2,1,1} - 98304S_{5,1,-2,1} - 65536S_{5,1,1,-2} \\
 & -81920S_{5,1,1,2} - 81920S_{5,1,2,1} - 81920S_{5,2,1,1} - 45056S_{6,1,1,1} + 118784S_{-4,-2,1,1,1}
 \end{aligned}$$

Five-loop ABA

$$\begin{aligned} &+86016S_{-4,1,-2,1,1} + 24576S_{-4,1,1,-2,1} + 4096S_{-4,1,1,1,-2} + 311296S_{-3,-3,1,1,1} \\ &+180224S_{-3,-2,-2,1,1} + 180224S_{-3,-2,1,-2,1} + 131072S_{-3,-2,1,1,-2} + 163840S_{-3,-2,1,1,2} \\ &+163840S_{-3,-2,1,2,1} + 163840S_{-3,-2,2,1,1} + 245760S_{-3,1,-3,1,1} + 196608S_{-3,1,-2,-2,1} \\ &+122880S_{-3,1,-2,1,-2} + 147456S_{-3,1,-2,1,2} + 147456S_{-3,1,-2,2,1} + 122880S_{-3,1,1,-3,1} \\ &+81920S_{-3,1,1,-2,-2} + 98304S_{-3,1,1,-2,2} + 24576S_{-3,1,1,1,-3} + 24576S_{-3,1,2,-2,1} \\ &+98304S_{-3,2,-2,1,1} + 24576S_{-3,2,1,-2,1} + 118784S_{-2,-4,1,1,1} + 167936S_{-2,-3,-2,1,1} \\ &+172032S_{-2,-3,1,-2,1} + 131072S_{-2,-3,1,1,-2} + 163840S_{-2,-3,1,1,2} + 163840S_{-2,-3,1,2,1} \\ &+163840S_{-2,-3,2,1,1} + 159744S_{-2,-2,-3,1,1} + 24576S_{-2,-2,-2,-2,1} \\ &+24576S_{-2,-2,-2,1,-2} + 77824S_{-2,-2,-2,1,2} + 77824S_{-2,-2,-2,2,1} + 163840S_{-2,-2,1,-3,1} \\ &+24576S_{-2,-2,1,-2,-2} + 81920S_{-2,-2,1,-2,2} + 147456S_{-2,-2,1,1,-3} + 36864S_{-2,-2,1,1,3} \\ &+65536S_{-2,-2,1,2,-2} + 81920S_{-2,-2,1,2,2} + 36864S_{-2,-2,1,3,1} + 81920S_{-2,-2,2,-2,1} \\ &+65536S_{-2,-2,2,1,-2} + 81920S_{-2,-2,2,1,2} + 81920S_{-2,-2,2,2,1} + 36864S_{-2,-2,3,1,1} \\ &+86016S_{-2,1,-4,1,1} + 192512S_{-2,1,-3,-2,1} + 122880S_{-2,1,-3,1,-2} \\ &+147456S_{-2,1,-3,1,2} + 147456S_{-2,1,-3,2,1} + 176128S_{-2,1,-2,-3,1} + 24576S_{-2,1,-2,-2,-2} \\ &+86016S_{-2,1,-2,-2,2} + 155648S_{-2,1,-2,1,-3} + 36864S_{-2,1,-2,1,3} + 65536S_{-2,1,-2,2,-2} \end{aligned}$$

