Strings in the AdS light-cone gauge and the generalized scaling function

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based on work with
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From Matthias Staudacher's talk:

Gauge Theory Meets String Theory

The asymptotic Bethe ansatz yields an integral equation for an interpolating scaling function $f(g)$ at arbitrary values of g. [Beisert, Eden, Staudacher '06]

At weak coupling this equation was (numerically) tested up to four loop order in gauge theory: [Bern, Czakon, Dixon, Kosower, Smirnov, '06; Cachazo, Spradlin, Volovich '06].

$$\boxed{f(g) = 8\,g^2 - \tfrac{8}{3}\pi^2 g^4 + \tfrac{88}{45}\pi^4 g^6 - 16\left(\tfrac{73}{630}\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots\,.}$$

A five-loop test is under way [Bourjaily, Henn, Spradlin, work in progress].

At strong coupling the scaling function agrees with string theory to the three known orders [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02], [Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] as was analytically shown by [Basso, Korchemsky, Kotański '07]

$$\boxed{f(g) = 4\,g - \frac{3\log 2}{\pi} - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \ldots}$$

$\rightarrow$ The AdS/CFT correspondence appears to be exactly true !

The issue: the anomalous dimension of high twist operators $\mathrm{Tr}[D^S Z^J]$
in the scaling limit $S, J \to \infty \quad J/\ln S - \mathrm{fixed}$

ABA vs. direct worldsheet calculations

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AdS dual is a generalization of the GKP string

Frolov, Tseytlin

- homogeneous in the scaling limit ➡ generalized scaling function

$$\frac{\pi}{\sqrt{\lambda}} \frac{(E-S)}{\ln S} = f(\sqrt{\lambda}, \ell) = f_0(\ell) + \frac{1}{\sqrt{\lambda}} f_1(\ell) + \frac{1}{\sqrt{\lambda}^2} f_2(\ell) + \ldots$$

$$= f(\lambda) + \ell^2 \left(q_0(\lambda) + q_1(\lambda)\ln\ell + q_2(\lambda)(\ln\ell)^2 + \ldots\right) + \ell^4 \left(q_0(\lambda) + q_1(\lambda)\ln\ell + q_2(\lambda)(\ln\ell)^2 + \ldots\right) + \mathcal{O}(\ell^6)$$

$$\ell = \frac{\pi J}{\sqrt{\lambda}\ln S}$$

$$= f(\lambda) + \ell^2 \sum_n \frac{1}{(\sqrt{\lambda})^{n-1}} \left(c_n(\ln\ell)^n + d_n(\ln\ell)^{n-1} + \ldots\right) + \mathcal{O}(\ell^4)$$

$O(6)$ renorm. group

Alday, Maldacena

$$f_2(\ell) = -K + \ell^2(8(\ln\ell)^2 - 6\ln\ell + q_{02}) + O(\ell^4)$$

$$q_{02\,\text{string?}} = -\frac{3}{2}\ln 2 + \frac{7}{4} - 2K$$

RR, Tseytlin

Highly successful Asymptotic Bethe Ansatz

Beisert, Staudacher

- equation for universal scaling function

Freyhult, Rej, Staudacher

$$q_{02} = -\frac{3}{2}\ln 2 + \frac{11}{4}$$

Gromov; Basso, Korchemsky; Volin

Outline

- Quantum corrections in the semiclassical approximation
- Strings in AdS light-cone gauge
- Open and closed string classical solutions with winding for high twist operators in the scaling limit
- Quantum corrections to their energies
    - 1 loop; equivalence of 2 approaches
    - 2 loops; comparison with ABA
- Resummation of leading logarithms
- On finite size corrections
- Summary and a question

Quantum corrections to conserved charges

Options:

1. (a) For a given classical solution evaluate:

$$\langle E\rangle = \frac{1}{Z}\int D\Phi\, E[\Phi]e^{-S[\Phi]} \qquad \langle Q_i\rangle = \frac{1}{Z}\int D\Phi\, Q_i[\Phi]e^{-S[\Phi]}$$

Integral of a density

$$\langle T_{\alpha\beta}\rangle = \frac{1}{Z}\int D\Phi\, T_{\alpha\beta}[\Phi]e^{-S[\Phi]}$$

Integrals of densities

(b) Eliminate the parameters of the classical solution between

$$E = \langle E\rangle \qquad Q_i = \langle Q_i\rangle \qquad \langle T_{ij}\rangle = 0$$

Essentially partition function with delta function constraint

2. Extract $\langle E\rangle,\ \langle Q_i\rangle$ from partition function with chemical potentials

$$Z = e^{-\beta\Sigma(\kappa,h_i)} = \mathrm{Tr}[e^{-\beta\widetilde{H}_{2d}}] \qquad \widetilde{H}_{2d} = H_{2d} + \kappa E - \sum_i h_i Q_i$$

$$[H_{2d}, E] = 0 = [H_{2d}, Q_i]$$

Perturbation theory + ground state + Virasoro: $V(\kappa, h_i, \sqrt{\lambda}) = 0$

Equivalence?

