Q-Operator Demystified

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Zur Theorie der Metalle.

I. Eigenwerte und Eigenfunktionen der linearen Atomkette.

Von **H. Bethe** in Rom.

(Eingegangen am 17. Juni 1931.)

Es wird eine Methode angegeben, um die Eigenfunktionen nullter und Eigenwerte erster Näherung (im Sinne des Approximationsverfahrens von London und Heitler) für ein „eindimensionales Metall" zu berechnen, bestehend aus einer linearen Kette von sehr vielen Atomen, von denen jedes außer abgeschlossenen Schalen ein *s*-Elektron mit Spin besitzt. Neben den „Spinwellen" von Bloch treten Eigenfunktionen auf, bei denen die nach einer Richtung weisenden Spins möglichst an dicht benachbarten Atomen zu sitzen suchen; diese dürften für die Theorie des Ferromagnetismus von Wichtigkeit sein.

§ 1. In der Theorie der Metalle hat man sich bis vor einiger Zeit darauf beschränkt, die Bewegung der einzelnen Leitungselektronen im Potentialfeld der Metallatome zu untersuchen (Sommerfeld, Bloch). Von der Wechselwirkung der Elektronen untereinander wurde abgesehen, wenigstens soweit sie nicht summarisch in dem auf die Elektronen wirkenden Potential untergebracht werden kann. Dieses Verfahren war für die Probleme der metallischen Leitfähigkeit (mit Ausnahme der Supraleitung) sehr fruchtbar, ließ aber ein tieferes Eindringen etwa in das Problem des Ferromagnetismus nicht zu[1]) und machte z. B. die Berechnung der Kohäsionskräfte im Metall zu einem ganz hoffnungslosen Unternehmen: Die für die Störungsenergie erster Näherung maßgebenden Austauschkräfte zwischen den Leitungselektronen sind von gleicher Größenordnung wie die Nullpunktsenergie des

Bethes Ansatz of 1931

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{K}\frac{u_k-u_j+i}{u_k-u_j-i},$$

$$-\,-\,-\,-\,-\quad --\quad -\,-\,-\,-\,-\,-\,-\,-$$

$$E = \sum_{j=1}^{K}\frac{2}{u_j^2+\frac{1}{4}}.$$

The Asymptotic All-Loop AdS/CFT Bethe Equations

[Beisert, Staudacher '05,'06]

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}},$$

$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right) = \frac{1}{g^2}\sum_{j=1}^{K_4}\left(\sqrt{1+16\,g^2\,\sin\frac{p_j}{2}}-1\right), \qquad \Delta=\Delta_0+g^2\,E(g)\,, \qquad K_4=K\,.$$

$$1 = \prod_{j=1}^{K_4}\left(\frac{x^+_{4,j}}{x^-_{4,j}}\right) = \prod_{j=1}^{K_4} e^{ip_j}\,, \qquad u_k = x_k+\frac{g^2}{x_k}\,, \qquad u_k\pm\frac{i}{2}=x^\pm_k+\frac{g^2}{x^\pm_k}\,.$$

Gauge Theory Meets String Theory

The asymptotic Bethe ansatz yields an integral equation for an interpolating scaling function $f(g)$ at arbitrary values of g. [Beisert, Eden, Staudacher '06]

At weak coupling this equation was (numerically) tested up to four loop order in gauge theory: [Bern, Czakon, Dixon, Kosower, Smirnov, '06; Cachazo, Spradlin, Volovich '06].

$$f(g) = 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots .$$

A five-loop test is under way [Bourjaily, Henn, Spradlin, work in progress].

At strong coupling the scaling function agrees with string theory to the three known orders [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02], [Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] as was analytically shown by [Basso, Korchemsky, Kotański '07]

$$f(g) = 4\,g - \frac{3\log 2}{\pi} - \frac{\mathrm{K}}{4\pi^2}\,\frac{1}{g} - \ldots$$

$\longrightarrow$ The AdS/CFT correspondence appears to be exactly true !

Planar AdS/CFT Appears to be Exactly True

- Therefore, independently of all attempts to use string theory as a "theory of everything", it has thus been established that string theory can be a "theory of something": A $4D$ Yang-Mills theory.

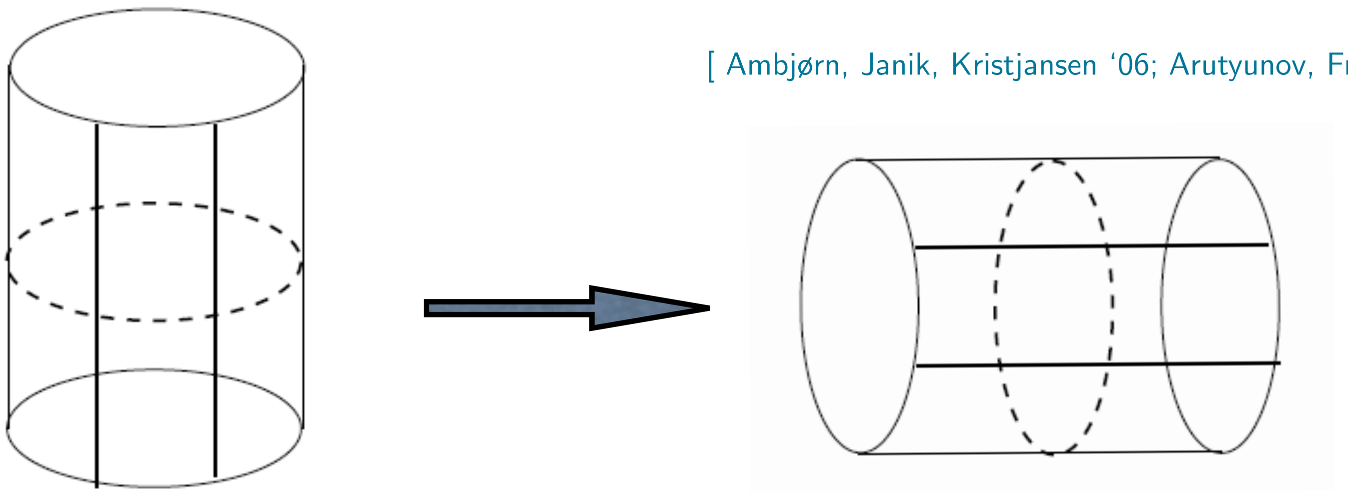
- Turning this around, it has therefore also been established that theories with no apparent trace of gravity (i.e. Yang-Mills theories) can in a hidden way contain quantum gravity.