$$\begin{aligned}
 &+81920S_{-2,1,-2,2,2} + 36864S_{-2,1,-2,3,1} + 40960S_{-2,1,1,-4,1} + 86016S_{-2,1,1,-3,-2} \\
 &+98304S_{-2,1,1,-3,2} + 159744S_{-2,1,1,-2,-3} + 24576S_{-2,1,1,-2,3} + 16384S_{-2,1,1,1,-4} \\
 &+28672S_{-2,1,1,1,4} + 102400S_{-2,1,1,4,1} + 32768S_{-2,1,2,-3,1} + 28672S_{-2,1,2,-2,-2} \\
 &+32768S_{-2,1,2,-2,2} - 8192S_{-2,1,3,-2,1} + 86016S_{-2,1,4,1,1} + 98304S_{-2,2,-3,1,1} \\
 &+102400S_{-2,2,-2,-2,1} + 57344S_{-2,2,-2,1,-2} + 65536S_{-2,2,-2,1,2} + 65536S_{-2,2,-2,2,1} \\
 &+32768S_{-2,2,1,-3,1} + 28672S_{-2,2,1,-2,-2} + 32768S_{-2,2,1,-2,2} + 4096S_{-2,3,-2,1,1} \\
 &-8192S_{-2,3,1,-2,1} + 77824S_{-2,4,1,1,1} + 118784S_{1,-4,-2,1,1} + 49152S_{1,-4,1,-2,1} \\
 &+8192S_{1,-4,1,1,-2} + 311296S_{1,-3,-3,1,1} + 192512S_{1,-3,-2,-2,1} + 139264S_{1,-3,-2,1,-2} \\
 &+163840S_{1,-3,-2,1,2} + 163840S_{1,-3,-2,2,1} + 221184S_{1,-3,1,-3,1} + 118784S_{1,-3,1,-2,-2} \\
 &+147456S_{1,-3,1,-2,2} + 81920S_{1,-3,1,1,-3} + 8192S_{1,-3,1,2,-2} + 73728S_{1,-3,2,-2,1} \\
 &+8192S_{1,-3,2,1,-2} + 118784S_{1,-2,-4,1,1} + 184320S_{1,-2,-3,-2,1} + 131072S_{1,-2,-3,1,-2} \\
 &+163840S_{1,-2,-3,1,2} + 163840S_{1,-2,-3,2,1} + 184320S_{1,-2,-2,-3,1} + 24576S_{1,-2,-2,-2,-2} \\
 &+94208S_{1,-2,-2,-2,2} + 155648S_{1,-2,-2,1,-3} + 36864S_{1,-2,-2,1,3} + 65536S_{1,-2,-2,2,-2} \\
 &+81920S_{1,-2,-2,2,2} + 36864S_{1,-2,-2,3,1} + 81920S_{1,-2,1,-4,1} + 118784S_{1,-2,1,-3,-2} \\
 &+147456S_{1,-2,1,-3,2} + 159744S_{1,-2,1,-2,-3} + 36864S_{1,-2,1,-2,3} + 40960S_{1,-2,1,1,-4} \\
 &+53248S_{1,-2,1,1,4} + 24576S_{1,-2,1,2,-3} - 4096S_{1,-2,1,3,-2} + 94208S_{1,-2,1,4,1} \\
 &+90112S_{1,-2,2,-3,1} + 53248S_{1,-2,2,-2,-2} + 65536S_{1,-2,2,-2,2} + 24576S_{1,-2,2,1,-3} \\
 &-4096S_{1,-2,3,1,-2} + 94208S_{1,-2,4,1,1} - 32768S_{1,1,-5,1,1} + 77824S_{1,1,-4,-2,1} \\
 &+12288S_{1,1,-4,1,-2} + 8192S_{1,1,-4,1,2} + 8192S_{1,1,-4,2,1} + 278528S_{1,1,-3,-3,1} \\
 &+139264S_{1,1,-3,-2,-2} + 163840S_{1,1,-3,-2,2} + 147456S_{1,1,-3,1,-3} - 8192S_{1,1,-3,1,3}
 \end{aligned}$$