For garden-variety sigma models: $\int_{-i\infty}^{+i\infty} d\kappa [dh_i] e^{-\kappa E} e^{-\sum_i h_i Q_i} Z[\kappa, h_i]$

integral over chemical potentials yields delta function constraints for $\langle E \rangle, \ \langle Q_i \rangle$ and Virasoro

Standard approach:

$$\langle E \rangle = -\frac{1}{\beta}\frac{\partial \Sigma}{\partial \kappa} \qquad \langle Q_i \rangle = +\frac{1}{\beta}\frac{\partial \Sigma}{\partial h_i}$$

& eliminate $\kappa, \ h_i$ repeating 1.(b)

$\Downarrow$

1. and 2. are equivalent if the integral over chemical potentials can be treated in a saddle-point approximation

- Care is needed for finite size systems

Information independent of worldsheet details:

$$F\left(E,\, Q_i,\, \sqrt{\lambda}\right) = 0$$

For (high) twist operators: $\mathrm{Tr}[D^S Z^J]$

- 3 chemical potentials: $\kappa,\ \omega,\ \nu$
- scaling limit leading to generalized scaling function:

  $E \sim S \ll \ln S \ \rightarrow\ \kappa = \omega$ chemical potential for $E - S$

- Virasoro constraint : $\kappa = \kappa(\nu)$

From "grand potential" $\Sigma(\nu)$:

$$\frac{d\Sigma(\nu)}{d\nu} = \frac{d\kappa(\nu)}{d\nu}\langle E - S\rangle - \langle J\rangle$$

$$\Sigma(\nu) = \langle H_{2d}\rangle + \kappa(\nu)\langle E - S\rangle - \nu\langle J\rangle$$

$\mathrm{Tr}[H_{2d}e^{-\beta\widetilde{H}_{2d}}]$ may first be nonzero at 2 loops

$$E - S \equiv \langle E - S\rangle \quad = \quad -\left[\nu\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\right]^{-1}\left[\left(\Sigma(\nu) - \langle H_{2d}\rangle\right) - \nu\frac{d\Sigma(\nu)}{d\nu}\right]$$

$$J \equiv \langle J\rangle \quad = \quad -\left[\nu\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\right]^{-1}\left[\left(\Sigma(\nu) - \langle H_{2d}\rangle\right)\frac{d\kappa(\nu)}{d\nu} - \kappa(\nu)\frac{d\Sigma(\nu)}{d\nu}\right]$$

A conclusion:

In general, “grand potential” Σ and the expectation value of the 2-d Hamiltonian $\langle H_{2d} \rangle$ are needed to determine the quantum corrections to conserved charges.

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- it is the worldsheet Hamiltonian before reparametrization gauge fixing, which is set to zero by a Virasoro constraint.

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guaranteed to all orders if Virasoro constraint is solved $H_{2d} = 0$

→ (some) light-cone gauge

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$\longrightarrow$ (some) light-cone gauge

■ H_{2d} is not the light-cone Hamiltonian $H_{lc} = -P^-$

$$\mathcal{L} = \dot{x}^- P^+ + \dot{x}^+ P^- + \dot{x}_i P^i - H_{2d}$$

$$\Big\downarrow\, x^+ = \tau$$

$$\mathcal{L} = \dot{x}_i P^i - (-P^-) = \dot{x}_i P^i - H_{lc}$$

Close relation between folded string and minimal surface with a cusp

Kruczenski, Tirziu, RR, Tseytlin; Alday, Maldacena

$$\tau' = -i\tau \qquad X'_5 = -iX_2 \qquad X'_2 = -iX_5$$

and discrete $SO(2,4)$ in planes $(0,5)$ and $(1,2)$ and $SO(2)$ rotations and projection to local coordinates

- Expect the same for the string with S and J
- Use light-cone gauge natural to local coordinates

Further technical simplifications — Giombi, Ricci, RR, Tseytlin, Vergu

Classical Virasoro: $\hat{\kappa} = \sqrt{1+\hat{\nu}^2}$ $\qquad \hat{\kappa} = \frac{\kappa}{\mu} \quad \hat{\nu} = \frac{\nu}{\mu}$

- Generalized scaling function i.t.o. the free parameter of the solution

$$f(\ell) \equiv \frac{\pi}{\sqrt{\lambda}} \frac{E-S}{\ln S} = \sqrt{1+\hat{\nu}^2}\,\left[\mathcal{F}(\hat{\nu}) - \hat{\nu}\frac{d\mathcal{F}(\hat{\nu})}{d\hat{\nu}}\right]$$

$$\ell \equiv \frac{\pi}{\sqrt{\lambda}} \frac{J}{\ln S} = \hat{\nu}\mathcal{F}(\hat{\nu}) - (1+\hat{\nu}^2)\frac{d\mathcal{F}(\hat{\nu})}{d\hat{\nu}}$$

- Eliminate $\hat{\nu}$: $\qquad f = f_0 + \frac{1}{\sqrt{\lambda}} f_1 + \frac{1}{\sqrt{\lambda}} f_2 + \dots$

$$f_0 = \sqrt{1+\ell^2}\,, \qquad f_1 = \frac{\mathcal{F}_1(\ell)}{\sqrt{1+\ell^2}}$$

$$f_2 = \frac{1}{\sqrt{1+\ell^2}}\left[\mathcal{F}_2(\ell) + \frac{1}{2}\left(\frac{\ell}{\sqrt{1+\ell^2}}\,\mathcal{F}_1(\ell) - \sqrt{1+\ell^2}\,\frac{d\mathcal{F}_1(\ell)}{d\ell}\right)^2\right]$$

$$= \frac{\mathcal{F}_2(\ell)}{\sqrt{1+\ell^2}} + \frac{1}{2}(1+\ell^2)^{3/2}\left(\frac{df_1}{d\ell}\right)^2$$

- Similar expressions if additional parameters are present

The $AdS_5 \times S^5$ Green-Schwarz string in AdS light-cone gauge

Metsaev, Tseytlin, Thorn

Unique 2-derivative action with κ – symmetry based on the coset

$$PSU(2,2|4)/SO(4,1) \times SO(5)$$

$$\mathcal{L} = \int \mathrm{STr}\left[L_{(-1)} \wedge * L_{(-1)} + L_{(-i)} \wedge L_{(i)} \right]$$