- Planar Feynman diagrams of a $4D$ Yang-Mills theory can really be summed to all orders, and analytically continued to strong coupling.

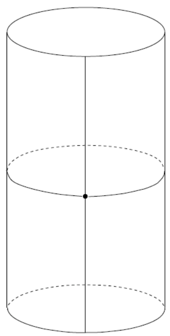
Lüscher Corrections and Thermodynamic Bethe Ansatz

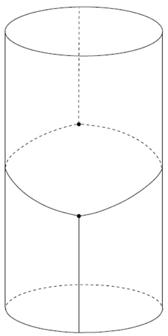
[Lüscher '86; A.B. Zamolodchikov '90]

[Ambjørn, Janik, Kristjansen '06; Arutyunov, Frolov, '07, '08; Bajnok, Janik '08]



In the TBA, one "turns around" the world sheet cylinder of the string σ-model, and considers scattering in the cross channel. This takes into account virtual field-theoretic corrections.



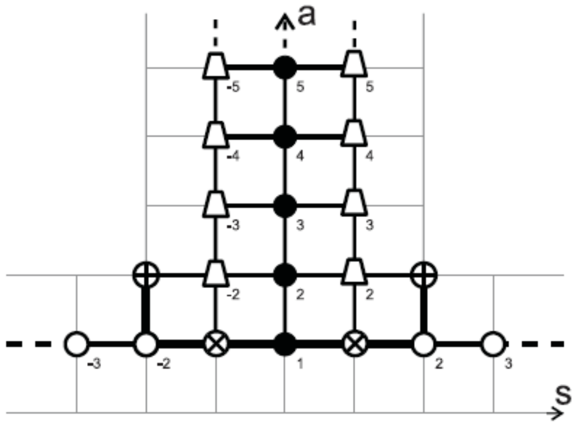


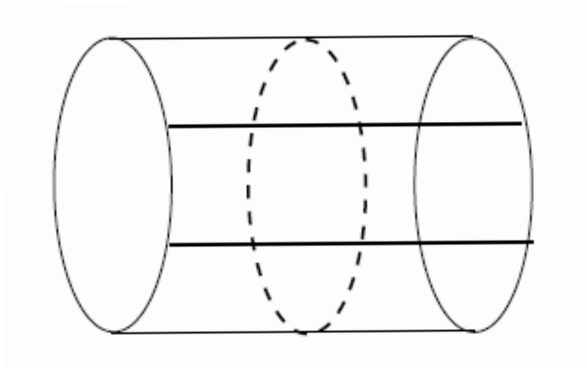
All-loop TBA equations, Y-System

Ground State TBA: [Bombardelli, Fioravanti, Tateo '09, Arutyunov, Frolov '09, Gromov, Kazakov, Kozak, Vieira '09]

Bold claim: Exact spectrum of planar $\mathcal{N} = 4.$ [Gromov, Kazakov, Kozak, Vieira '09]

Infinite system of integral equations for "Y-functions" living on a lattice:

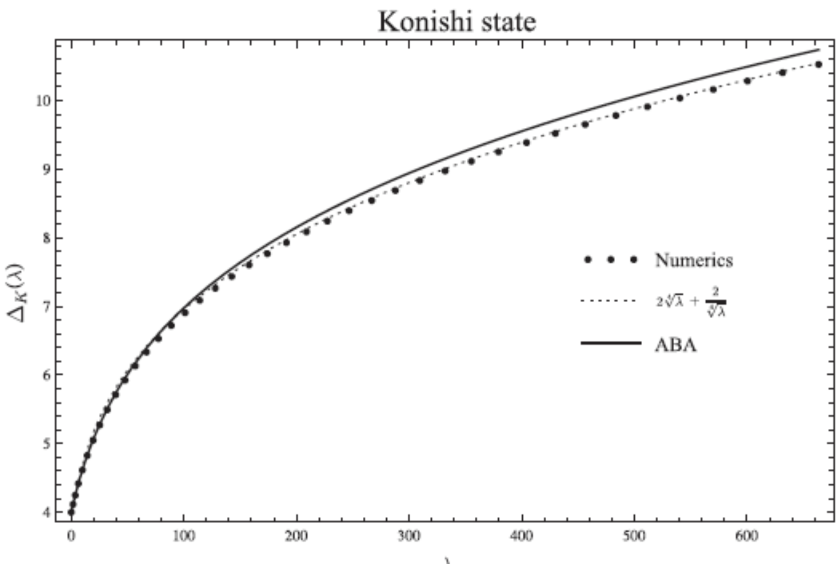




Supposed to take into account infinitely many virtual corrections!

Is the Y-system correct? If so, why? In integrable models, it frequently much easier to guess the exact solution than to prove and understand it!

Predictions from the All-Loop TBA and the Y-System



[Gromov, Kazakov, Vieira '09]

- A numerical plot for Konishi was obtained. Very recently largely confirmed. [Frolov '10] Structure of the expansion not quite clear though.

- It fits very well weak coupling. Analytical results up to five loops.

- It predicts the strong coupling behavior $2\,\lambda^{\frac{1}{4}} + 2\,\lambda^{-\frac{1}{4}} + \ldots$. Currently there is a discrepancy with string theory: $2\,\lambda^{\frac{1}{4}} + 1\,\lambda^{-\frac{1}{4}} + \ldots$. [Roiban, Tseytlin '09] Analytical approach as in the BES case direly needed.

Weak Coupling Challenges for AdS/CFT Integrability

- It is very important to explore the predictions of integrability for short operators in $\mathcal{N}=4$ gauge theory to much higher orders.

- Analytic result for Konishi up to 5 loops: [Bajnok, Hegedus, Janik, Łukowski '09]

$$\gamma = 12\,g^2 - 48\,g^4 + 336\,g^6 - 96\,\left(26 - 6\,\zeta(3) + 15\,\zeta(5)\right)\,g^8 + +96\,\Big(158 + 72\,\zeta(3) - 54\,\zeta(3)^2 - 90\,\zeta(5) + 315\,\zeta(7)\Big) + \ldots$$

- The dots are NOT boring!

