# Quantum Integrable Spin Systems and Generalized Schur-Weyl duality

Stockholm - 2010

- Integrable systems & representation theor.

  oscill. Hydr.

- Schur-Weyl duality of $sl(2)$ & $S_N$ irreps on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$ (mult. free decomposion)

- Heisenberg $XXX_{1/2}$ spin chain (bound. cond.)

  $sl(2)$-invar. $H = \sum_k (\vec{\sigma}_k, \vec{\sigma}_{k+1}) = \sum_k (2P_{k\,k+1} - I) \in \mathbb{C}[S_N]$

  QISM ~ 3d algebra yangian $Y(sl(2))$

- Generalizations: anisotropy $XXZ_{1/2}$

  higher rank $sl(n)$, $so$...

  other irreps $XXX_{s=1,3/2,\dots}$

- $XXZ_{1/2}$ spin chain

  $sl_q(2)$-invar. & Hecke $H_N(q)$ algebra ($TL_N(q)$)

- $U_q(n)$-inv. spin chains $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$, — " — " —

  The same spectrum & structure of rep. ring

- $U_q(so(2n+1))$ spin chain

  e.g. $U_q(so(3))$ & $BMW_N(q, 1/q^2)$

# Schur-Weyl duality (Examples)

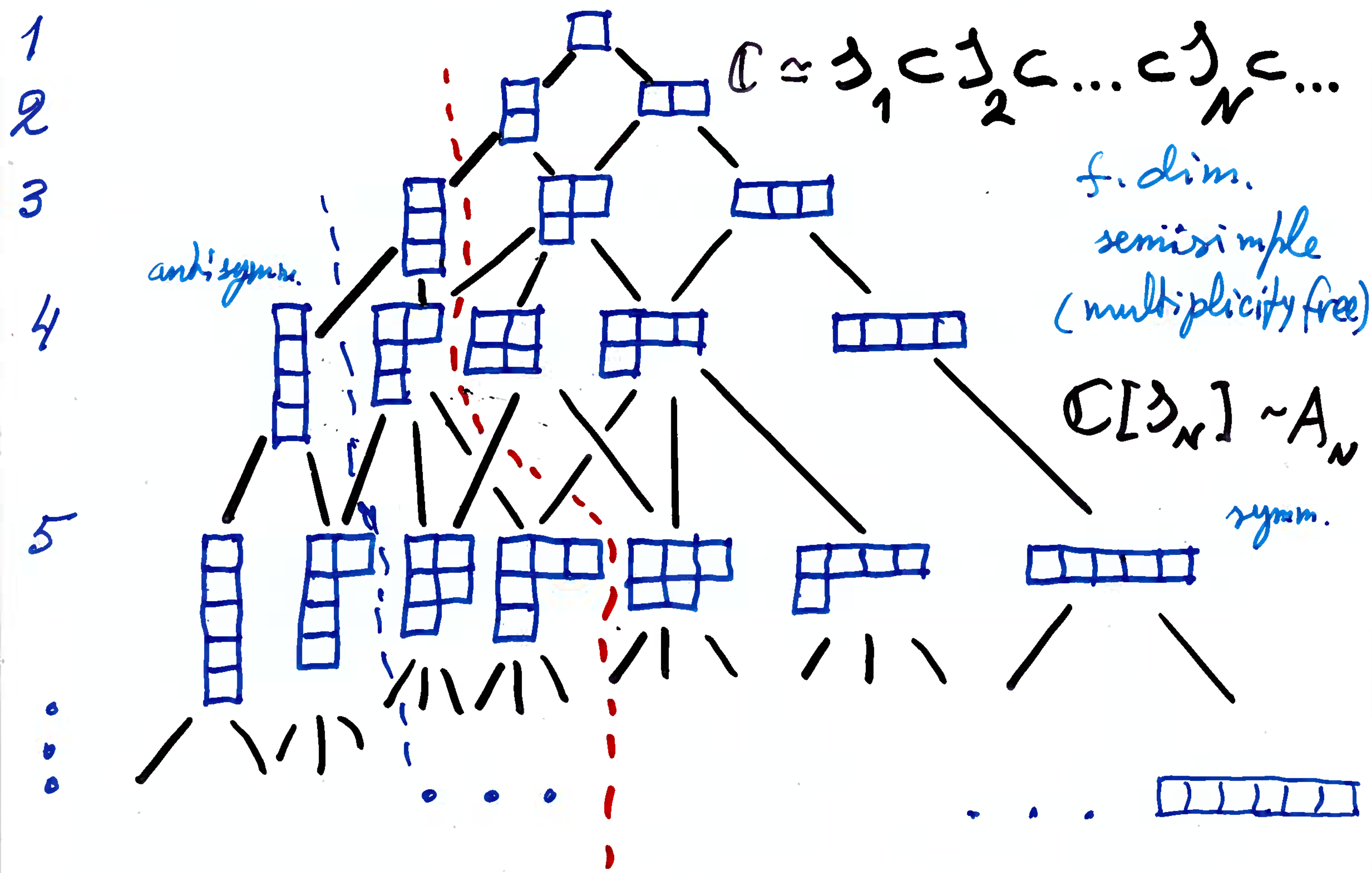
- $XXX_{1/2}$, $\mathcal{U}(sl(2))$ & $\mathbb{C}[S_N]$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$
  (symm. alg.) $H \simeq \sum_{k=1}^{N-1} P_{k\,k+1}$
- Sulh.m. $\mathcal{U}(sl(n))$ & $\mathbb{C}[S_N]$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$
- $XXZ_{1/2}$, $\mathcal{U}_q(sl(2))$ & $\mathcal{H}_N(q)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$
  $H \simeq \sum_k \check{R}_{k\,k+1}(q) \in (TL_N(q))$
- $XXZ_{TL}(n)$, $\mathcal{U}_q(n)$ & $TL_N(q)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$
  (param. $b, b^{-1} \in Mat(\mathbb{C}^n)$) $H \simeq \sum_k X_{k\,k+1}$ (rank 1)
- $XXZ_1$, $\mathcal{U}_q(o(3))$ & $W_N(q, 1/q^2)$, $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$
  BMW-alg.
- $q$-$A_2^{(2)}$ model, $\mathcal{U}_q(O(3))$ & $W_N(q, 1/q^2)$, $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$