Metsaev, Tseytlin

Bosonic group element for Poincare coordinates: $g_{AdS} = \begin{pmatrix} \mathbf{1} & 0 \\ \mathbf{x} & z\mathbf{1} \end{pmatrix}$

$$ds^2 = \frac{1}{z^2}(dx^+ dx^- + dx^* dx + dz^M dz^M) = \frac{1}{z^2}(dx^m dx_m + dz^2) + du^M du^M$$

$$x^\pm = x^3 \pm x^0 \;, \qquad x, x^* = x^1 \pm \mathrm{i} x^2 \;, \qquad u^M u^M = 1$$

Gauge fixing: κ, diffeomorphisms, consistency

$$\Gamma^+ \theta^I = 0 \;, \quad x^+ = p^+ \tau \;, \quad \sqrt{-g} g^{\alpha\beta} = \mathrm{diag}(-z^2, z^{-2})$$

Our purpose: further Wick rotation

… and solve the Virasoro constraint

$$I = \frac{1}{2} T \int d\tau \int d\sigma \; \mathcal{L}_E \; , \qquad T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$

Metsaev, Tseytlin, Thorn

$$\mathcal{L}_E = \dot{x}^* \dot{x} + (\dot{z}^M + \mathrm{i} z^{-2} z_N \eta_i \rho^{MN\,i}{}_j \eta^j)^2 + \mathrm{i}(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - h.c.) - z^{-2}(\eta^2)^2$$
$$+ z^{-4}(x'^* x' + z'^M z'^M) + 2\mathrm{i}\Big[\, z^{-3} \eta^i \rho^M_{ij} z^M (\theta'^j - \mathrm{i} z^{-1} \eta^j x') + h.c. \Big]$$

- $p^+ = 1$
- ρ^M off-diag components of 6-d Dirac matrices
- Non-polynomial; perturbation theory requires bosonic background
- Quartic in fermions
- Power-counting renormalizable ($[z] = 0 = [x] \qquad [\theta] = 1/2 = [\eta]$)
- Need regularization preserving classical (local) symmetries
- cancellation of power-like divergences requires careful analysis of path integral measure and κ-symmetry ghosts

The light-cone state dual to $\mathrm{Tr}[D^S Z^J]$

$$\mathcal{L} = \dot{z}^2 + \frac{1}{z^4} z'^2 + z^2 \dot{\varphi}^2 + \frac{1}{z^2} \varphi'^2$$

(a) direct construction

(b) map from global coordinates

Model after the cusp

(from Virasoro constraint)

$$z = \sqrt{\frac{\tau}{\sigma}}, \qquad x^+ = \tau \qquad x^- = -\frac{1}{2\sigma} \qquad \varphi = 0$$

Kruczenski

Giombi, Ricci, RR, Tseytlin, Vergu

Separated variable solution with $\varphi \neq 0$

winding

$$z = \sqrt{\hat{\kappa}} \sqrt{\frac{\tau}{\sigma}} \qquad x^+ = \tau \qquad x^- = -(\hat{\kappa} - \hat{\nu}_e \hat{m}) \frac{1}{2\sigma} \qquad \varphi = \frac{\hat{\nu}_e}{2\hat{\kappa}} \ln \tau + \frac{\hat{m}}{2} \ln \sigma$$

$$\hat{\kappa}^2 = 1 - \hat{\nu}_e^2 + \hat{m}^2$$

$$Y_0 Y_2 - Y_1 Y_5 = \frac{\hat{\nu}_e \hat{m}}{2\hat{\kappa}}$$

$$Y_0^2 - Y_1^2 + Y_2^2 - Y_5^2 = 1$$

Giombi, Ricci, RR, Tseytlin, Vergu

- 2d coord. transf. to conformal gauge $\qquad \mathcal{F}_0 = \frac{\sqrt{\lambda}}{2\pi}(1 + \hat{m}^2)$
- Interpretation of parameters: from comparison w/ closed string
  (an additional parameter compared to the usual (SJ) string)

$$ds^2 = -\cosh^2\!\rho\, dt^2 + d\rho^2 + \sinh^2\!\rho\, d\theta^2 + d\varphi^2$$

(S,J) string and flat space closed string solution w/ winding suggest:

$$t = \kappa\tau \qquad \varphi = \nu\tau + m\sigma \qquad \rho = \rho(\sigma) \qquad \theta = \omega\tau + \vartheta(\sigma)$$

bending in AdS_3

Virasoro constraints:

$$\rho'^2 - \kappa^2\cosh^2\rho + \omega^2\sinh^2\rho + \sinh^2\rho\,\vartheta'^2 + \nu^2 + m^2 = 0$$

$$\omega\ \sinh^2\rho\ \vartheta' + \nu m = 0$$

Same equation as for $S_1 = S_2$ in AdS_3

Tirziu, Tseytlin

- solution: elliptic; periodicity condition: relations btw parameters
- glue several arcs to cover complete range of ϑ
- change in topology: string no longer folded ("rounded spiky string")

$$t = \kappa\tau \qquad \varphi = \nu\tau + m\sigma \qquad \rho = \rho(\sigma) \qquad \theta = \omega\tau + \vartheta(\sigma)$$

Scaling limit: (1) relax periodicity condition

(2) $\kappa = \omega \to \infty\ , \quad m, \nu \to \infty\ , \quad \mu^2 = \kappa^2 - \nu^2 - m^2 \to \infty$

$$\cosh\,\rho = \sqrt{1+\gamma^2}\cosh(\mu\sigma) \qquad \tan\,\vartheta = \gamma\coth(\mu\sigma) \qquad \gamma \equiv \frac{\nu m}{\kappa\mu}$$

