On The Mirror TBA

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The logic of the TBA construction

String Hypothesis

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Canonical TBA Equations (vacuum)

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Simplified TBA Equations (vacuum)

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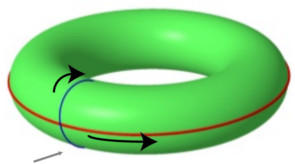
TBA for Excited States

Outline

Mirror theory and TBA
Inspired by Yang+Yang '69, Zamolodchikov '90

Frolov and G.A. '07



- Two Minkowski theories related by the double Wick rotation:

$$\tilde{\sigma} = -i\tau\,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ with respect to $\tilde{\tau}$ defines the *mirror theory*

- Ground state energy is related to the free energy of its mirror

$$E(L) = \lim_{R\to\infty} \frac{L}{R} F(L) = L\mathcal{F}$$

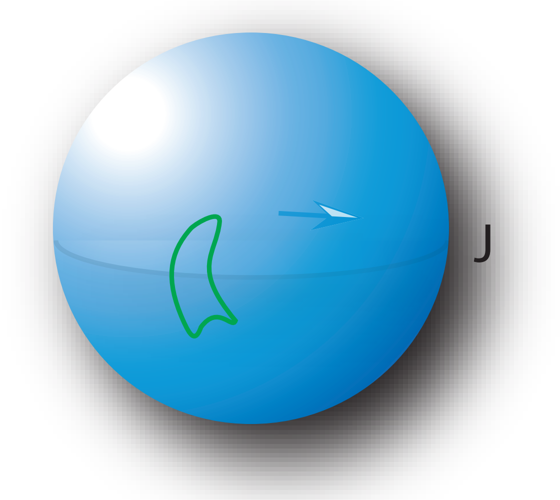
Free energy per unit length $\mathcal{F}$ is found from the Bethe ansatz for the mirror model

Define

- λ – 't Hooft coupling, g – string tension

$$\lambda = 4\pi^2 g^2 \,, \qquad g = \frac{\sqrt{\lambda}}{2\pi}$$

- p – momentum of a string particle
- $\mathcal{E}$ – energy of a string particle
- $\tilde{p}$ – momentum of a mirror particle
- $\tilde{\mathcal{E}}$ – energy of a mirror particle



J is an angular momentum of string rotating around the equator of S^5.
In the light-cone gauge the string world-sheet is a cylinder of circumference J.

L – the “length” of TBA will be related to J

BYE of the mirror model

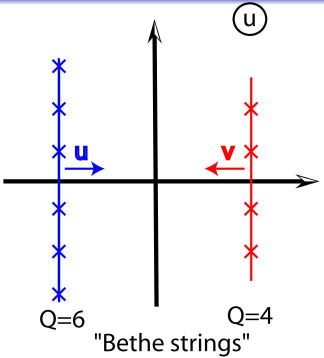
Construct the mirror Bethe-Yang equations for fundamental particles ($\alpha = 1, 2$)

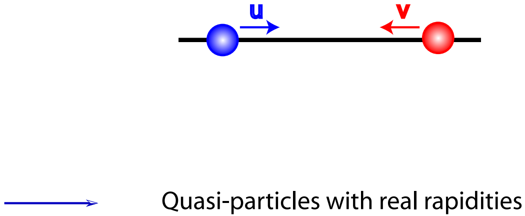
$$
\begin{aligned}
1 &= e^{i\tilde{p}_k R} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\rm I}} S(u_k, u_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}} \\
-1 &= \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}} \\
1 &= \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1 \\ l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}
\end{aligned}
$$

Double Wick rotation (analytic continuation) of the Beisert-Staudacher S-matrix.
Subtleties related to unitarity and the choice of the reference state (vacuum)

Frolov and G.A. '07

Bethe strings





$$u_j = u + (Q + 1 - 2j)\frac{i}{g}\,, \quad j = 1, \ldots, Q, \quad u \in \mathbf{R}$$

Auxiliary roots $v = y + 1/y$ and w participate in building up Bethe strings!