$$\begin{aligned}
 &+32768S_{1,1,-3,2,-2} - 8192S_{1,1,-3,3,1} + 110592S_{1,1,-2,-4,1} + 139264S_{1,1,-2,-3,-2} \\
 &+163840S_{1,1,-2,-3,2} + 172032S_{1,1,-2,-2,-3} + 36864S_{1,1,-2,-2,3} + 61440S_{1,1,-2,1,-4} \\
 &+65536S_{1,1,-2,1,4} + 57344S_{1,1,-2,2,-3} - 12288S_{1,1,-2,3,-2} + 98304S_{1,1,-2,4,1} \\
 &-32768S_{1,1,1,-5,1} + 4096S_{1,1,1,-4,-2} - 8192S_{1,1,1,-4,2} + 221184S_{1,1,1,-3,-3} \\
 &-8192S_{1,1,1,-3,3} + 69632S_{1,1,1,-2,-4} + 57344S_{1,1,1,-2,4} - 16384S_{1,1,1,1,-5} + 65536S_{1,1,1,1,5} \\
 &+73728S_{1,1,1,4,-2} + 81920S_{1,1,1,4,2} + 131072S_{1,1,1,5,1} + 32768S_{1,1,2,-3,-2} \\
 &+16384S_{1,1,2,-3,2} + 131072S_{1,1,2,-2,-3} + 98304S_{1,1,2,4,1} - 16384S_{1,1,3,-3,1} \\
 &-20480S_{1,1,3,-2,-2} - 16384S_{1,1,3,-2,2} + 98304S_{1,1,4,-2,1} + 73728S_{1,1,4,1,-2} \\
 &+81920S_{1,1,4,1,2} + 81920S_{1,1,4,2,1} + 163840S_{1,1,5,1,1} - 8192S_{1,2,-4,1,1} + 163840S_{1,2,-3,-2,1} \\
 &+57344S_{1,2,-3,1,-2} + 16384S_{1,2,-3,1,2} + 16384S_{1,2,-3,2,1} + 147456S_{1,2,-2,-3,1} \\
 &+73728S_{1,2,-2,-2,-2} + 81920S_{1,2,-2,-2,2} + 90112S_{1,2,-2,1,-3} - 8192S_{1,2,-2,1,3} \\
 &+24576S_{1,2,-2,2,-2} - 8192S_{1,2,-2,3,1} + 32768S_{1,2,1,-3,-2} + 16384S_{1,2,1,-3,2} \\
 &+131072S_{1,2,1,-2,-3} + 98304S_{1,2,1,4,1} + 81920S_{1,2,4,1,1} - 8192S_{1,3,-3,1,1} + 28672S_{1,3,-2,-2,1} \\
 &+8192S_{1,3,-2,1,2} + 8192S_{1,3,-2,2,1} - 16384S_{1,3,1,-3,1} - 20480S_{1,3,1,-2,-2} - 16384S_{1,3,1,-2,2}
 \end{aligned}$$

$$\begin{aligned}
 &+61440S_{1,4,-2,1,1} + 90112S_{1,4,1,-2,1} + 65536S_{1,4,1,1,-2} + 81920S_{1,4,1,1,2} + 81920S_{1,4,1,2,1} \\
 &+ 81920S_{1,4,2,1,1} + 163840S_{1,5,1,1,1} + 8192S_{2,-4,1,1,1} + 163840S_{2,-3,-2,1,1} \\
 &+ 114688S_{2,-3,1,-2,1} + 32768S_{2,-3,1,1,-2} + 163840S_{2,-2,-3,1,1} + 98304S_{2,-2,-2,-2,1} \\