- Results from mathematical Feynman graph theory and algebraic geometry indicate that at some order multiple zeta functions are expected. 6 loops in ϕ^4-theory. [Bloch, Broadhurst, Brown, Kreimer] 8 loops? Double-Wrapping?

- A non-Tate graph appears in planar ϕ^4-theory at 9 loops. [Brown, Schnetz '10] Leads to functions not expressible as (multi)-zeta functions!

Solvable Structures in the (Planar) AdS/CFT System

- Spectral Problem
- High Energy Scattering (BFKL)
- Gluon Amplitudes
- Wilson Loops

These are all related! (E.g. recall the universal scaling function.)

- Recently much progress with $\mathcal{N} = 4$ gluon amplitudes.
- Exciting hints at Yangian structures in planar $\mathcal{N} = 4$ amplitudes.

[Drummond, Henn, Plefka '09; Drummond, Ferro '10; Beisert, Henn, McLoughlin, Plefka '10]

What are we Solving?

- However, it remains utterly unclear what system we are "diagonalizing".

- Why is planar AdS/CFT integrable? What is its full quantum Yangian?

$\Rightarrow$ Back to the drawing board!

Q-Operators

- One of the most powerful methods to solve quantum integrable models involves Baxter's Q-operator.
- The eigenvalues of the Q-operator appear as boundary values of the Y-system.
- The XXX spin chain appears as the one-loop approximation of the $\mathfrak{su}(2)$ sector of $\mathcal{N} = 4$ gauge theory.
- In contrast to the $\mathfrak{su}(1,1) \simeq \mathfrak{sl}(2)$ sector [Derkachov, Korchemsky, Manashov] the Q-operator was not known for the compact case.

Bethes Ansatz of 1931, Revisited

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i},$$

$$E = \sum_{j=1}^{M} \frac{2}{u_j^2 + \frac{1}{4}}.$$

As we shall see, there are still new things to be discovered here. In fact ...

The Q-operator of the XXX chain has never been properly constructed.

"Wick Rotation"

I will use throughout

$$z := -i\,u\,.$$

This will make most formulas look much more beautiful. Bethe equations:

$$\left(\frac{z_k + \frac{1}{2}}{z_k - \frac{1}{2}}\right)^L \quad = \quad \prod_{\substack{j=1 \\ j \neq k}}^{M} \frac{z_k - z_j + 1}{z_k - z_j - 1}\,.$$

It is well known, that these can be turned into a TQ-equation.

TQ-Equation for Eigenvalues

The Bethe equations are equivalent to the solutions of the equation

$$T(z)\,Q(z) = \left(z+\tfrac{1}{2}\right)^L Q(z-1) + \left(z-\tfrac{1}{2}\right)^L Q(z+1),$$

if we make the ansatz

$$Q(z) \sim \prod_{k=1}^{M} (z - z_k).$$

What about the second solution of this equation? [Pronko, Stroganov '98]

$$P(z) \sim \prod_{k=1}^{L-M+1} (z - \tilde{z}_k).$$

$T(z)$ is the eigenvalue of the transfer matrix. What about $Q(z)$, $P(z)$?

The Transfer Matrix as a Trace

The transfer matrix is constructed from the Lax operator

$$\mathcal{L}_l(z) = \begin{pmatrix} z + \mathcal{S}_l^3 & \mathcal{S}_l^- \\ \mathcal{S}_l^+ & z - \mathcal{S}_l^3 \end{pmatrix},$$

by building a monodromy matrix and taking a trace

$$\mathbf{T}(z) = \mathrm{Tr}\; \mathcal{L}_L(z) \cdot \mathcal{L}_{L-1}(z) \cdot \ldots \cdot \mathcal{L}_2(z) \cdot \mathcal{L}_1(z)\,.$$

The transfer matrix generates the Hamiltonian of the Heisenberg chain:

$$\mathbf{H} = 2\,L - 2\,\frac{d}{dz} \log \mathbf{T}(z) \Big|_{z=\frac{1}{2}} = 4 \sum_{l=1}^{L} \left(\frac{1}{4} - \vec{\mathcal{S}}_l \cdot \vec{\mathcal{S}}_{l+1} \right).$$

TQ-Equation for Operators ?

Since $T(z)$ is the eigenvalue of an operator $\mathbf{T}(z)$, we should have

$$\mathbf{T}(z)\,\mathbf{Q}(z) = \left(z+\tfrac{1}{2}\right)^L \mathbf{Q}(z-1) + \left(z-\tfrac{1}{2}\right)^L \mathbf{Q}(z+1)\,.$$

- What is the operator $\mathbf{Q}(z)$?
- Can it be constructed as the trace of a suitable monodromy?
- If so, what is the Lax operator generating it?

In principle the answer should be in Baxter's work on the XYZ chain. In practise, it remained obscure until now.

Important hints in earlier work by [Bazhanov, Lukyanov, Zamolodchikov] and [Boos, Jimbo, Miwa, Smirnov, Takeyama].

L-Operators as Solutions of the Yang-Baxter Equation

Consider the Yang-Baxter equation:

$$\mathbb{R}(x-y)\left(\mathbb{L}_{\mathcal{A}}(x)\otimes 1\right)\left(1\otimes\mathbb{L}_{\mathcal{A}}(y)\right)=\left(1\otimes\mathbb{L}_{\mathcal{A}}(y)\right)\left(\mathbb{L}_{\mathcal{A}}(x)\otimes 1\right)\mathbb{R}(x-y),$$

where $\mathbb{R}(z)$ is the rational 4×4 R-matrix

$$\mathbb{R}(z):\qquad \mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}^2\otimes\mathbb{C}^2,\qquad \mathbb{R}(z)=z+\mathbb{P}\,,$$

and $\mathbb{P}$ is the permutation operator. The L-operator $\mathbb{L}_{\mathcal{A}}(z)$ is a 2×2 matrix, acting in the quantum space of a single spin-$\frac{1}{2}$

$$\mathbb{L}_{\mathcal{A}}(z)=\begin{pmatrix}\mathbb{L}_{11}(z) & \mathbb{L}_{12}(z)\\ \mathbb{L}_{21}(z) & \mathbb{L}_{22}(z)\end{pmatrix}.$$

whereas its matrix elements act in an auxiliary space $\mathcal{A}$.