The spectrum of TBA particles

String hypothesis suggests the existence of
nine types of TBA vacuum particles ($\alpha = 1, 2$):

Frolov and G.A. '09(a)

- Q-particles (Q-particle bound states) carrying momentum $\tilde{p}_Q$
- $y^{\pm(\alpha)}$-particles corresponding to fermionic Bethe roots
- $M|vw^{(\alpha)}$-strings
- $M|w^{(\alpha)}$-strings

Derivation of the canonical TBA equations follows a text book

Essler, Frahm, Göhmann, Klümper, Korepin, "The One-Dimensional Hubbard Model"

Frolov and G.A. '09(b); Bombardelli, Fioravanti, Tateo '09; Gromov, Kazakov, Kozak and Vieira '09

TBA equations

- Thermodynamic limit: $R \to \infty$, $N_i \to \infty$ with a ratio N_i/R kept fixed

$$\rho_i(u) + \bar{\rho}_i(u) = K_{ij} \star \rho_j(u) \qquad K_{ij}(u,v) = \frac{1}{2\pi i}\frac{d}{du}\log S_{ij}(u,v)\,,$$

where ρ_i and $\bar{\rho}_i$ are the densities of the particles/holes

- Mirror free energy $\mathcal{F}(L) = \int \mathrm{d}u \sum_k \Big[\underbrace{\widetilde{\mathcal{E}}_k\, \rho_k}_{\text{energy}} - \underbrace{\frac{1}{L}\mathfrak{s}(\rho_k)}_{T\times\text{entropy}} \Big]$

- The TBA equations

$$\frac{\delta\mathcal{F}}{\delta\rho_k} = 0 \quad \Longrightarrow \quad \epsilon_k = L\,\widetilde{\mathcal{E}}_k - \log\left(1+e^{-\epsilon_j}\right) \star K_{jk} \quad \Longleftarrow \quad \text{canonical TBA}$$

The pseudo-energies ϵ_k are $e^{-\epsilon_k} = \frac{\rho_k}{\bar{\rho}_k} \equiv Y_k$

$$E(L) = -\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, \frac{d\widetilde{p}^Q}{du}\log(1+Y_Q)$$

Simplified TBA equations for the vacuum

Frolov and G.A. '09(b)

- $M|w$-strings: $M \geq 1$ $\quad \log Y^{(\alpha)}_{M|w} = \log(1 + Y^{(\alpha)}_{M-1|w})(1 + Y^{(\alpha)}_{M+1|w}) \star s + \delta_{M1} \log \frac{1 - \frac{1}{Y^{(\alpha)}_{-}}}{1 - \frac{1}{Y^{(\alpha)}_{+}}} \star s$

- $M|vw$-strings: $M \geq 1$

$$\log Y^{(\alpha)}_{M|vw} = \log(1 + Y^{(\alpha)}_{M-1|vw})(1 + Y^{(\alpha)}_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y^{(\alpha)}_{-}}{1 - Y^{(\alpha)}_{+}} \star s - \log(1 + Y_{M+1}) \star s$$

- y-particles:

$$\log \frac{Y_+}{Y_-}(v) = \log(1 + Y_Q) \star K_{Qy} \,,$$

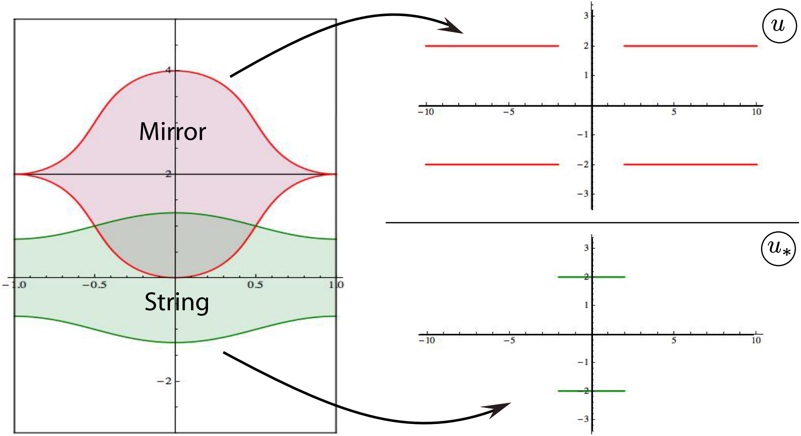
$$\log Y_+ Y_-(v) = 2 \log \frac{1 + Y_{1|vw}}{1 + Y_{1|w}} \star s - \log\left(1 + Y_Q\right) \star K_Q + 2 \log(1 + Y_Q) \star K^{Q1}_{xv} \star s \,,$$

- Q-particles for $Q \geq 2$: $\quad \log Y_Q = \log \frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{(1+\frac{1}{Y_{Q-1}})(1+\frac{1}{Y_{Q+1}})} \star s$