Piece together n arcs to have a closed string

Parameters vs. energy, spin and momentum

$$\mathcal{E}_0 = 2n\kappa \int_0^{\frac{\pi}{n}} \frac{d\sigma}{2\pi}\cosh^2\rho \qquad \mathcal{S} = 2n\kappa \int_0^{\frac{\pi}{n}} \frac{d\sigma}{2\pi}\sinh^2\rho \qquad \mathcal{J} = \nu$$

$$\mathcal{E}_0 - \mathcal{S} = \kappa \qquad \mathcal{S} \simeq \frac{n\kappa}{8\pi\mu}(1+\gamma^2)e^{\frac{2\pi\mu}{n}} \qquad \to \mu \simeq \frac{n}{2\pi}\ln\mathcal{S}$$

$$\mathcal{E}_0 - \mathcal{S} \approx \sqrt{\frac{n^2}{4\pi^2}\ln^2\mathcal{S} + \nu^2 + m^2}$$

$$\mathcal{E}_0 - \mathcal{S} \approx \sqrt{\frac{n^2}{4\pi^2}\ln^2 S + \nu^2 + m^2}$$

Also recovered from ABA in presence of nontrivial winding Kruczenski, Tirziu

The equivalence

- Embedding coordinates

$$X_0 + \mathrm{i}X_5 = \cosh\rho\; e^{\mathrm{i}t} \qquad X_1 + \mathrm{i}X_2 = \sinh\rho\; e^{\mathrm{i}\theta} \qquad X_3 = X_4 = 0$$

- boost in (02) and (15) plans w/ parameter: $\tanh(2v) = \dfrac{\gamma}{\sqrt{1+\gamma^2}}$

$$\downarrow$$

$$Y_0Y_2 - Y_1Y_5 = \frac{\nu m}{2\kappa\mu} \qquad Y_0^2 + Y_5^2 - Y_1^2 - Y_2^2 = 1$$

$$\Big\downarrow \quad \begin{array}{lllll} Y_0 \to Y_0, & Y_1 \to Y_1, & Y_2 \to \mathrm{i}Y_2, & Y_5 \to \mathrm{i}Y_5 & \\ \nu_e \to \mathrm{i}\nu, & \mu \to \mu, & \kappa \to \kappa, & \frac{1}{2}\hat{\kappa}t \to -\mathrm{i}\kappa\tau, & \frac{1}{2}s \to \mu\sigma \end{array}$$

$$Y_0Y_2 - Y_1Y_5 = \frac{\hat{\nu}_e\hat{m}}{2\hat{\kappa}} \qquad Y_0^2 - Y_1^2 + Y_2^2 - Y_5^2 = 1$$

▶ Interested in dependence on closed string param's or S and J

▲ Solution is still homogeneous, even in the presence of $m = i\hat{w}$

$$X_0 \pm X_3 = \cosh r\; \sin\beta\, e^{\pm p}\;, \quad X_5 \pm X_4 = \cosh r\; \cos\beta\, e^{\pm q}\;, \quad X_1 \pm \mathrm{i}X_2 = \sinh r\, e^{\pm \mathrm{i}h}$$

Fluctuation action:

$$
\begin{aligned}
\mathcal{L} \;=\;& \left|\partial_t\tilde{x}+\frac{1}{2}\hat{\kappa}\,\tilde{x}\right|^2 \\
&+\Big(\partial_t(R\tilde{z})^M-[((\partial_t R)R^{-1})(R\tilde{z})]^M+\frac{1}{2}\hat{\kappa}\,(R\tilde{z})^M+\frac{\mathrm{i}}{\tilde{z}^2}\tilde{\eta}_i\rho^{MNi}{}_j\tilde{\eta}^j(R\tilde{z})_N\Big)^2 \\
&+\frac{1}{\tilde{z}^4}\Big[\left|\partial_s\tilde{x}-\frac{1}{2}\tilde{x}\right|^2+\Big(\partial_s R\tilde{z}^M-[((\partial_s R)R^{-1})(R\tilde{z})]^M-\frac{1}{2}R\tilde{z}^M\Big)^2\Big] \\
&\qquad+\mathrm{i}\Big(\tilde{\theta}^k(\partial_t\delta^k_l-\frac{1}{2}\partial_t\check{\varphi}(\rho^{[5}\rho^{\dagger 6]})_k{}^l)\tilde{\theta}_l+\tilde{\eta}^k(\partial_t\delta^k_l-\frac{1}{2}\partial_t\check{\varphi}(\rho^{[5}\rho^{\dagger 6]})_k{}^l)\tilde{\eta}_l \\
&\qquad\quad+\tilde{\theta}_k(\partial_t\delta^k_l-\frac{1}{2}\partial_t\check{\varphi}(\rho^{\dagger[5}\rho^{6]})^k{}_l)\tilde{\theta}^l+\tilde{\eta}_k(\partial_t\delta^k_l-\frac{1}{2}\partial_t\check{\varphi}(\rho^{\dagger[5}\rho^{6]})^k{}_l)\tilde{\eta}^l\Big)-\frac{1}{\tilde{z}^2}(\tilde{\eta}^2)^2\Big] \\
&+\frac{2\mathrm{i}}{\tilde{z}^3}\Big[\tilde{\eta}^k\rho^M_{kl}(R\tilde{z})^M\Big(\partial_s\tilde{\theta}^l-\frac{1}{2}\partial_s\check{\varphi}(\rho^{\dagger[5}\rho^{6]})^l{}_u\tilde{\theta}^u-\frac{1}{2}\tilde{\theta}^l-\frac{\mathrm{i}}{\tilde{z}}\tilde{\eta}^l(\partial_s\tilde{x}-\frac{1}{2}\tilde{x})\Big) \\
&\quad+\tilde{\eta}_k(\rho^\dagger_M)^{kl}(R\tilde{z})^M\Big(\partial_s\tilde{\theta}_l-\frac{1}{2}\partial_s\check{\varphi}(\rho^{[5}\rho^{\dagger 6]})_l{}^u\tilde{\theta}_u-\frac{1}{2}\tilde{\theta}_l+\frac{\mathrm{i}}{\tilde{z}}\tilde{\eta}_l(\partial_s\tilde{x}^*-\frac{1}{2}\tilde{x}^*)\Big)\Big]
\end{aligned}
$$