Type I Solution of the Yang-Baxter Equation

Let us make the ansatz

$$\mathbb{L}_{\mathcal{A}}(z) = \begin{pmatrix} z + A & B \\ C & z + D \end{pmatrix}.$$

From the YBE, the algebra of the A, B, C, D leads to $\mathfrak{sl}(2)$.

To be precise,

$$\mathbb{L}_{\mathcal{A}}(z) = \mathcal{L}(z) = \begin{pmatrix} z + \mathfrak{J}^3 & \mathfrak{J}^- \\ \mathfrak{J}^+ & z - \mathfrak{J}^3 \end{pmatrix} = z\,\mathbb{I} + 2\sum_{k=1}^{3} \mathcal{S}^k\,\mathfrak{J}^k.$$

$\mathfrak{J}^\pm = \mathfrak{J}^1 \pm i\mathfrak{J}^2$ and $\mathfrak{J}^3$ generate $\mathfrak{sl}(2)$ and act on the auxiliary space $\mathcal{A}$:

$$[\mathfrak{J}^3, \mathfrak{J}^\pm] = \pm\mathfrak{J}^\pm\,, \qquad\qquad [\mathfrak{J}^+, \mathfrak{J}^-] = 2\,\mathfrak{J}^3\,.$$

Type II Solution of the Yang-Baxter Equation

Let us now make a different ansatz:

$$\mathbb{L}_{\mathcal{A}}(z) = \begin{pmatrix} z + A & B \\ C & D \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & B \\ C & z + D \end{pmatrix}.$$

Now the algebra of the A, B, C, D leads to the harmonic oscillator algebra!

To be precise, one finds that $\mathbb{L}_{\mathcal{A}}(z)$ is either

$$L_1(z) = \begin{pmatrix} z - \mathbf{a}_{12}^{+}\,\mathbf{a}_{12}^{-} - \frac{1}{2} & \mathbf{a}_{12}^{+} \\ -\mathbf{a}_{12}^{-} & 1 \end{pmatrix} \text{or } L_2(z) = \begin{pmatrix} 1 & \mathbf{a}_{21}^{+} \\ \mathbf{a}_{21}^{-} & z + \mathbf{a}_{21}^{+}\,\mathbf{a}_{21}^{-} + \frac{1}{2} \end{pmatrix}$$

with two sets of mutually commuting oscillators acting on $\mathcal{A}$:

$$[\mathbf{a}_{12}^{-}, \mathbf{a}_{12}^{+}] = 1, \qquad [\mathbf{a}_{21}^{-}, \mathbf{a}_{21}^{+}] = 1\,.$$

First Factorization

The type I and type II solutions of the YBE must be related. The reason is that $L_2(z_2)\,L_1(z_1)$ is also a solution of, after some redefinitions, type I! The precise statement is that with

$$z = \frac{z_1 + z_2}{2} \quad \text{and} \quad j + \tfrac{1}{2} = \frac{z_1 - z_2}{2}\,,$$

one finds

$$L_2(z_2)\cdot L_1(z_1) = e^{\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}} \begin{pmatrix} 1 & 0 \\ \mathbf{a}_{21}^{-} & 1 \end{pmatrix} \begin{pmatrix} z + \mathfrak{J}_j^3 & \mathfrak{J}_j^- \\ \mathfrak{J}_j^+ & z - \mathfrak{J}_j^3 \end{pmatrix} e^{-\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}}\,.$$

where the $\mathfrak{sl}(2)$ generators are realized in Holstein-Primakoff form as

$$\mathfrak{J}_j^- = \mathbf{a}_{12}^+\,, \qquad \mathfrak{J}_j^+ = \left(2\,j - \mathbf{a}_{12}^+\,\mathbf{a}_{12}^-\right)\mathbf{a}_{12}^-\,, \quad \mathfrak{J}_j^3 = j - \mathbf{a}_{12}^+\,\mathbf{a}_{12}^-\,.$$

Second Factorization

Likewise, there is a similar expression for the opposite order

$$L_1(z_1)\cdot L_2(z_2) \;=\; \begin{pmatrix} z_1 - \mathbf{a}_{12}^{+}\,\mathbf{a}_{12}^{-} - \frac{1}{2} & \mathbf{a}_{12}^{+} \\ -\mathbf{a}_{12}^{-} & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathbf{a}_{21}^{+} \\ \mathbf{a}_{21}^{-} & z_2 + \mathbf{a}_{21}^{+}\,\mathbf{a}_{21}^{-} + \frac{1}{2} \end{pmatrix}$$

$$= \; e^{\mathbf{a}_{12}^{+}\mathbf{a}_{21}^{-}}\begin{pmatrix} z + \mathfrak{J}_j^{3} & \mathfrak{J}_j^{-} \\ \mathfrak{J}_j^{+} & z - \mathfrak{J}_j \end{pmatrix}\begin{pmatrix} 1 & \mathbf{a}_{21}^{+} \\ 0 & 1 \end{pmatrix} e^{-\mathbf{a}_{12}^{+}\mathbf{a}_{21}^{-}}\,,$$

where the $\mathfrak{sl}(2)$ algebra is now realized with $\mathbf{a}_{12}^{\pm}$

$$\mathfrak{J}_j^{-} = \mathbf{a}_{12}^{+}\left(2\,j - \mathbf{a}_{12}^{+}\,\mathbf{a}_{12}^{-}\right)\;,\quad \mathfrak{J}_j^{+} = \mathbf{a}_{12}^{-}\,,\quad \mathfrak{J}_j^{3} = j - \mathbf{a}_{12}^{+}\,\mathbf{a}_{12}^{-}\,.$$

In a more compact form

$$e^{\mathbf{a}_{12}^{+}\mathbf{a}_{21}^{-}}\,\mathcal{L}_{12}^{\mathrm{j}}(z)\cdot\mathcal{B}_{21}^{+}\,e^{-\mathbf{a}_{12}^{+}\mathbf{a}_{21}^{-}} = L_1(z_1)\cdot L_2(z_2)\,.$$

Taking Traces, I

Let us built monodromies from either side of our factorization formula:

$$e^{\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}}\,\mathcal{B}_{21}^{-}\;\cdot\mathcal{L}_{12}^{\mathrm{j}}(z)\,e^{-\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}}=L_2(z_2)\cdot L_1(z_1)\,.$$