 &+ 73728S_{2,-2,-2,1,-2} + 81920S_{2,-2,-2,1,2} + 81920S_{2,-2,-2,2,1} + 131072S_{2,-2,1,-3,1} \\
 &+ 65536S_{2,-2,1,-2,-2} + 81920S_{2,-2,1,-2,2} + 57344S_{2,-2,1,1,-3} + 8192S_{2,-2,1,2,-2} \\
 &+ 49152S_{2,-2,2,-2,1} + 8192S_{2,-2,2,1,-2} - 8192S_{2,1,-4,1,1} + 163840S_{2,1,-3,-2,1} \\
 &+ 57344S_{2,1,-3,1,-2} + 16384S_{2,1,-3,1,2} + 16384S_{2,1,-3,2,1} + 147456S_{2,1,-2,-3,1} \\
 &+ 73728S_{2,1,-2,-2,-2} + 81920S_{2,1,-2,-2,2} + 90112S_{2,1,-2,1,-3} - 8192S_{2,1,-2,1,3} \\
 &+ 24576S_{2,1,-2,2,-2} - 8192S_{2,1,-2,3,1} + 32768S_{2,1,1,-3,-2} + 16384S_{2,1,1,-3,2} \\
 &+ 131072S_{2,1,1,-2,-3} + 98304S_{2,1,1,4,1} + 81920S_{2,1,4,1,1} + 16384S_{2,2,-3,1,1} \\
 &+ 98304S_{2,2,-2,-2,1} + 32768S_{2,2,-2,1,-2} + 16384S_{2,2,-2,1,2} + 16384S_{2,2,-2,2,1} \\
 &- 16384S_{2,3,-2,1,1} + 81920S_{2,4,1,1,1} + 8192S_{3,-3,1,1,1} + 36864S_{3,-2,-2,1,1} \\
 &+ 16384S_{3,-2,1,-2,1} + 4096S_{3,-2,1,1,-2} - 8192S_{3,1,-3,1,1} + 28672S_{3,1,-2,-2,1} \\
 &+ 8192S_{3,1,-2,1,2} + 8192S_{3,1,-2,2,1} - 16384S_{3,1,1,-3,1} - 20480S_{3,1,1,-2,-2} - 16384S_{3,1,1,-2,2} \\
 &- 16384S_{3,2,-2,1,1} + 77824S_{4,-2,1,1,1} + 49152S_{4,1,-2,1,1} + 81920S_{4,1,1,-2,1} + 65536S_{4,1,1,1,-2} \\
 &+ 81920S_{4,1,1,1,2} + 81920S_{4,1,1,2,1} + 81920S_{4,1,2,1,1} + 81920S_{4,2,1,1,1} + 163840S_{5,1,1,1,1} \\
 &- 327680S_{-3,-2,1,1,1,1} - 294912S_{-3,1,-2,1,1,1} - 196608S_{-3,1,1,-2,1,1} - 49152S_{-3,1,1,1,-2,1} \\
 &- 327680S_{-2,-3,1,1,1,1} - 155648S_{-2,-2,-2,1,1,1} - 163840S_{-2,-2,1,-2,1,1} \\
 &- 163840S_{-2,-2,1,1,-2,1} - 131072S_{-2,-2,1,1,1,-2} - 163840S_{-2,-2,1,1,1,2} - 163840S_{-2,-2,1,1,2,1} \\
 &- 163840S_{-2,-2,1,2,1,1} - 163840S_{-2,-2,2,1,1,1} - 294912S_{-2,1,-3,1,1,1} - 172032S_{-2,1,-2,-2,1,1}
 \end{aligned}$$