Symmetry alg. $\mathcal{U}$ & central. alg. $\mathcal{J}_N$

$$\mathcal{H} = \bigotimes_1^N V = \bigoplus_\lambda V_\lambda \otimes W^\lambda$$

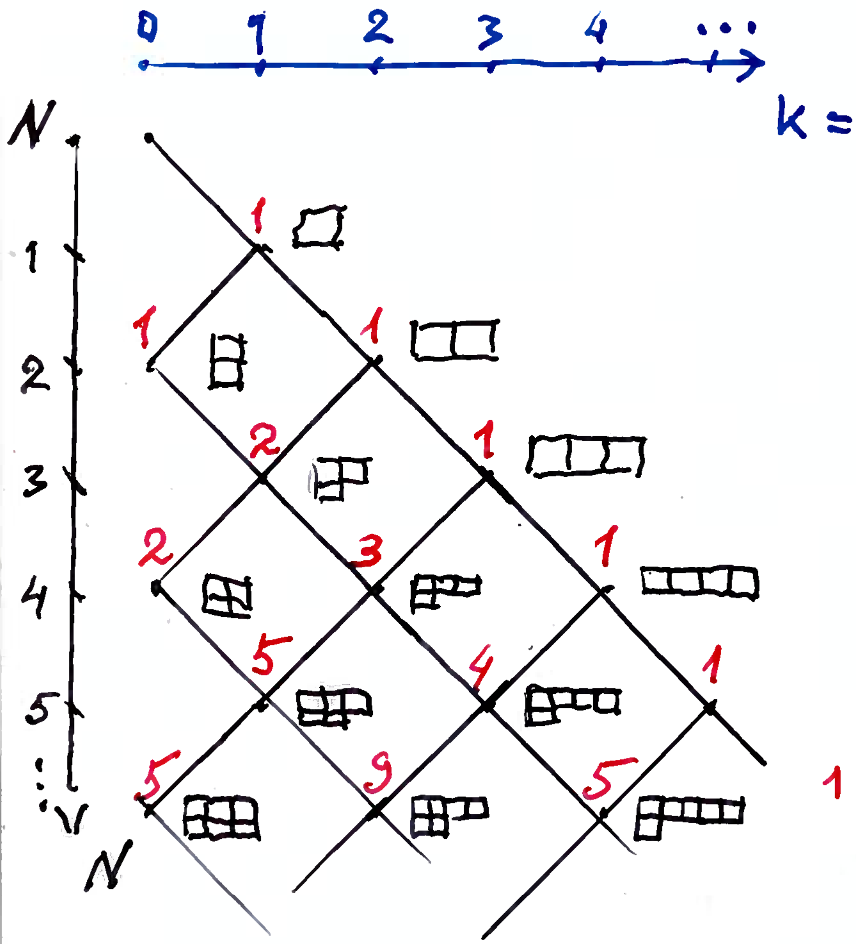
$$H = \sum_1^{N-1} h_k = \bigoplus_\lambda H|_{W^\lambda}$$