Take the tensor product $\otimes$ of all 2×2 matrices acting on the L local quantum spaces. Here $\cdot$ denotes 2×2 matrix multiplication.

$$e^{\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}}\,\mathcal{B}_{L,21}^{-}\cdot\mathcal{L}_{L,12}^{\mathrm{j}}(z)\otimes\,\cdots\,\otimes\mathcal{B}_{1,21}^{-}\cdot\mathcal{L}_{1,12}^{\mathrm{j}}(z)\,e^{-\mathbf{a}_{21}^{+}\mathbf{a}_{12}^{-}}=$$
$$=L_{L,2}(z_2)\cdot L_{L,1}(z_1)\otimes\,\cdots\,\otimes L_{1,2}(z_2)\cdot L_{1,1}(z_1).$$

Take the trace over the $\mathcal{F}_{21}\times\mathcal{F}_{12}$ double-oscillator space:

$$\mathrm{Tr}_{21}\left(\mathcal{B}_{L,21}^{-}\otimes\cdots\otimes\mathcal{B}_{1,21}^{-}\right)\,\mathrm{Tr}_{12}\left(\mathcal{L}_{L,12}^{\mathrm{j}}(z)\otimes\cdots\otimes\mathcal{L}_{1,12}^{\mathrm{j}}(z)\right)=$$
$$=\mathrm{Tr}_{21}\left(L_{L,2}(z_2)\otimes\cdots\otimes L_{1,2}(z_2)\right)\,\mathrm{Tr}_{12}\left(L_{L,1}(z_1)\otimes\cdots\otimes L_{1,1}(z_1)\right).$$

Taking Traces, II

The traces are actually infinite, so we put a regulator inside the trace:

$$e^{-i\phi(\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}+\mathbf{a}_{12}^{+}\mathbf{a}_{12}^{-})}\,.$$

The insertion of this operator does not spoil the cyclicity argument!

$$\mathrm{Tr}_{21}\left(e^{-i\phi\,\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}}\mathcal{B}_{L,21}^{-}\otimes\cdots\otimes\mathcal{B}_{1,21}^{-}\right) \\ \times\,\mathrm{Tr}_{12}\left(e^{-i\phi\,\mathbf{a}_{12}^{+}\mathbf{a}_{12}^{-}}\mathcal{L}_{L,12}^{\mathrm{j}}(z)\otimes\cdots\otimes\mathcal{L}_{1,12}^{\mathrm{j}}(z)\right)$$

$$=\mathrm{Tr}_{21}\left(e^{-i\phi\,\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}}L_{L,2}(z_2)\otimes\cdots\otimes L_{1,2}(z_2)\right) \\ \times\,\mathrm{Tr}_{12}\left(e^{-i\phi\,\mathbf{a}_{12}^{+}\mathbf{a}_{12}^{-}}L_{L,1}(z_1)\otimes\cdots\otimes L_{1,1}(z_1)\right).$$

Taking Traces, III

Dividing the last equation by

$$\mathsf{Tr}_{\mathcal{F}_{21}\times\mathcal{F}_{12}}\big(e^{-i\phi(\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}+\mathbf{a}_{12}^{+}\mathbf{a}_{12}^{-})}\big),$$

we may prove in conjunction with

$$\frac{1}{\mathsf{Tr}_{\mathcal{F}}\, e^{-i\phi\,\mathbf{a}^{+}\mathbf{a}^{-}}} = 2\,i\,\sin\frac{\phi}{2}\,e^{-i\,\frac{\phi}{2}}\,,$$

as well as the formula

$$\frac{1}{\mathsf{Tr}_{21}(e^{-i\,\phi\,\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}})}\mathrm{Tr}_{21}\left(e^{-i\phi\,\mathbf{a}_{21}^{+}\mathbf{a}_{21}^{-}}\mathcal{B}_{L,21}^{-}\otimes\cdots\otimes\mathcal{B}_{1,21}^{-}\right) = \mathbb{I}_{2^L\times 2^L}\,,$$

a relation we term fundamental operator relation.

Fundamental Operator Relation

It reads, with $z_2 = z + j + \frac{1}{2}$ and $z_1 = z - j - \frac{1}{2}$,

$$f(\phi)\,\mathbf{T}_j^+(z) = \mathbf{Q}_2(z_2)\,\mathbf{Q}_1(z_1), \quad \text{where} \quad f(\phi) = 2\,i\,\sin\frac{\phi}{2}.$$

Here the spin-j transfer matrix (over an infinite $\mathfrak{sl}(2)$ Verma module) is

$$\mathbf{T}_j^+(z;\phi) \equiv \mathsf{Tr}_{\mathcal{F}_{21}\times\mathcal{F}_{12}}(e^{i\phi\mathfrak{J}^3}\cdot\mathcal{L}_L^{\mathrm{j}}(z)\otimes\ldots\otimes\mathcal{L}_1^{\mathrm{j}}(z))\,,$$

The Q-operators are explicitly defined as

$$\mathbf{Q}_1(z_1) \equiv \frac{1}{\mathsf{Tr}_{12}(e^{-i\phi\,\mathbf{a}_{12}^+\mathbf{a}_{12}^-})}\mathsf{Tr}_{12}\left(e^{-i\phi\,\mathbf{a}_{12}^+\mathbf{a}_{12}^-}\,L_{L,1}(z_1)\otimes\cdots\otimes L_{1,1}(z_1)\right),$$

$$\mathbf{Q}_2(z_2) \equiv \frac{1}{\mathsf{Tr}_{21}(e^{-i\phi\,\mathbf{a}_{21}^+\mathbf{a}_{21}^-})}\mathsf{Tr}_{21}\left(e^{-i\phi\,\mathbf{a}_{21}^+\mathbf{a}_{21}^-}\,L_{L,2}(z_2)\otimes\cdots\otimes L_{1,2}(z_2)\right).$$

Commutativity of the Q-Operators

Since we have

$$f(\phi)\,\mathbf{T}_j^+(z) = \mathbf{Q}_2(z_2)\,\mathbf{Q}_1(z_1), \quad \text{and} \quad f(\phi)\,\mathbf{T}_j^+(z) = \mathbf{Q}_1(z_1)\,\mathbf{Q}_2(z_2),$$

we have

$$[\mathbf{Q}_1(z_1)\,,\mathbf{Q}_2(z_2)] = 0\,.$$

The two Q-operators commute!