R - rotation in (56) plane due to non-trivial φ

Next steps:

find partition function; any all loop information accessible?

finite size effects in this approach?

- Essential feature of the fluctuation action for $\hat{\nu} = 0 = \hat{w}$

quadratic fluctuation operator is diagonal

- $\hat{\nu} \neq 0 \neq \hat{w}$ $\quad$ z & φ mix

$$K_B^{-1}(p) = \begin{pmatrix} 0 & \frac{2}{p^2+\frac{1}{4}(1+\hat{\kappa}^2)} & 0 & 0 & \mathbf{0}_{1\times4} \\ \frac{2}{p^2+\frac{1}{4}(1+\hat{\kappa}^2)} & 0 & 0 & 0 & \mathbf{0}_{1\times4} \\ 0 & 0 & \frac{p^2}{\mathcal{D}_B(p)} & \frac{\hat{\nu}p_0-\hat{w}p_1}{\mathcal{D}_B(p)} & \mathbf{0}_{1\times4} \\ 0 & 0 & \frac{-\hat{\nu}p_0+\hat{w}p_1}{\mathcal{D}_B(p)} & \frac{\hat{\kappa}^2-\hat{\nu}^2+p^2}{\mathcal{D}_B(p)} & \mathbf{0}_{1\times4} \\ \mathbf{0}_{4\times1} & \mathbf{0}_{4\times1} & \mathbf{0}_{4\times1} & \mathbf{0}_{4\times1} & \frac{\mathbf{1}_{4\times4}}{p^2+\frac{1}{4}(\hat{\nu}^2+\hat{w}^2)} \end{pmatrix}$$

- 1-loop free energy: $\mathcal{F}_1 = 4\pi \int \frac{d^2p}{(2\pi)^2} \left(\ln\det K_B - \ln\det K_F\right)$

$$\det K_B = \left[p^2 + \tfrac{1}{4}(1+\hat{\kappa}^2)\right]^2 \left(p^4 + \hat{\kappa}^2 p_0^2 + p_1^2 - 2\hat{\nu}\hat{w}\, p_0 p_1\right) \left[p^2 + \tfrac{1}{4}(\hat{\nu}^2+\hat{w}^2)\right]^4$$

$$\det K_F = \left[\left(p^2 + \frac{4\hat{\kappa}^2 - \hat{\nu}^2 + 3\hat{w}^2}{16}\right)^2 + \tfrac{1}{4}(\hat{\nu}p_0 - \hat{w}p_1)^2\right]^4$$

- Equivalent to sum of fluctuation energies

1-loop free energy:

Giombi, Ricci, RR, Tseytlin, Vergu

$$\mathcal{F}_1(\hat\nu,\hat w) = -1 + \hat w^2 + \sqrt{(1+\hat\nu^2)(1-\hat w^2)} - (\hat\nu^2+\hat w^2)\ln(\hat\nu^2+\hat w^2) + 2(1+\hat\nu^2)\ln(1+\hat\nu^2)$$
$$- (2+\hat\nu^2-\hat w^2)\ln\left[\sqrt{2+\hat\nu^2-\hat w^2}\left(\sqrt{1+\hat\nu^2}+\sqrt{1-\hat w^2}\right)\right]$$

- $\hat w = 0$ reproduces know results

Frolov, Tirziu, Tseytlin

- $\hat w \neq 0$ same symmetries as the classical solution

$$\nu \leftrightarrow w \qquad \kappa \leftrightarrow \mu \qquad \int dtds \leftrightarrow \kappa^2 \int dtds$$

1-loop generalized scaling function:

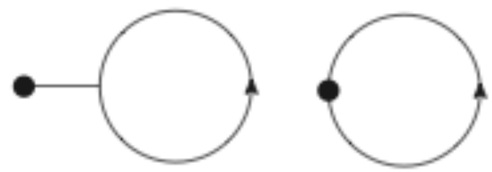
$$\mathrm{f}_1 = \frac{1}{\sqrt{1+\ell^2+\hat m^2}}\mathcal{F}_1(\ell, -\mathrm{i}\hat m)$$

What about the direct evaluation of $\langle E - S\rangle_1$ and $\langle J\rangle_1$?

$$J = \frac{\sqrt{\lambda}}{2\pi}\int ds\left[\hat{\nu} + \left(2\hat{\nu}\,\tilde{\phi} - 2\mathrm{i}\,\partial_t\tilde{\varphi}\right) + \left(2\hat{\nu}\,\tilde{\phi}^2 - \hat{\nu}\tilde{y}^a\tilde{y}^a - 4\mathrm{i}\tilde{\phi}\partial_t\tilde{\varphi} - \tilde{\theta}_i(\rho^{\dagger[5}\rho^{6]})^i{}_j\tilde{\theta}^j - 3\tilde{\eta}_i(\rho^{\dagger[5}\rho^{6]})^i{}_j\tilde{\eta}^j\right)\right]$$

nontrivial contributions due to existence of a tadpole for $\tilde{\phi}$

$$E - S = \frac{\sqrt{\lambda}}{2\pi}\int ds\ \left[\hat{\kappa} + 2\hat{\kappa}\tilde{\phi} + \hat{\kappa}(2\tilde{\phi}^2 + |\tilde{x}|^2)\right]$$