# Bratteli diagram of $S_N$




realization on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$ only two row Y-diagram

— " — " on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$ only three row Y-diagr.




$$\mathcal{H} = \bigotimes_{1}^{N} \mathbb{C}^2$$

Sch-W. duality

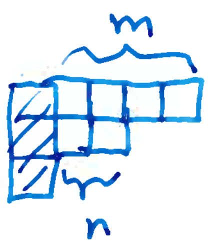
3 multiplicity

$\lambda \vdash N$, $(\lambda_1, \lambda_2 \,|\, \lambda_1+\lambda_2=N)$

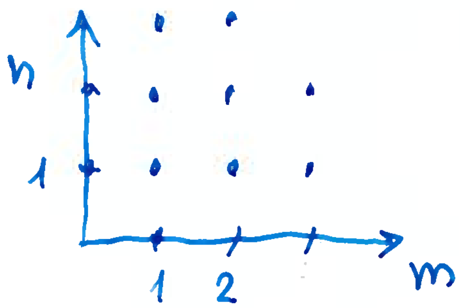
$C_N^n - C_N^{n-1}$ $n=\lambda_2$, b.c. Catalan

$$P_k \cdot P_1 = P_{k+1} + P_{k-1}$$

sl(3) case




$d_{m,n}$ $\square$ $d_{1,0}$, $\boxminus$ $d_{0,1}$




$$\begin{cases} d_{m,n}\cdot d_{1,0} = d_{m+1,n} + d_{m,n+1} + d_{m-1,n-1} \\ d_{m,n}\cdot d_{0,1} = \ldots\ldots \end{cases}$$