An intertwiner for each of our two partonic Lax operators exists. Thus,

$$[\mathbf{Q}_1(z_1)\,,\mathbf{Q}_1(z_1')] = 0, \quad \text{and} \quad [\mathbf{Q}_2(z_2)\,,\mathbf{Q}_2(z_2')] = 0\,.$$

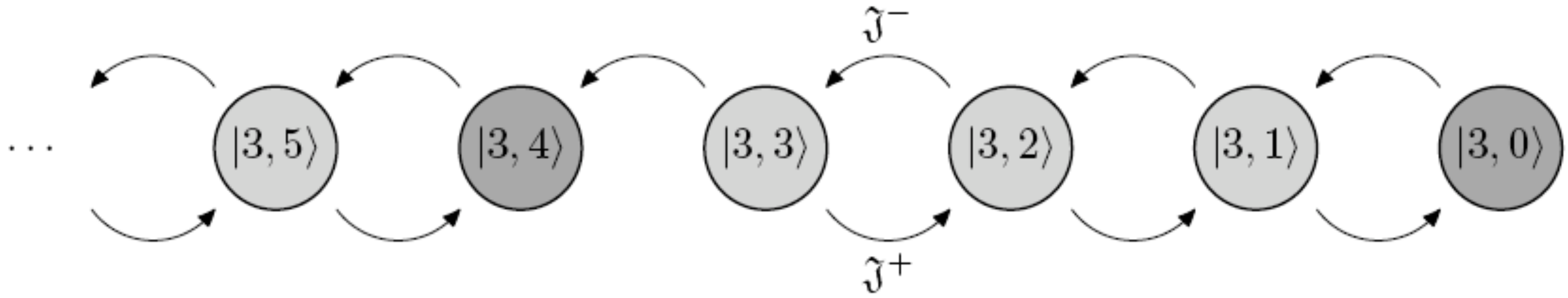
Therefore we also have

$$\left[\mathbf{T}_j^+(z)\,,\mathbf{Q}_{1,2}(z_{1,2})\right] = 0, \quad \text{and} \quad \left[\mathbf{T}_j^+(z)\,,\mathbf{T}_j^+(z')\right] = 0\,.$$

$\Rightarrow$ Everything commutes with everything!

The Compact Transfer Matrix

The infinite $\mathfrak{sl}(2)$ Verma module is reducible, and splits into a finite dimensional part and an infinite dimensional part:



[Picture from Niklas Beisert's PhD thesis, '05]

This leads to the usual spin-j transfer matrix

$$\mathbf{T}_j(z) \equiv \mathbf{T}_j^+(z) - \mathbf{T}_{-j-1}^+(z)\,,$$

if we put $2j \in \mathbb{Z}_{\geq 0}$. The $\phi \to 0$ limit exists.

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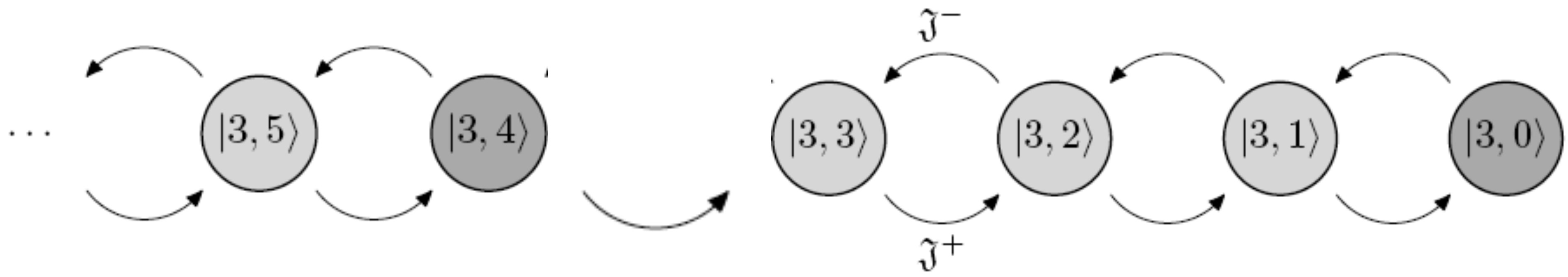

[Picture (modified) from Niklas Beisert's PhD thesis, '05]

This leads to the usual spin-j transfer matrix

$$\mathbf{T}_j(z) \equiv \mathbf{T}^+_j(z) - \mathbf{T}^+_{-j-1}(z)\,,$$

if we put $2j \in \mathbb{Z}_{\geq 0}$. The $\phi \to 0$ limit exists.

Transfer Matrix and the Generalized Wronskian

The fundamental relation turns into the generalized Wronskian relation

$$f(\phi)\,\mathbf{T}_j(z) = \mathbf{Q}_2(z_2)\,\mathbf{Q}_1(z_1) - \mathbf{Q}_2(z_1)\,\mathbf{Q}_1(z_2) = \det_{ij}\mathbf{Q}_i(z_j).$$

where $z_2 = z + j + \frac{1}{2}$ and $z_1 = z - j - \frac{1}{2}$.

From our point of view, this is the basis for all other functional relations.

In particular, the TQ-equation is not fundamental, but a derived concept.

A close analogy is Q-operator = quark and transfer matrix = meson.

The Bethe equations also easily follow ...

Bethe Equations

The eigenvalues of our Q-operators take the form

$$Q_2(z) = e^{-i\,z\,\frac{\phi}{2}} \prod_{k=1}^{M} (z - z_k) \qquad Q_1(z) = e^{+i\,z\,\frac{\phi}{2}} \prod_{k=1}^{L-M} (z - \tilde{z}_k)$$

The Bethe equations then follow from the fundamental relation:

$$\left(\frac{z_k + \frac{1}{2}}{z_k - \frac{1}{2}}\right)^L e^{i\phi} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{z_k - z_j + 1}{z_k - z_j - 1}, \quad \left(\frac{\tilde{z}_k + \frac{1}{2}}{\tilde{z}_k - \frac{1}{2}}\right)^L e^{-i\phi} = \prod_{\substack{j=1\\ j\neq k}}^{L-M} \frac{\tilde{z}_k - \tilde{z}_j + 1}{\tilde{z}_k - \tilde{z}_j - 1}$$

No “ansatz” of any kind!