$$\begin{aligned}
 & -180224S_{-2,1,-2,1,-2,1} - 131072S_{-2,1,-2,1,1,-2} - 163840S_{-2,1,-2,1,1,2} - 163840S_{-2,1,-2,1,2,1} \\
 & -163840S_{-2,1,-2,2,1,1} - 196608S_{-2,1,1,-3,1,1} - 204800S_{-2,1,1,-2,-2,1} - 114688S_{-2,1,1,-2,1,-2} \\
 & -131072S_{-2,1,1,-2,1,2} - 131072S_{-2,1,1,-2,2,1} - 65536S_{-2,1,1,1,-3,1} - 57344S_{-2,1,1,1,-2,-2} \\
 & -65536S_{-2,1,1,1,-2,2} + S_2((1024S_{-3} + 4096S_3)S_{-2}^2 + (11264S_{-5} + 5120S_5 - 8192S_{-4,1} \\
 & -6144S_{-3,2} - 8192S_{-2,3} + 2048S_{4,1} + 12288S_{-3,1,1} - 4096S_{-2,-2,1} + 12288S_{-2,1,2} \\
 & + 12288S_{-2,2,1} - 24576S_{-2,1,1,1})S_{-2} + 8192S_{-7} + 9216S_7 - 16384S_{-6,1} - 6144S_{-5,-2} \\
 & -16384S_{-5,2} - 1024S_{-4,-3} - 17408S_{-4,3} - 15360S_{-3,4} - 18432S_{-2,5} - 5120S_{4,3} \\
 & + 4096S_{5,2} + 6144S_{6,1} + 32768S_{-5,1,1} - 6144S_{-4,-2,1} + 36864S_{-4,1,2} + 36864S_{-4,2,1} \\
 & -4096S_{-3,-3,1} - 2048S_{-3,-2,-2} - 4096S_{-3,-2,2} + 36864S_{-3,1,3} + 40960S_{-3,2,2} \\
 & + 36864S_{-3,3,1} + 2048S_{-2,-4,1} - 8192S_{-2,-3,2} + 10240S_{-2,-2,3} + S_{-2,1}(-4096S_{-4} \\
 & -8192S_4 + 12288S_{-3,1} + 16384S_{-2,2} - 8192S_{-2,1,1}) + S_{-3}(6144S_{-4} + 3072S_4 \\
 & -6144S_{-3,1} - 4096S_{-2,2} + 12288S_{-2,1,1}) + S_3(10240S_{-4} + 3072S_4 - \frac{47104S_{-3,1}}{3}
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{40960S_{-2,2}}{3} + \frac{69632}{3}S_{-2,1,1}) + 34816S_{-2,1,4} + 36864S_{-2,2,3} + 36864S_{-2,3,2} \\
 & + 32768S_{-2,4,1} - 4096S_{4,1,2} - 4096S_{4,2,1} - 73728S_{-4,1,1,1} - 81920S_{-3,1,1,2} \\
 & - 81920S_{-3,1,2,1} - 81920S_{-3,2,1,1} + 24576S_{-2,-3,1,1} + 4096S_{-2,-2,-2,1} + 8192S_{-2,-2,1,2} \\
 & + 8192S_{-2,-2,2,1} - 8192S_{-2,1,1,-3} - 73728S_{-2,1,1,3} - 81920S_{-2,1,2,2} - 73728S_{-2,1,3,1} \\
 & - 8192S_{-2,2,-2,1} - 81920S_{-2,2,1,2} - 81920S_{-2,2,2,1} - 73728S_{-2,3,1,1} + 24576S_{4,1,1,1} \\
 & + 163840S_{-3,1,1,1,1} - 49152S_{-2,-2,1,1,1} - 16384S_{-2,1,-2,1,1} + 163840S_{-2,1,1,1,2} \\
 & + 163840S_{-2,1,1,2,1} + 163840S_{-2,1,2,1,1} + 163840S_{-2,2,1,1,1} - 327680S_{-2,1,1,1,1,1}) \\
 & - 65536S_{-2,1,2,-2,1,1} - 131072S_{-2,2,-2,1,1,1} - 65536S_{-2,2,1,-2,1,1} - 327680S_{1,-3,-2,1,1,1} \\
 & - 294912S_{1,-3,1,-2,1,1} - 147456S_{1,-3,1,1,-2,1} - 16384S_{1,-3,1,1,1,-2} - 327680S_{1,-2,-3,1,1,1} \\
 & - 188416S_{1,-2,-2,-2,1,1} - 180224S_{1,-2,-2,1,-2,1} - 131072S_{1,-2,-2,1,1,-2} \\
 & - 163840S_{1,-2,-2,1,1,2} - 163840S_{1,-2,-2,1,2,1} - 163840S_{1,-2,-2,2,1,1} - 294912S_{1,-2,1,-3,1,1} \\
 & - 188416S_{1,-2,1,-2,-2,1} - 131072S_{1,-2,1,-2,1,-2} - 163840S_{1,-2,1,-2,1,2} - 163840S_{1,-2,1,-2,2,1} \\
 & - 180224S_{1,-2,1,1,-3,1} - 106496S_{1,-2,1,1,-2,-2} - 131072S_{1,-2,1,1,-2,2} - 49152S_{1,-2,1,1,1,-3} \\
 & - 49152S_{1,-2,1,2,-2,1} - 131072S_{1,-2,2,-2,1,1} - 49152S_{1,-2,2,1,-2,1} - 16384S_{1,1,-4,1,1,1} \\
 & - 327680S_{1,1,-3,-2,1,1} - 229376S_{1,1,-3,1,-2,1} - 65536S_{1,1,-3,1,1,-2} - 327680S_{1,1,-2,-3,1,1} \\
 & - 196608S_{1,1,-2,-2,-2,1} - 147456S_{1,1,-2,-2,1,-2} - 163840S_{1,1,-2,-2,1,2} - 163840S_{1,1,-2,-2,2,1} \\
 & - 262144S_{1,1,-2,1,-3,1} - 131072S_{1,1,-2,1,-2,-2} - 163840S_{1,1,-2,1,-2,2} - 114688S_{1,1,-2,1,1,-3}
 \end{aligned}$$