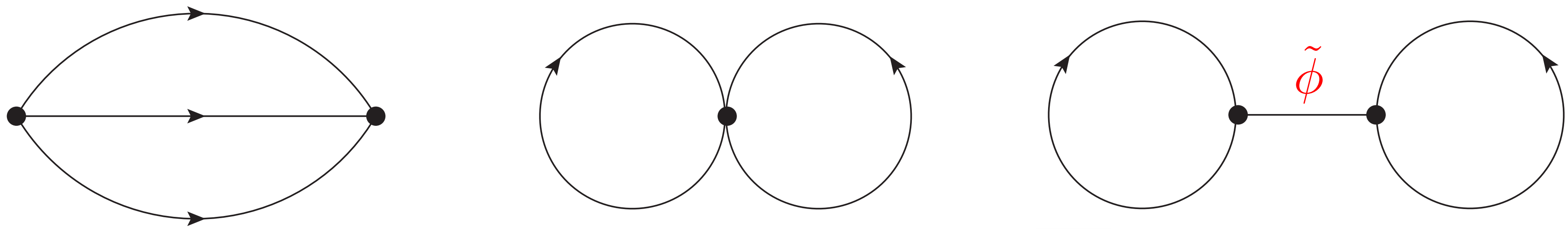
Direct evaluation of the graphs  yields:

$$\langle E - S\rangle_1 = \frac{1}{\pi}\ln S\,\frac{\hat{\kappa}}{1-\hat{w}^2}\left[\mathcal{F}_1(\hat{\nu},\hat{w}) - \hat{\nu}\frac{\partial\mathcal{F}_1(\hat{\nu},\hat{w})}{\partial\hat{\nu}}\right]$$

$$\langle \ell\rangle_1 = \frac{1}{1-\hat{w}^2}\ \left[\hat{\nu}\mathcal{F}_1(\hat{\nu},\hat{w}) - (1+\hat{\nu}^2-\hat{w}^2)\frac{d\mathcal{F}_1(\hat{\nu},\hat{w})}{d\hat{\nu}}\right]$$

Comforting that it is same as partition function approach

▲ 2-loop free energy: three types of diagramatic contributions



Both for $J = 0$ and for $J \neq 0$

- All fields and interaction terms enter these diagrams
- for $J = 0$ quadratic terms are 2d Lorentz invariant; expand in J
- Regularization:
    - Dimensional regularization is not an option
    - Algebraic manipulations in two dimensions
    - 2d conformal invariance should take care of logarithmic divergences
    - Consistent with flat space limit

- Regularization: stay in $d=2$
- The calculation: reduce to scalar integrals through manipulations valid either at the level of integrand or only under the integral sign, e.g.

- no external lines:

$$p_1^\alpha p_2^\beta \mapsto \frac{1}{2} p_1 \cdot p_2 \delta^{\alpha\beta} + \frac{1}{2} p_1 \times p_2 \epsilon^{\alpha\beta}$$

- two external lines with momentum q:

$$p_1^\alpha p_2^\beta \mapsto \left(-\frac{p_1 \cdot p_2}{q^2} + 2\frac{(p_1 \cdot q)(p_2 \cdot q)}{(q^2)^2}\right)\delta^{\alpha\beta} + \left(p_1 \cdot p_2 - \frac{(p_1 \cdot q)(p_2 \cdot q)}{q^2}\right) q^\alpha q^\beta$$

- All divergent integrals are 1-loop bubbles; express all of them i.t.o. a single one, e.g. $I[1]$

$$I[m^2] = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2+m^2} \qquad I[m^2] - I[1] = -\frac{1}{4\pi} \ln m^2$$

- Propagator for 0-momentum $\tilde{\phi}$: $\langle \tilde{\phi}\tilde{\phi} \rangle = \dfrac{1}{1-\hat{w}^2}$

2-loop free energy to fourth order:

Giombi, Ricci, RR, Tseytlin, Vergu

$$\mathcal{F}_2(\hat{\nu},\hat{w}) = -K + \tfrac{1}{4}\Big[(9-2K-6\ln 2)\hat{\nu}^2 + (9+2K-6\ln 2)\hat{w}^2 - 4(\hat{\nu}^2+\hat{w}^2)\ln(\hat{\nu}^2+\hat{w}^2)\Big]$$
$$+\frac{1}{576}\Big[\big(126K - 449 + 72(17-9\ln 2)\ln 2\big)\hat{\nu}^4 + 6\big(18K - 55 + 72\ln 2(-1+3\ln 2)\big)\hat{\nu}^2\hat{w}^2$$
$$+\big(126K - 1025 + 72(17-9\ln 2)\ln 2\big)\hat{w}^4 - 48\ln(\hat{\nu}^2+\hat{w}^2)\Big((-17+18\ln 2)\hat{\nu}^4$$
$$+6(11-6\ln 2)\hat{\nu}^2\hat{w}^2 + (-17+18\ln 2)\hat{w}^4 + 6(\hat{\nu}^2-\hat{w}^2)^2\ln(\hat{\nu}^2+\hat{w}^2)\Big)\Big] + \mathcal{O}(\hat{\nu}^6,\dots)$$