- $Q = 1$-particle: $\quad \log Y_1 = \log \frac{\left(1-\frac{1}{Y^{(1)}_{-}}\right)\left(1-\frac{1}{Y^{(2)}_{-}}\right)}{1+\frac{1}{Y_2}} \star s - \Delta(L) \star s \,, \qquad s(u) = \frac{1}{\cosh \frac{g\pi u}{2}}$

z-torus

$$x(u) = \frac{1}{2}(u - i\sqrt{4-u^2}), \quad \text{Im}\, x(u) < 0$$



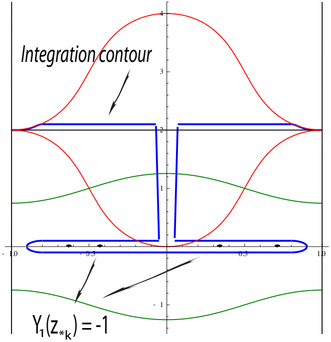
$$x_s(u) = \frac{u}{2}\left(1 + \sqrt{1 - \frac{4}{u^2}}\right) \quad |x_s(u)| \geq 1$$

Mirror and string regions on the z-torus. The boundaries are mapped onto the cuts on the u- and u_*-planes, respectively

Excited States

Excited states

P. Dorey, Tateo '96; Bazhanov, Lukyanov, Zamolodchikov '96; NLIE



- **TBA equations for excited states differ from each other only by a choice of the integration contour**
- Taking the contour back to the real mirror line produces extra contributions $-\log S(z_*, z)$ from $\log(1+Y_1)\star K$, where $K(w,z) = \frac{1}{2\pi i}\frac{d}{dw}\log S(w,z)$

The $\mathfrak{sl}(2)$-sector

- The Bethe-Yang equations have a rank one sector called $\mathfrak{sl}(2)$
- States are dual to the gauge theory operators

$$\mathrm{Tr}(D^N Z^J) + \dots$$

- Y-functions $Y^{(\alpha)}_{M|vw}$, $Y^{(\alpha)}_{M|w}$, $Y^{(\alpha)}_{\pm}$ satisfy

$$Y^{(1)} = Y^{(2)} = Y$$

- Two-particle states are dual to the gauge theory operators

$$\mathrm{Tr}(D^2 Z^J) + \dots$$

Bethe-Yang equations

BY equations for two-particle states in the $\mathfrak{sl}(2)$ sector with $p_1 + p_2 = 0$

$$ip(J+1) - \log \frac{1 + \frac{1}{x_s^{+2}}}{1 + \frac{1}{x_s^{-2}}} - 2i \underbrace{\theta(p,-p)}_{\text{dress phase}} = 2\pi i\, n\,, \quad n > 0$$

At small g the momentum is

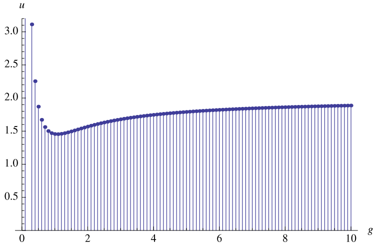
$$p^o_{J,n} = \frac{2\pi n}{J+1}\,, \quad n = 1\,,\ldots\,,\left[\frac{J+1}{2}\right]\,,$$

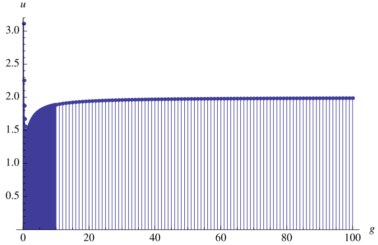
The corresponding rapidity

$$u_{J,n} \to \frac{1}{g} u^o_{J,n}\,, \quad u^o_{J,n} = \cot \frac{\pi n}{J+1}$$

At large g the integer n will coincide with the string level

Solving BY equation for the Konishi state





For large values of $g \sim 100$ the momentum is well approximated by

$$p_{\rm AFS} = \sqrt{\frac{2\pi}{g}} - \frac{1}{g}$$

For the following asymptotic expression (Rej-Spill)

$$p_{\rm RS} = \sqrt{\frac{2\pi}{g}} - \frac{1}{g} + \frac{0.931115 + 0.199472\log(g)}{g^{3/2}}\,,$$

and for $g = 100$ the difference with the numerical solution is 0.00016

How to choose the contour?