$$sl_2(3) \sim d_{1,0} = d_{0,1} = 3 \sim \mathbb{C}^3$$

$$d_{1,0} = d_{0,1} = n > 3\ ?$$

# Tensor category of $sl(2)$ finite dimensional representations

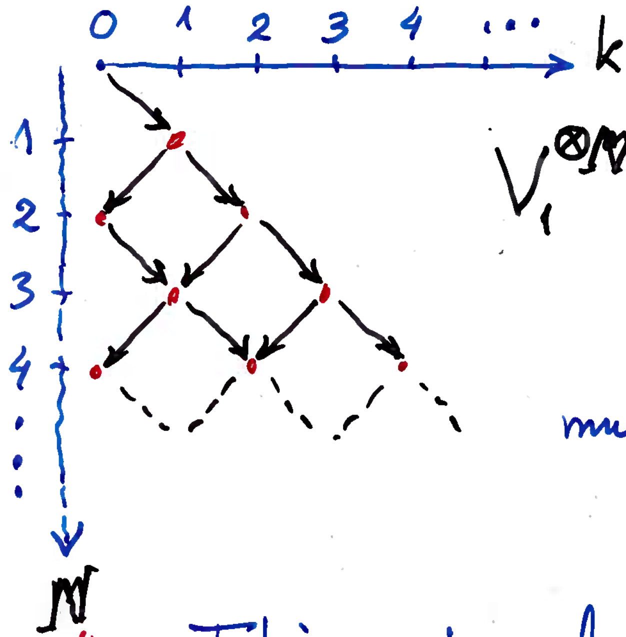


$sl(2)$, $\{S^z, X^+, X^-\}$, $s = 0, 1/2, 1, \ldots$ irreps $V_{2s}$

$2s := k \in \mathbb{Z}_{\geq 0}$

$\dim V_k = k+1$

$V_1 \otimes V_1 = V_0 \oplus V_2$




$$V_1^{\otimes N} = \bigoplus_{k=0,1}^{N} V_k \otimes \mathbb{C}^{\nu_k}$$

$m \sim e$, odd

$V_1 \otimes V_k = V_{k+1} \oplus V_{k-1}$

$$\boxed{V_\ell \otimes V_k = \bigoplus_{|\ell-k|}^{k+\ell} V_m}$$

(shift by 2)

multiplicity free C-G decomposition

This „ring of representations" corresponds to some quantum algebra $b_n$ if one starts from $V_0 \simeq \mathbb{C}^1$, $V_1 \simeq \mathbb{C}^n$ (e.g. $\dim V_2 = n^2 - 1$)

$$\boxed{n \cdot p_k = p_{k+1} + p_{k-1}}$$

$\dim V_k = p_k(n)$

Tchebyshev polynomials 2 kind

$p_{-1} = 0$, $p_0 = 1$

$V_k$ as corepresentations of dual Hopf algebra and FRT-formalism $R_{12} T_1 T_2 = T_2 T_1 R_{12}$

Gurevich $\geq 1990$

Dubois-Violette, Bichon, Etingof, Ostrik ...

q-spaces...

# Spin systems and Temperley-Lieb algebra

BrGr $\quad \check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}$ $\quad$ notation

$\check{R} = qI + X \,, \quad X^2 = -(q+\frac{1}{q})X \qquad \check{R}^2 = \omega(q)\check{R} + I$

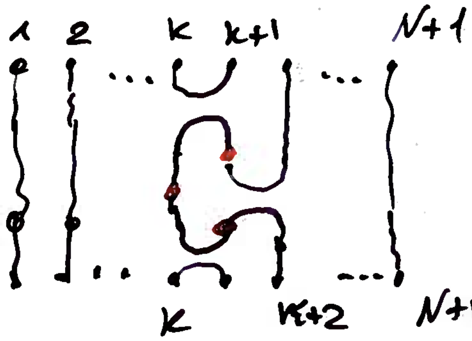
(invertible)

$\omega(q) = q - \frac{1}{q} \,, \qquad \nu(q) = q + \frac{1}{q}$

TL-algebra is Hecke algebra with condition (quotient)

$\check{R}_{j\,j+1} = \text{rep}(\sigma_j) \,, \quad \check{R}_j \to X_j$ s.t.

$X_k X_{k\pm1} X_k = X_k$




$H_N(q)$

$N!$

Catalan $C_N$

or

$\check{R}_{12}(\frac{1}{q})\check{R}_{23}(\frac{1}{q^2})\check{R}_{12}(\frac{1}{q}) = 0$

higher „antisymmetrizer"

$\check{R}(u) = u\check{R} - \frac{1}{u}\check{R}^{-1} \rightsquigarrow \mathcal{P}\check{R}(u) = R(u)$ YBE

(Baxterization) $\quad \check{R}(1) \simeq I$ (regularity property)

$R(u)$ gives rise to a particular spin system

$L$-operator:

$$L(u) = R_{an}(u) = (uq - \tfrac{1}{uq})\mathcal{P} + (u - \tfrac{1}{u})\mathcal{P}X$$

$\underset{\cap}{}$

$\mathrm{Mat}(V \otimes V)$ $\quad$ with hamiltonian $H = \sum_{k=1}^{N} X_k$ (+ b.c.)