2-loop generalized scaling function to fourth order at zero winding:

$$\mathrm{f}_2 = \frac{1}{\sqrt{1+\ell^2}}\left[\mathcal{F}_2(\ell) + \frac{1}{2}\left(\frac{\ell}{\sqrt{1+\ell^2}}\mathcal{F}_1(\ell) - \sqrt{1+\ell^2}\,\frac{d\mathcal{F}_1(\ell)}{d\ell}\right)^2\right]$$

$$\mathrm{f}_2 = -K + \ell^2\left(8\ln^2\ell - 6\ln\ell - \frac{3}{2}\ln 2 + \frac{11}{4}\right) + \ell^4\left(-6\ln^2\ell - \frac{7}{6}\ln\ell + 3\ln 2\ln\ell - \frac{9}{8}\ln^2 2 + \frac{11}{8}\ln 2 + \frac{3}{32}K - \frac{233}{576}\right) + \mathcal{O}(\ell^6)$$

Same as q_{02} from ABA

Also reproduces ABA results of Gromov

Some comments:

- framework of higher-loop calculations is consistent with ABA

  nontrivial relation between worldsheet Z with chemical potentials and anomalous dimensions E

- AdS-lc calculations are technically simpler than in other gauges
- nontrivial cancelations between graphs of different topology
- leading logarithms in ℓ not generated by 1PI 2-loop graphs
- leading logarithms in ℓ generated by non-1PI 2-loop graphs

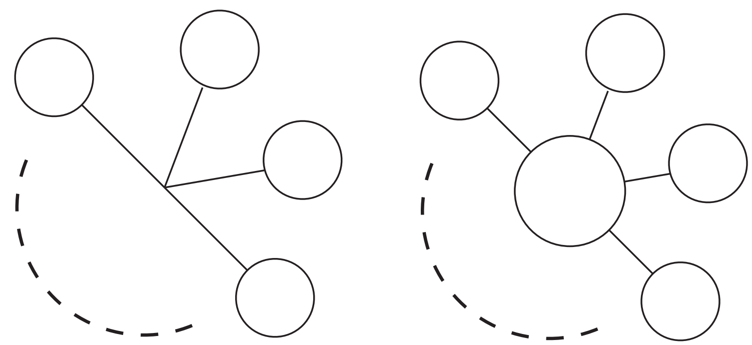
  generalizes to higher loops

- Extension to include $\hat{w}$ is straightforward

Is it possible to access leading logarithms to all orders?

All-loop resummation of leading logarithms: Giombi, Ricci, RR, Tseytlin, Vergu

1. in free energy: leading logs from maximally disconnected graphs
2. the only field with tadpole is ϕ
3. logs arise from fields with mass $\hat{\nu}$
   - compute 1-loop effective action for ϕ
   - integrate out ϕ at the classical level



- Only S^5 fields yield logarithms:

$$L = \frac{1}{4}\cosh(2\phi) + e^{2\phi}(\partial_t y)^2 + e^{-2\phi}(\partial_s y)^2 + \frac{1}{4}\hat{\nu}^2 e^{2\phi} y^2$$

- Integrate them out:

$$S_{eff}(\phi) = \frac{\sqrt{\lambda}}{2\pi}\cosh(2\phi) - \frac{e^{2\phi}}{2\pi}\hat{\nu}^2 \ln \hat{\nu}^2$$

- The leading-logarithm free energy to all orders:

$$\mathcal{F} = \sqrt{1 + \frac{2}{\sqrt{\lambda}}\mathcal{F}_1}$$ with $$\mathcal{F}_1 = -\hat{\nu}^2 \ln \hat{\nu}^2$$

also $\ln \hat{\nu}$ dependence of the 1-loop free energy

Eliminate $\hat{\nu}$; restrict to leading logarithms

$$\mathrm{f}(\ell) \equiv \frac{\pi}{\sqrt{\lambda}} \frac{E-S}{\ln S} = \sqrt{1+\hat{\nu}^2}\ \left[\mathcal{F}(\hat{\nu}) - \hat{\nu}\frac{d\mathcal{F}(\hat{\nu})}{d\hat{\nu}}\right]$$

$$\ell \equiv \frac{\pi}{\sqrt{\lambda}} \frac{J}{\ln S} = \hat{\nu}\mathcal{F}(\hat{\nu}) - (1+\hat{\nu}^2)\frac{d\mathcal{F}(\hat{\nu})}{d\hat{\nu}}$$

Out[11]= $-\dfrac{\ell^2 \,\mathrm{Log}\left[\ell^2\right]}{\sqrt{1+\ell^2}}$

Out[12]= $\dfrac{2\,\ell^2\left(4+3\,\ell^2\right)\mathrm{Log}[\ell]^2}{\left(1+\ell^2\right)^{3/2}}$

Out[13]= $-\dfrac{4\,\ell^2\left(8+12\,\ell^2+5\,\ell^4\right)\mathrm{Log}[\ell]^3}{\left(1+\ell^2\right)^{5/2}}$

Out[14]= $\dfrac{2\,\ell^2\left(64+144\,\ell^2+120\,\ell^4+35\,\ell^6\right)\mathrm{Log}[\ell]^4}{\left(1+\ell^2\right)^{7/2}}$

Out[15]= $-\dfrac{4\,\ell^2\left(128+384\,\ell^2+480\,\ell^4+280\,\ell^6+63\,\ell^8\right)\mathrm{Log}[\ell]^5}{\left(1+\ell^2\right)^{9/2}}$

Out[16]= $\dfrac{1}{\left(1+\ell^2\right)^{11/2}}4\,\ell^2\left(512+1920\,\ell^2+3200\,\ell^4+2800\,\ell^6+1260\,\ell^8+231\,\ell^{10}\right)\mathrm{Log}[\ell]^6$

Eliminate ν; restrict to leading logarithms

$$\mathrm{f}(\ell,\sqrt{\lambda})\Big|_{\text{leading log}} = \sqrt{1+\frac{\ell^2}{1+\frac{2}{\sqrt{\lambda}}\ln\ell^2}}$$

- agreement with ABA

Gromov

- pole at $\ln \ell^2 = -\frac{1}{2}\sqrt{\lambda}$ is probably an artifact of LL truncation
- subleading logs are in principle accessible

Can this approach be used in finite volume?