Five-loop ABA

$$\begin{aligned} & -16384S_{1,1,-2,1,2,-2} - 98304S_{1,1,-2,2,-2,1} - 16384S_{1,1,-2,2,1,-2} + 16384S_{1,1,1,-4,1,1} \\ & -327680S_{1,1,1,-3,-2,1} - 114688S_{1,1,1,-3,1,-2} - 32768S_{1,1,1,-3,1,2} - 32768S_{1,1,1,-3,2,1} \\ & -294912S_{1,1,1,-2,-3,1} - 147456S_{1,1,1,-2,-2,-2} - 163840S_{1,1,1,-2,-2,2} - 180224S_{1,1,1,-2,1,-3} \\ & + 16384S_{1,1,1,-2,1,3} - 49152S_{1,1,1,-2,2,-2} + 16384S_{1,1,1,-2,3,1} - 65536S_{1,1,1,-3,-3,-2} \\ & -32768S_{1,1,1,1,-3,2} - 262144S_{1,1,1,1,-2,-3} - 196608S_{1,1,1,1,4,1} - 163840S_{1,1,1,4,1,1} \\ & -32768S_{1,1,2,-3,1,1} - 196608S_{1,1,2,-2,-2,1} - 65536S_{1,1,2,-2,1,-2} - 32768S_{1,1,2,-2,1,2} \\ & -32768S_{1,1,2,-2,2,1} + 32768S_{1,1,3,-2,1,1} - 163840S_{1,1,4,1,1,1} - 32768S_{1,2,-3,1,1,1} \\ & -163840S_{1,2,-2,-2,1,1} - 131072S_{1,2,-2,1,-2,1} - 49152S_{1,2,-2,1,1,-2} - 32768S_{1,2,1,-3,1,1} \\ & -196608S_{1,2,1,-2,-2,1} - 65536S_{1,2,1,-2,1,-2} - 32768S_{1,2,1,-2,1,2} - 32768S_{1,2,1,-2,2,1} \\ & -16384S_{1,3,-2,1,1,1} + 32768S_{1,3,1,-2,1,1} - 163840S_{1,4,1,1,1,1} - 163840S_{2,-2,-2,1,1,1} \\ & -163840S_{2,-2,1,-2,1,1} - 98304S_{2,-2,1,1,-2,1} - 16384S_{2,-2,1,1,1,-2} - 32768S_{2,1,-3,1,1,1} \\ & -163840S_{2,1,-2,-2,1,1} - 131072S_{2,1,-2,1,-2,1} - 49152S_{2,1,-2,1,1,-2} - 32768S_{2,1,1,-3,1,1} \\ & -196608S_{2,1,1,-2,-2,1} - 65536S_{2,1,1,-2,1,-2} - 32768S_{2,1,1,-2,1,2} - 32768S_{2,1,1,-2,2,1} \\ & -32768S_{2,2,-2,1,1,1} - 16384S_{3,1,-2,1,1,1} + 32768S_{3,1,1,-2,1,1} - 163840S_{4,1,1,1,1,1} \\ & + 327680S_{-2,-2,1,1,1,1,1} + 327680S_{-2,1,-2,1,1,1,1} + 262144S_{-2,1,1,-2,1,1,1} \end{aligned}$$

Five-loop ABA

$$\begin{aligned}
 &+131072S_{-2,1,1,1,-2,1,1} + 327680S_{1,-2,-2,1,1,1,1} + 327680S_{1,-2,1,-2,1,1,1} \\
 &+262144S_{1,-2,1,1,-2,1,1} + 98304S_{1,-2,1,1,1,-2,1} + 327680S_{1,1,-2,-2,1,1,1} \\
 &+327680S_{1,1,-2,1,-2,1,1} + 196608S_{1,1,-2,1,1,-2,1} + 32768S_{1,1,-2,1,1,1,-2} + 65536S_{1,1,1,-3,1,1,1} \\
 &+327680S_{1,1,1,-2,-2,1,1} + 262144S_{1,1,1,-2,1,-2,1} + 98304S_{1,1,1,-2,1,1,-2} + 65536S_{1,1,1,1,-3,1,1} \\
 &+393216S_{1,1,1,1,-2,-2,1} + 131072S_{1,1,1,1,-2,1,-2} + 65536S_{1,1,1,1,-2,1,2} + 65536S_{1,1,1,1,-2,2,1} \\
 &+65536S_{1,1,2,-2,1,1,1} + 65536S_{1,2,1,-2,1,1,1} + 65536S_{2,1,1,-2,1,1,1} - 131072S_{1,1,1,1,-2,1,1,1} \\
 &+512 \left(4S_{-2,1}S_{-3} - S_{-3}^2 + S_3^2 - 4S_{-2,1}^2 + S_1^2 \left(2S_{-2}^2 - 4S_{-4} + 6S_4 + 16S_{-3,1} + 12S_{-2,2} \right. \right. \\
 &\quad \left. \left. - 16S_{-2,1,1} \right) + S_1 \left(-2S_{-5} - 4S_{-3}S_2 + 4S_{-2}S_3 + 4S_2S_3 + 6S_5 + 8S_{-4,1} - 4S_{-3,-2} \right. \right. \\
 &\quad \left. \left. + 12S_{-3,2} + 8S_{-2}S_{-2,1} + 8S_2S_{-2,1} + 8S_{-2,3} + 4S_{4,1} - 24S_{-3,1,1} - 8S_{-2,-2,1} - 24S_{-2,1,2} \right. \right. \\
 &\quad \left. \left. - 24S_{-2,2,1} + 48S_{-2,1,1,1} \right) \right) \zeta(3) \\
 &+2560 S_1(S_3 - S_{-3} + 2 S_{-2,1}) \zeta(5)
 \end{aligned}$$