$L = 2$ Example: $\mathrm{Q}_2(z)$ Operator

$$e^{-i\,z\,\frac{\phi}{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z-\frac{i}{2}\cot\frac{\phi}{2} & -\frac{i}{2}\cot\frac{\phi}{2}+\frac{1}{2} & 0 \\ 0 & -\frac{i}{2}\cot\frac{\phi}{2}-\frac{1}{2} & z-\frac{i}{2}\cot\frac{\phi}{2} & 0 \\ 0 & 0 & 0 & z^2-z\,i\cot\frac{\phi}{2}-\frac{1}{2\sin^2\frac{\phi}{2}}+\frac{1}{4} \end{pmatrix}$$

Take the $\phi \to 0$ limit:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \infty & \infty & 0 \\ 0 & \infty & \infty & 0 \\ 0 & 0 & 0 & \infty \end{pmatrix}$$

These divergences are very natural!

$L=2$ Example: Diagonalized $\mathrm{Q}_2(z)$ Operator

$$e^{-i\,z\,\frac{\phi}{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z-\frac{i}{2}\cot\frac{\phi}{4} & 0 & 0 \\ 0 & 0 & z^2 - z\,i\cot\frac{\phi}{2} - \frac{1}{2\sin^2\frac{\phi}{2}} + \frac{1}{4} & 0 \\ 0 & 0 & 0 & z+\frac{i}{2}\tan\frac{\phi}{4} \end{pmatrix}$$

Take the $\phi \to 0$ limit:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \infty & 0 & 0 \\ 0 & 0 & \infty & 0 \\ 0 & 0 & 0 & z \end{pmatrix}$$

We see that finite eigenvalues correspond to $\mathfrak{su}(2)$ highest weight states, while infinite eigenvalues correspond to $\mathfrak{su}(2)$ descendants!

Removal of the Twist

As we saw, the divergences of many of the matrix elements of the Q-operators have a very natural interpretation.

This still leaves us to find finite operators whose eigenvalues are the Baxter polynomials $Q(z)$ and $P(z)$ of degree, respectively M and $L-M+1$.

Here is the answer:

$$\mathbf{Q}_{z_0}(z) \equiv \lim_{\phi \to 0} \mathbf{Q}_{1,2}(z;\phi)\, \mathbf{Q}_{1,2}^{-1}(z_0;\phi).$$

$$\mathbf{P}_{z_0}(z) \equiv \lim_{\phi \to 0} f^{-1}(\phi)\left(\mathbf{Q}_2(z;\phi)\, \mathbf{Q}_1(z_0;\phi) - \mathbf{Q}_2(z_0;\phi)\, \mathbf{Q}_1(z;\phi)\right),$$

We can show that these operators are $\mathfrak{su}(2)$ invariant.
However, they are "composite" operators.

From $\mathfrak{sl}(2)$ to $\mathfrak{sl}(n)$

Recall that the "usual" rational Lax operator of $\mathfrak{sl}(2)$ takes the form

$$\mathbb{L}(z) = \begin{pmatrix} z + A & B \\ C & z + D \end{pmatrix} = z\,\mathbb{I} + \sum_{i,j} l_{ij}\, e_{ij}\,.$$

The second way of writing this immediately generalizes to $\mathfrak{sl}(n)$.

The two "partonic" rational Lax operators of $\mathfrak{sl}(2)$ took the form

$$\mathbb{L}_1(z) = \begin{pmatrix} z + A & B \\ C & D \end{pmatrix} \quad \text{and} \quad \mathbb{L}_2(z) = \begin{pmatrix} A & B \\ C & z + D \end{pmatrix}.$$

These may also be written in a novel form which lifts to $\mathfrak{sl}(n)$!

$$\mathbb{L}_k(z) = z\, e_{kk} + \sum_{i,j} l_{ij}^{(k)}\, e_{ij}\,.$$

The $\mathfrak{sl}(n)$ Partonic Lax Operators

Solving the YBE for the $l_{ij}^{(k)}$ one finds [Bazhanov, Frassek, Łukowski, Meneghelli, MS to appear]

$$L_k(z_k) = \begin{pmatrix} 1 & & & \mathbf{a}^+_{k,1} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & \mathbf{a}^+_{k,k-1} & & & \\ \mathbf{a}^-_{k,1} & \cdots & \mathbf{a}^-_{k,k-1} & z_k + \mathbf{h}_k & \mathbf{a}^+_{k,k+1} & \cdots & \mathbf{a}^+_{k,n} \\ & & & -\mathbf{a}^-_{k,k+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -\mathbf{a}^-_{k,n} & & & 1 \end{pmatrix}.$$

Again, the oscillators $\mathbf{a}^{\pm}$ satisfy canonical commutation relations and

$$\mathbf{h}_k := \sum_{k>i} \mathbf{h}_{ki} - \sum_{i>k} \mathbf{h}_{ki}\,, \quad \text{with} \quad \mathbf{h}_{ki} := \mathbf{a}^+_{ki}\,\mathbf{a}^-_{ki} + \tfrac{1}{2}\,.$$

Factorization of the $\mathfrak{sl}(n)$ Partonic Lax Operators

[Bazhanov, Frassek, Łukowski, Meneghelli, MS to appear]

Excitingly, the "usual" Lax operator of $\mathfrak{sl}(n)$ again factorizes as $(i < j)$

$$L_1(z_1) \dots L_n(z_n) = S\, \mathfrak{L}\left(z, \lambda_1, \dots, \lambda_{n-1}; \{\mathbf{a}_{ij}^{\pm}\}\right) \mathcal{B}\left(\{\mathbf{a}_{ji}^{+}\}\right) S^{-1} .$$

Here

$$z = \frac{z_1 + \dots + z_n}{n} \quad \text{and} \quad \lambda_i = z_i - z_{i+1} - 1 .$$

$\Lambda = \{\lambda_1, \dots, \lambda_{n-1}\}$ are the Dynkin labels of $\mathfrak{sl}(n)$.
Consistent with $\mathfrak{sl}(2)$ before, where the Dynkin label is twice the spin j.