A test: compare the two possible approaches at 1-loop at $J = 0$

- 1-loop universal scaling function:

$$f_1 = \frac{1}{\pi}\left[-3\ln 2 - \frac{5\pi^2}{12\ln^2 S} + \mathcal{O}(\ln^{-3} S)\right] \quad E_1 = -\frac{3\ln 2}{\pi}\ln S - \frac{5\pi}{12\ln S} + \mathcal{O}(\ln^{-2} S)$$

- Non-wrapping finite size correction from ABA — Schafer-Nameki, Zamaklar
- Exact 1-loop worldsheet calculation — Beccaria, Dunne, Forini, Pawellek, Tseytlin

Captured in partition function approach by simply discretizing space-like momentum on a worldsheet of length $L = 4\ln S$

- Scale of masses of fluctuations becomes relevant: use closed string scale

$$\mathcal{F}_1 = \int \frac{d\omega}{(2\pi)^2}\frac{2\pi}{L}\sum_n \ln \frac{(\omega^2 + (2\pi n/L)^2 + 2)^2 (\omega^2 + (2\pi n/L)^2)^5 (\omega^2 + (2\pi n/L)^2 + 4)}{(\omega^2 + (2\pi n/L)^2 + 1)^8}$$

Leading $\ln S$ (continuum) limit: $\quad \frac{2\pi}{L}\sum_n \mapsto \int dp$

$$-\frac{5\pi^2}{12\ln^2 S}$$

Equivalent with the evaluation of $\langle E - S \rangle_1$?

$$\mathcal{E} - \mathcal{S} = \int ds \ \left[\hat{\kappa} + 2\hat{\kappa}\tilde{\phi} + \hat{\kappa}(2\tilde{\phi}^2 + |\tilde{x}|^2)\right]$$

1-loop expectation value on worldsheet of length $L = 4 \ln S$

$$\langle E - S \rangle_1 = \mathcal{E}_1 + \mathcal{E}_2$$

$$\begin{aligned}\mathcal{E}_1 = -2 \int \frac{d\omega}{(2\pi)^2} \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \Bigg[& \boxed{4 \frac{\omega^2 - \left(\frac{2\pi n}{L}\right)^2}{\omega^2 + \left(\frac{2\pi n}{L}\right)^2}} - \frac{4 + 4\left(\frac{2\pi n}{L}\right)^2}{\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 2} \\ & + \boxed{2 \frac{(\omega^2 - \left(\frac{2\pi n}{L}\right)^2)(\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 2)}{(\omega^2 + \left(\frac{2\pi n}{L}\right)^2)(\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 4)}} + 4 \frac{4 + 4\left(\frac{2\pi n}{L}\right)^2}{\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 1} \Bigg]\end{aligned}$$

$$\mathcal{E}_2 = \int \frac{d\omega}{(2\pi)^2} \frac{2\pi}{L} \sum_{n=-\infty}^{\infty} 2 \left[\frac{4}{\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 2} + \frac{4}{\omega^2 + \left(\frac{2\pi n}{L}\right)^2 + 4} \right]$$

Extract leading finite size correction:

$$E_1 = -\frac{3 \ln 2}{\pi} \ln S - \frac{5\pi}{12 \ln S} + \mathcal{O}(\ln^{-2} S)$$

Same as from partition function

… yet curiously different from vanishing NLIE result Fioravanti, Grinza, Rossi

What about higher loops?

- Tempted to expect that the leading $1/\ln^2 S$ correction to the universal scaling function at 2-loops may be obtained by simply discretizing space-like momenta
- only integrals containing at least one massless field contribute

Summary

- ☮: ABA and worldsheet agree at 2-loops for high twist
- Consistent approach to higher-loop comp's i.t.o. 2d free energy
- AdS light-cone gauge leads to important technical simplifications
  - a curious feature is the existence of 2d tadpoles; treated perturbatively
- Discussed states with winding; (N)NLO ABA results to be determined
- Leading logarithms can be resummed to all orders; same as ABA
- Free energy approach appears to capture leading finite size correction at 1-loop; expect the same holds at 2-loops; only massless contrib's

A question:

Thermodynamic Bethe Ansatz:

$$e^{-LE_0(R)} = Z[L,R] = Z_{\mathrm{mirror}}[R,L] = e^{-LRf(R)}$$

With chemical potentials:

$$E_0(R) \mapsto E_0(R, h_i)$$

"Ground state" energy extracted as "grand potential" of the ws theory

However, we used a different relation: just eliminate h_i between

$$\frac{d\Sigma}{dh_i} = \frac{d\kappa}{dh_i}\langle E\rangle - \langle Q_i\rangle \qquad \Sigma = \kappa\langle E\rangle - \sum_i h_i\langle Q_i\rangle \qquad -\beta\,\Sigma = \ln Z$$

Is there a contradiction? Not visible directly at 1-loop, as there $\langle E\rangle$ is given in terms of the grand potential