Generalized Lüscher formulae give the large J asymptotic solution

$$Y_Q^o(v) = e^{-J\widetilde{\mathcal{E}}_Q(v)}\, T_{Q,1}(v|\vec{u})^2 \prod_{i=1}^{N} S_{\mathfrak{sl}(2)}^{Q1_*}(v,u_i) \quad \longleftarrow \quad \text{exp suppressed}$$

Bajnok and Janik '08

- $\{u_1, \ldots, u_N\}$ is the set of rapidities of string theory particles (solutions of BYE)
- The BY equations then follow from the fact that $\widetilde{\mathcal{E}}_{1_*}(u_k) = -ip_k$ and

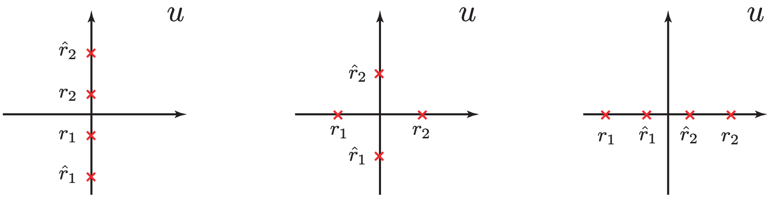
$$T_{1_*,1}(u_k|\vec{u}) = 1 \implies -1 = e^{iJp_k} \prod_{i=1}^{N} S_{\mathfrak{sl}(2)}^{1_*1_*}(u_k,u_i)$$

- The remaining Y-functions are given in terms of $T_{a,s}$-transfer matrices

$$Y_-^o = -\frac{T_{2,1}}{T_{1,2}}\,, \quad Y_+^o = -\frac{T_{2,3}\,T_{2,1}}{T_{3,2}\,T_{1,2}}\,, \quad Y_{Q|vw}^o = \frac{T_{Q+2,1}\,T_{Q,1}}{T_{Q+1,2}}\,, \quad Y_{Q|w}^o = \frac{T_{1,Q+2}\,T_{1,Q}}{T_{2,Q+1}\,T_{0,Q+1}}$$

Kuniba,Nakanishi,Suzuku '93; Tsuboi '97; Gromov, Kazakov and Vieira '09

Zeroes of $Y_{k|vw}$



- For finite values of g any function $Y_{k|vw}$ has four zeroes
- It turns out that $\hat{r}_j^{(k-1)} = r_j^{(k+1)}$, *i.e.* $Y_{k|vw}$ has zeroes

$$\{r_j^{(k)}, r_j^{(k+2)}\}$$

- Positions of zeroes of $Y_{k|vw}$ govern the analytic structure of other Y-functions:

$$Y_{k|vw}(r_j^{(k+1)} \pm \frac{i}{g}) = -1\,, \quad Y_{k+1}(r_j^{(k+1)}) = 0\,, \quad k = 1, \ldots, \infty\,, \quad Y_{\pm}(r_j^{(2)}) = 0$$

Types of states

At $g \sim 0$ the following classification of two-particle states in the $\mathfrak{sl}(2)$-sector takes place

Type of a state	Y-functions	Number of zeroes
I	$Y_{1\|vw}$	2
II	$Y_{1\|vw},\ Y_{2\|vw}$	2+2
III	$Y_{1\|vw},\ Y_{2\|vw},\ Y_{3\|vw}$	4+2+2
IV	$Y_{1\|vw},\ Y_{2\|vw},\ Y_{3\|vw},\ Y_{4\|vw}$	4+4+2+2
$\vdots$	$\vdots$	$\vdots$
$k \to \infty$	$Y_{1\|vw},\ Y_{2\|vw},\ \ldots$	4+4+ ...

Type of a state depends on how many Y-functions have zeroes in the rescaled physical strip $|\mathrm{Im}(u)| < 1$