As before, we now define n "elementary" Q-operators by

$$\mathbf{Q}_i(z) \sim \mathrm{Tr}\, \underbrace{L_i(z) \otimes L_i(z) \otimes \cdots \otimes L_i(z)}_{L-\text{ times}} .$$

Fundamental Functional Operator Equations for $\mathfrak{sl}(n)$

To achieve convergence, we now need $n-1$ angles $\Phi = \{\phi_1, \dots, \phi_{n-1}\}$. In generalization of the $\mathfrak{sl}(2)$ case one finds

$$f(\Phi)\, \mathbf{T}^{+}_{\Lambda}(z) = \mathbf{Q}_1(z_1)\ \dots\ \mathbf{Q}_n(z_n).$$

Substraction of the reducible infinite Verma modules leads again to the transfer matrices for all finite dimensional representations labelled by $\Lambda = \{\lambda_1, \dots, \lambda_{n-1}\}$:

$$f(\Phi)\, \mathbf{T}_{\Lambda}(z) = \det_{ij} \mathbf{Q}_i(z_j).$$

For the trivial representation $\Lambda = \{0, \dots, 0\}$ this is the Wronskian relation. Without "ansatz", it allows to obtain the $\mathfrak{sl}(n)$ nested Bethe equations.

From the Wronskian to the Nested Bethe Equations

The following hierarchy corresponds to the nested Bethe ansatz levels:

$$\mathbf{Q}_\emptyset(z) \equiv \mathbb{I}$$

$$\mathbf{Q}_i(z) \qquad \Leftarrow \qquad \text{Fundamental Q-operators}$$

$$g^{(2)}(\Phi)\,\mathbf{Q}_{ij}(z) \equiv \mathbf{Q}_i(z-\tfrac{1}{2})\,\mathbf{Q}_j(z+\tfrac{1}{2}) - i \leftrightarrow j$$

$$g^{(3)}(\Phi)\,\mathbf{Q}_{ijk}(z) \equiv \mathbf{Q}_i(z-1)\,\mathbf{Q}_j(z)\,\mathbf{Q}_k(z+1) - \mathsf{perm}(i,j,k)$$

$$\cdots$$

$$g^{(n)}(\Phi)\,\mathbf{Q}_{i_1\ldots i_n}(z) \equiv \cdots = \epsilon_{i_1\ldots i_n}\,f(\Phi)\,\mathbf{T}_0(z) \sim z^L$$

The functions $g^{(i)}(\Phi)$ are some (known) functions of the twist angles.

Restricting to eigenvalues, one easily derives the nested Bethe equations.

Factorization of the $\mathfrak{sl}(n|m)$ Partonic Lax Operators

From graded Yang-Baxter, we find

[Bazhanov, Frassek, Łukowski, Meneghelli, MS to appear]

$$
L_k(z_k) = \left(\begin{array}{ccccccc|ccc}
1 & & & \mathbf{a}^+_{k,1} & & & & & & \\
 & \ddots & & \vdots & & & & & & \\
 & & 1 & \mathbf{a}^+_{k,k-1} & & & & & & \\
\mathbf{a}^-_{k,1} & \cdots & \mathbf{a}^-_{k,k-1} & z_k + \mathbf{h}_k & \mathbf{a}^+_{k,k+1} & \cdots & \mathbf{a}^+_{k,n} & \mathbf{c}^+_{k,n+1} & \cdots & \mathbf{c}^+_{k,n+m} \\
 & & & -\mathbf{a}^-_{k,k+1} & 1 & & & & & \\
 & & & \vdots & & \ddots & & & & \\
 & & & -\mathbf{a}^-_{k,n} & & & 1 & & & \\
\hline
 & & & \mathbf{c}^-_{k,n+1} & & & & -1 & & \\
 & & & \vdots & & & & & \ddots & \\
 & & & \mathbf{c}^-_{k,n+m} & & & & & & -1
\end{array}\right),
$$

$$
L_k(z_k) = \left(\begin{array}{ccc|ccccccc}
1 & & & & & & \mathbf{c}^+_{k,1} & & & \\
 & \ddots & & & & & \vdots & & & \\
 & & 1 & & & & \mathbf{c}^+_{k,n} & & & \\
\hline
 & & & -1 & & & -\mathbf{a}^+_{k,n+1} & & & \\
 & & & & \ddots & & \vdots & & & \\
 & & & & & -1 & -\mathbf{a}^+_{k,k-1} & & & \\
\mathbf{c}^-_{k,1} & \cdots & \mathbf{c}^-_{k,n} & -\mathbf{a}^-_{k,n+1} & \cdots & -\mathbf{a}^-_{k,k-1} & z_k - \mathbf{h}_k & -\mathbf{a}^+_{k,k+1} & \cdots & -\mathbf{a}^+_{k,n+m} \\
 & & & & & & \mathbf{a}^-_{k,k+1} & -1 & & \\
 & & & & & & \vdots & & \ddots & \\
 & & & & & & \mathbf{a}^-_{k,n+m} & & & -1
\end{array}\right),
$$

Conclusions

- A new, simple, and explicit way to construct the Q-operators for compact $\mathfrak{su}(n)$ and $\mathfrak{su}(n|m)$ spin chains.

- Furthermore, leads to a transparent, fast, and entirely algebraic way to solve the associated models. No "ansatz" of any kind!

- The most intriguing feature of our construction is the appearance of "non-compact" oscillator representations when fully uncovering the algebraic structure of the XXX chain.

- I believe this puts spin chains en a par with sigma models, and seems very suggestive for AdS/CFT.

Remark on Fluxes

- It is crucial to regulate the Q-operators by fluxes.
- Physically interesting by itself, the flux may be removed from physical quantities whenever sensible.
- Slightly breaks the $\mathfrak{su}(n|m)$ symmetry. Recovering it is somewhat singular.
- Should be an interesting hint for AdS/CFT. We would like to suggest that the theory is "easier" if fluxes are present.
- In particular, it should prove crucial to study β-deformed AdS/CFT. Find TBA, Y-system. Four-loop Lüscher works. [Ahn, Bajnok, Bombardelli, Nepomechie '10]

Work in Progress

[Bazhanov, Frassek, Łukowski, Meneghelli, MS]

- Our theory has to be extended to more general representations, and especially the non-compact case.
- Apply methodology to the AdS/CFT integrable system!