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- For any M :

[T. Ł., A. Rej, V. Velizhanin, 2009]

Five-loop wrapping

$$\begin{aligned}\Delta_w^{(5)} = & 13440 \zeta(7) s_1^2 - 1536 \zeta(3)^2 s_1^3 + 2560 \zeta(5) s_1 (3 s_1 (2 s_{-2} + s_2) - s_1^3 + s_{-3} + s_3 - 2 s_{-2,1}) \\ & + 1024 \zeta(3) s_1 (-2 s_1^3 s_{-2} + 2 s_1^2 (2 s_{-3} + 3 s_3) + s_1 (4 s_{-2}^2 + 6 s_2 s_{-2} + 3 s_{-4} - s_4 \\ & - 2 (s_{-3,1} - 2 s_{-2,-2} + s_{-2,2} + s_{3,1} - 2 s_{-2,1,1})) + 2 s_{-2} (s_{-3} + s_3 - 2 s_{-2,1})) \\ & - 1024 s_1 ((s_1 (3 s_2 + 2 s_{-2}) + s_{-3} + s_3 - 2 s_{-2,1} - s_1^3) (s_{-5} - s_5 + 2 s_{-2,-3} - 2 s_{3,-2} \\ & + 2 s_{4,1} - 4 s_{-2,-2,1}) + 2 s_1^2 (2 s_{-6} - 2 s_6 - s_{-4,-2} + 2 s_{-3,-3} + 3 s_{-2,-4} + s_{-2,4} \\ & - 2 s_{3,-3} - 2 s_{4,-2} + s_{4,2} + 4 s_{5,1} - 4 s_{-3,-2,1} - 4 s_{-2,-3,1} - 2 s_{-2,-2,-2} - 2 s_{-2,-2,2}) \\ & + s_1 (5 s_{-7} - 5 s_7 - 4 s_{-6,1} + 4 s_{-5,-2} - s_{-5,2} + 3 s_{-4,-3} + s_{-3,-4} - s_{-3,4} + 8 s_{-2,-5} \\ & - 6 s_{-2,5} - 4 s_{3,-4} + 2 s_{3,4} - 8 s_{4,-3} + 3 s_{4,3} - 6 s_{5,-2} + s_{5,2} + 6 s_{6,1} + 2 s_{-5,1,1} \\ & - 6 s_{-4,-2,1} - 2 s_{-3,-3,1} + 2 s_{-3,-2,-2} - 2 s_{-3,1,-3} - 8 s_{-2,-4,1} + 6 s_{-2,-3,-2} - 2 s_{-2,-3,2} \\ & + 14 s_{-2,-2,-3} - 6 s_{-2,-2,3} - 2 s_{-2,1,-4} + 2 s_{-2,1,4} - 2 s_{-2,2,-3} - 4 s_{-2,3,-2} + 10 s_{-2,4,1} \\ & + 2 s_{3,-3,1} - 4 s_{3,-2,-2} + 2 s_{3,-2,2} + 2 s_{3,1,-3} + 2 s_{3,2,-2} + 10 s_{4,-2,1} + 6 s_{4,1,-2} - 2 s_{4,1,2} \\ & - 2 s_{4,2,1} - 2 s_{5,1,1} + 4 s_{-3,1,-2,1} + 4 s_{-2,-3,1,1} - 20 s_{-2,-2,-2,1} - 8 s_{-2,-2,1,-2} \\ & + 4 s_{-2,-2,1,2} + 4 s_{-2,-2,2,1} + 4 s_{-2,1,-3,1} - 4 s_{-2,1,-2,-2} + 4 s_{-2,1,1,-3} + 4 s_{-2,2,-2,1} \\ & - 4 s_{3,-2,1,1} - 4 s_{3,1,1,-2} + 4 s_{4,1,1,1} - 8 s_{-2,-2,1,1,1} - 8 s_{-2,1,1,-2,1}))\end{aligned}$$

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Thank you!