$\simeq e_k$

auxiliary & quantum spaces

# Local realization of TL-algebra

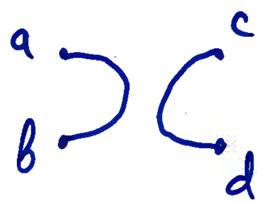
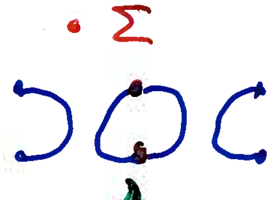
or $\otimes$ reps...

$$\mathcal{H}_N = \bigotimes_{k=1}^{N} \mathbb{C}^n \,, \quad \text{(quantum) space} \qquad M(\mathbb{C}^n \otimes \mathbb{C}^n)$$

generators of $TL_N$: $\check{R}_{k\,k+1} := R_k = q I + X_k$ dim

$$X = b \otimes \bar{b} \,, \quad X_{ab;\,cd} = b_{ab}\, \bar{b}_{cd} \quad \sim |b\rangle\langle\bar{b}|$$

$b = \{b_{ab}\}_1^n$ is an invertible matrix, $\underline{\bar{b} := b^{-1}}$

$X_{ab,cd}$  $\qquad X^2 \sim$  $= -\gamma(q) X$

$\cdot\,\Sigma$

$$b_{kl}\, \bar{b}_{ke} = -(q + \tfrac{1}{q})$$

$$\underline{\check{R}_{a_1 a_2}(u/w)\, L_{a_1 j}(u)\, L_{a_2 j}(w)} = \underline{L_{a_1 j}(w)\, L_{a_2 j}(u)\, \check{R}_{a_1 a_2}(u/w)}$$

$$\check{R}_{a_1 a_2}(u) = \omega(uq) I + \omega(u) X_{a_1 a_2} = \omega(uq)(I - P) - \omega\!\left(\tfrac{u}{q}\right) P \qquad \mathbb{C}^n_j$$

$$P^2 = P \,, \quad X = -\gamma(q) P \,.$$

**Points of degeneracy**: $u = q^{\pm 1}$, $\check{R} \to \begin{cases} I - P \\ P \end{cases}$

$$T(u) = L_{aN}(u) \dots L_{a1}(u)$$

$$t(u) = \mathrm{tr}_{(a)} \prod_{\overleftarrow{1}}^{N} L_{a_j}(u)$$

# Birman-Wenzl-Murakami algebra $W_N(q,\nu)$

generators $1,\ \sigma_i,\ \sigma_i^{-1},\ e_i\ ;\ i=1,2,\dots,N-1$

relations $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\qquad \sigma_i\sigma_j=\sigma_j\sigma_i\ ,\ |i-j|>1$

$$e_i\sigma_i=\sigma_i e_i=\nu\, e_i\ ,$$

$$e_i\sigma_{i-1}^{\pm1}e_i=\nu^{\mp1}e_i\ ,$$

$$\sigma_i-\sigma_i^{-1}=\omega(q)(1-e_i)\ ,\quad \omega(q)=q-1/q$$

$\dim W_N(q,\nu)=(2N-1)!!\qquad (1,3,15,\dots)$

consequences: $e_i^2=\mu e_i,\quad \mu=\dfrac{\omega-\nu+1/\nu}{\omega(q)},$

$$(\sigma_i-q)(\sigma_i+\tfrac{1}{q})(\sigma_i-\nu)=0$$

inclusion: $W_M\subset W_N,\ M<N$

Yang-Baxterization $\leadsto$ two(!) spectral parameter depend. elements $\sigma_i(u)$

$$\sigma_i^{(\pm)}(u)=\frac{1}{\omega}\left(u^{-1}\sigma_i-u\sigma_i^{-1}\right)+\frac{\nu\pm q^{\pm1}}{u\nu\pm q^{\pm1}u^{-1}}\,e_i$$

$$\sigma_i(u)\sigma_{i+1}(uw)\sigma_i(w)=\sigma_{i+1}(w)\sigma_i(uw)\sigma_{i+1}(u)$$