Frolov,Suzuku and G.A. '09

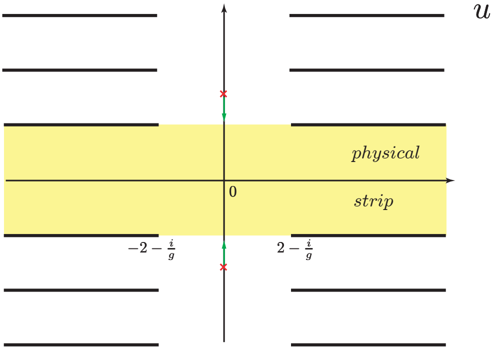
Evolution of an asymptotic state

Initial cond. $\longrightarrow$	$Y_{1\|vw}$, $Y_{2\|vw}$	2+2
	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$	4+2+2
g $\downarrow$	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$, $Y_{4\|vw}$	4+4+2+2
	$Y_{1\|vw}$, $Y_{2\|vw}$, $Y_{3\|vw}$, $Y_{4\|vw}$, $Y_{5\|vw}$	4+4+4+2+2
$\vdots$	$\vdots$	$\vdots$
	$Y_{1\|vw}$, $Y_{2\|vw}$, $\ldots$	4+4+ $\ldots$

The change of analytic properties of Y's must be incorporated by the TBA equations. This leads to the issue of critical values of the coupling

Critical values of g?

Frolov,Suzuku and G.A. '09



The (simplified) TBA equations must be modified upon crossing a critical value of g

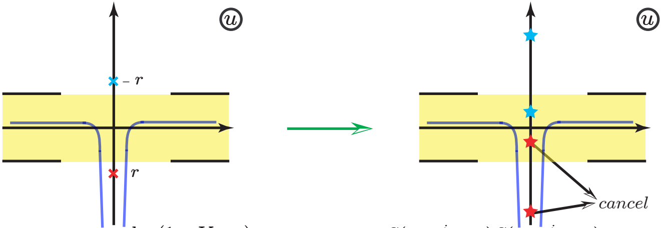
$$Y_{k+1|vw}(\pm\frac{i}{g_{\rm crit}}, g_{\rm crit}) = 0\,, \quad k = 1, 2, \dots\,,$$

At a critical value $Y_{k|vw}$ at $u = 0$ is

$$Y_{k|vw}(0, g_{\rm crit}) = -1$$

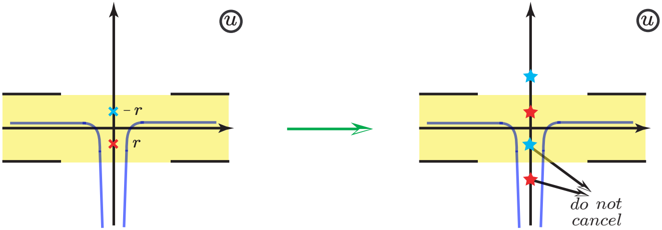
Singularities evolve and call to modify the TBA

Zeros of $Y_{k+1|vw}$ for $g < g_{cr}$ — Zeros of $1 + Y_{k|vw}$



$\log(1 + Y_{k|vw}) \star s \longrightarrow S(r - \frac{i}{g} - u)S(r + \frac{i}{g} - u) = 1$

Zeros of $Y_{k+1|vw}$ for $g_{cr} < g < \bar{g}_{cr}$



$\log(1 + Y_{k|vw}) \star s \longrightarrow S(-r - \frac{i}{g} - u)S(r - \frac{i}{g} - u) \neq 1$

Critical values of g?

For the Konishi state the corresponding *asymptotic* solution exhibits 7 critical values for $g < 100$

$$g_{\mathrm{crit}}^{\mathrm{asympt}} \approx \{4.4, 11.5, 21.6, 34.8, 51.2, 70.7, 93.3\}$$

What about an *exact* solution? There are two possibilities

- g_{crit} is finite ☹
- $g_{\mathrm{crit}} = \infty$ ☺

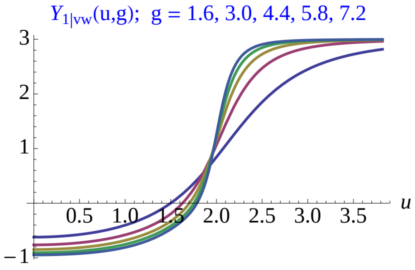
Locations of critical points are currently unknown.

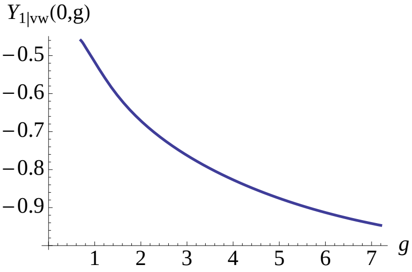
To find them is an important open problem for TBA at strong coupling!