Br Gr of YBE

$\dim W_N(q,\nu)=(2N-1)!!$

$\dim\mathbb{C}[S_N]=\dim H_N(q)=N!$

$\dim TL_N(q)=\dfrac{2N!}{(N+1)!\,N!}$

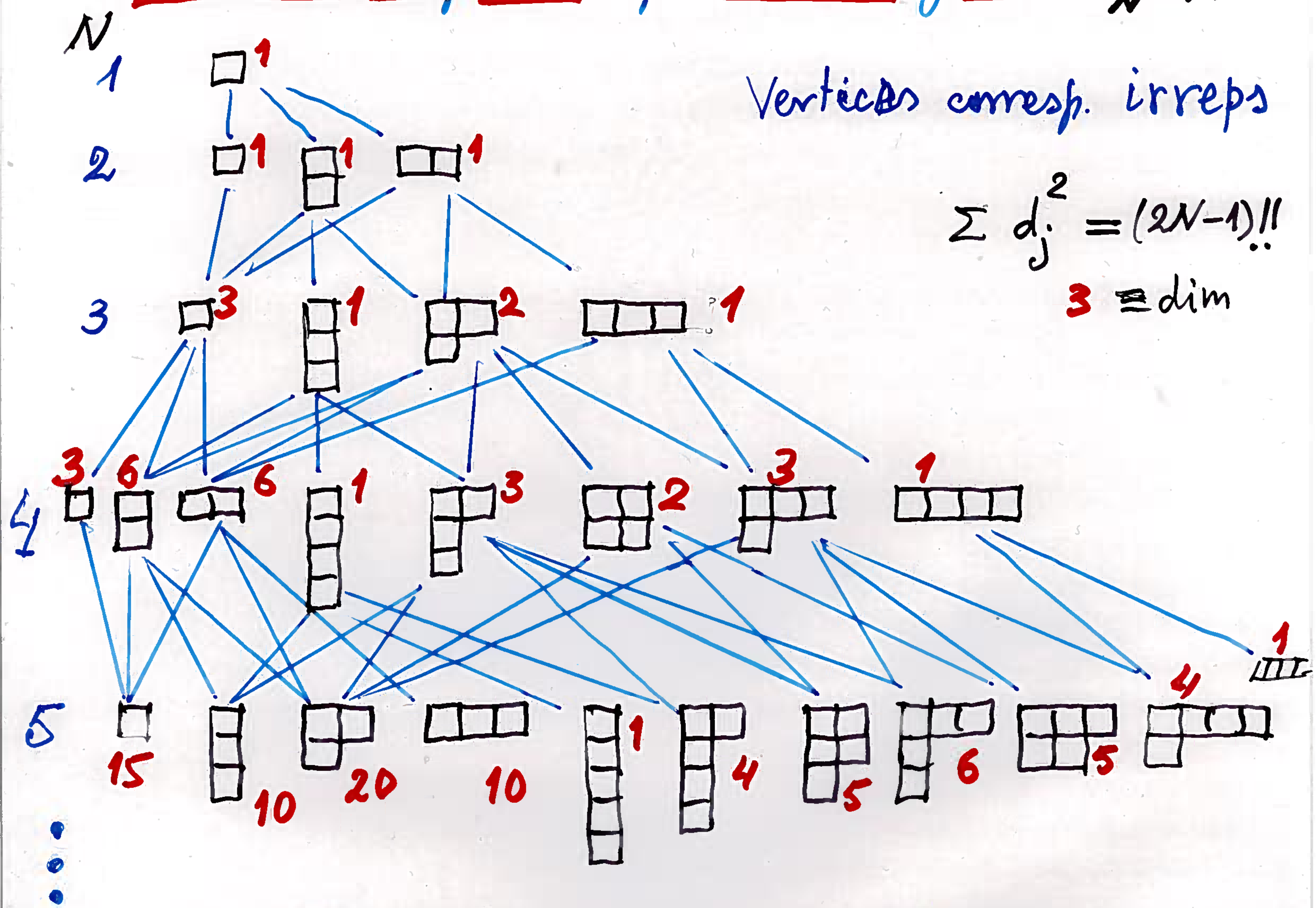
Catalan #

# Bratteli diagram of BMW-algebra $W_N(q,\nu)$

Vertices corresp. irreps

$$\sum d_j^2 = (2N-1)!!$$

3 ≡ dim




# Irreducible representations of $U_q(so(3))$

$$\mathcal{H} = \bigotimes_1^N \mathbb{C}^3 = \sum_0^N V_k \otimes \mathbb{C}^{m_k}$$

spin $k$, multiplicity $m_k$

$$\dim V_k = 2k+1$$

$$V_k \otimes V_1 = V_{k-1} \oplus V_k \oplus V_{k+1}$$

$$\# = N+1$$