Recent studies for the Konishi state

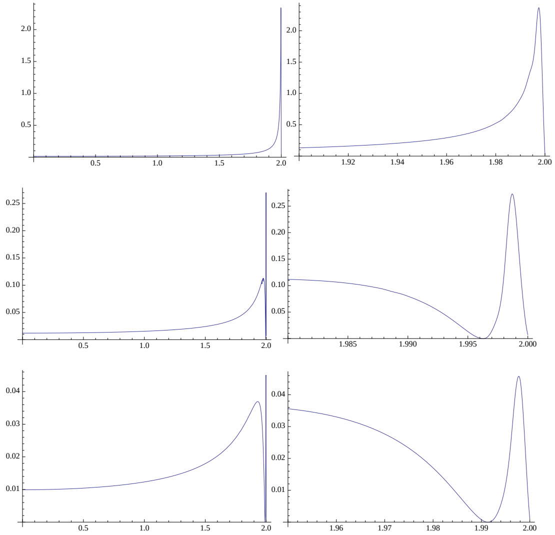
Frolov '10

- Recently the exact solution of the TBA equations have been obtained by iterations up to $g = 7.2$ ($\lambda \approx 2047$) — Frolov '10
- Up to $g = 4.1$ ($\lambda \approx 664$) the numerical data for the energy $E_K(g)$ found is in a good agreement with the result by — Gromov, Kazakov and Vieira '09
- The first g_{crit} (in case it exists) is pushed to much higher values of g than the asymptotic one $g_{\mathrm{crit}} \gg g_{\mathrm{crit}}^{\mathrm{asympt}} = 4.3$
- At the first critical value $Y_{1|vw}(0, g_{\mathrm{crit}}) = -1$





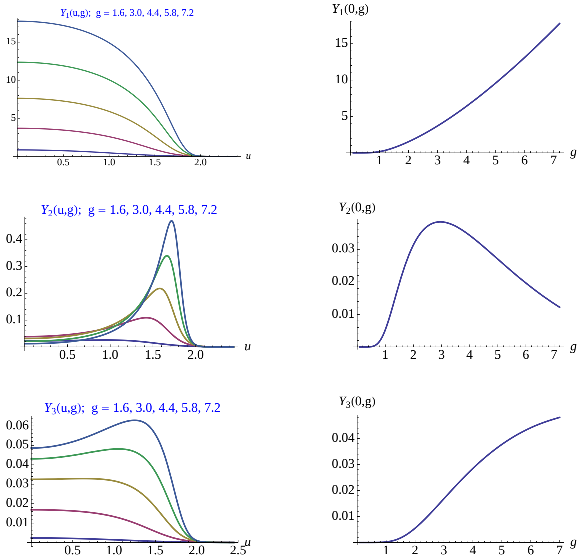
Asymptotic Y_Q-functions



The profiles of the asymptotic Y_Q, $Q = 1, 2, 3$ depicted for $g = 1591.6$, $\lambda = 10^8$

Y_Q-functions

Frolov '10



The profiles of the asymptotic Y_Q, $Q = 1, 2, 3$ depicted for $g = 1591.6$, $\lambda \sim 10^8$

Intermediate coupling?

Frolov '10

- Is $g = 7.2$ is still an intermediate coupling?!

  Numerics suggests

$$Y_1(0,g) = c_3 g^{3/2} + c_2 g + c_1 g^{1/2} + c_0 + \ldots$$

  so that $\log(1+Y_1) \sim \log g$

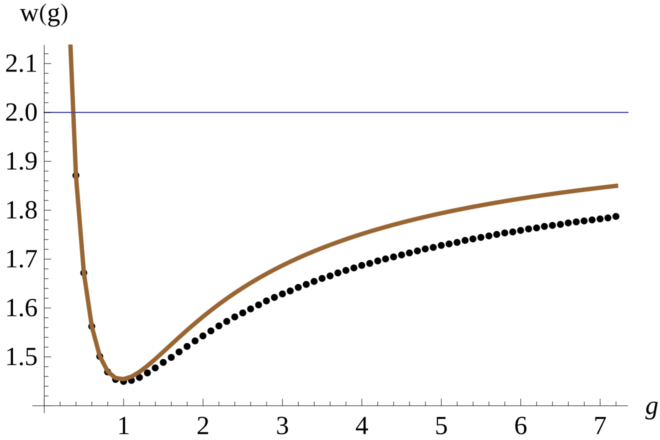
- If everything continues to behave as before beyond $g = 7.2$, then the critical value might be expected at $g_{\text{crit}} \approx 11 \div 12$

- If the behavior Y-functions will change for exponential one, then $Y_{1|vw}(0)$ will be reaching -1 exponentially slow as $g \to \infty$. This raises another puzzle:

  If a strong coupling solution is very different from the asymptotic one (and the critical value is never reached), how one obtains the spectrum of strings in flat space? Especially, its large degeneracy in J ?

  The existing data are not enough to make a reliable statement about the location of the critical point

The Bethe root



Scenarios: Exact Bethe root scaling at $g \to \infty$

$$E = J + \underbrace{\sum_{i=1}^{N} \mathcal{E}(p_i)}_{\mathrm{Bethe-Yang}} - \underbrace{\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \frac{d\tilde{p}^Q}{du} \log(1 + Y_Q)}_{\mathrm{finite-size\quad corr.}}$$

- For $g \to \infty$ roots scale as $u = 2 - \frac{w}{g}$ (the same as for the asymptotic solution). Then

$$\mathcal{E} = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}} = c_1 \sqrt{g} + c_2 + c_3 \frac{1}{\sqrt{g}} + \ldots$$

  If $w = w_{AFS}$ then $c_1 = \sqrt{2\pi}$. If not then the finite-size term must contribute to give $c_1 = \sqrt{2\pi}$

- Roots scale as $u = c_1 - \frac{c_2}{\sqrt{g}}$, $c_1 \neq 2$, as $g \to \infty$. Then

$$\mathcal{E} = b_1 g + b_2 \sqrt{g} + \ldots$$

  Thus, it is the finite-size term which must cancel the leading b_1-term and reproduce $\sqrt{2\pi}\sqrt{g}$

- Devil's scenario – scaling is something else...

Konishi at weak coupling

Five-loop Konishi from Lüscher's approach

The CDT provides the spectrum of excited states

$$E = \underbrace{J + \sum_{i=1}^{N} \mathcal{E}(p_i)}_{\text{Bethe}-\text{Yang}} - \underbrace{\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \frac{d\tilde{p}^Q}{du} \log(1 + Y_Q)}_{\text{finite}-\text{size corr.}}$$

The perturbative Lüscher approach yields

$$\Delta^{(10)} = \Delta^{(10)}_{\text{asympt}} + g^{10} \left\{ -\frac{81\zeta(3)^2}{16} + \frac{81\zeta(3)}{32} - \frac{45\zeta(5)}{4} + \frac{945\zeta(7)}{32} - \frac{2835}{256} \right\}$$

Bajnok, Hegedus, Janik and Lukowski '09

all twist two – Lukowski, Rej and Velizhanin '09

The five-loop correction comes from two sources

1. Position of Bethe roots shifts from their asymptotic value determined by BYE

$$\text{BYE}(u_k) = 0 \quad \neq \quad Y_{1*}(u_k) = -1$$

2. Finite-size correction to the BY energy (the same for Lüscher's and TBA)

Finite-size correction to BYE from Lüscher

As $\mathcal{E}(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$ it is enough to compute δu_k

$$u_k = u_k^o + g^8 \delta u_k$$

from

$$\sum_{j=1}^{N=2} \frac{\delta \mathrm{BYE}(u_k)}{\delta u_j} \, \delta u_j + \Phi_k^{(8)} = 0 \, .$$

Here $\Phi^{(8)}$ is the leading finite-size correction to the BYE of order g^8 for which the Lüscher approach yields

$$\Phi_k^{(8)} = - \sum_{M=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}u}{2\pi} \, \partial_{u_1} \, Y_M^o(u) \big|_{u_1+u_2=0} \, ,$$

Here Y_M^o is an asymptotic Y_M function taken at leading order g^8

Hybrid form of the TBA equations for Y_Q

Both canonical and simplified TBA equations lead to too complicated exact Bethe equations (to do five-loop analytic and numeric computations)

Canonical equations for Y_Q:

$$\log Y_Q = -L\,\widetilde{\mathcal{E}}_Q + \log\left(1+Y_{Q'}\right)\star K_{\mathfrak{sl}(2)}^{Q'Q}$$
$$+\,2\log\left(1+\frac{1}{Y_{M'|vw}}\right)\star K_{vwx}^{M'Q} + 2\log\left(1-\frac{1}{Y_-}\right)\,\hat{\star}\, K_-^{yQ} + 2\log\left(1-\frac{1}{Y_+}\right)\,\hat{\star}\, K_+^{yQ}$$

Simplified equations for Y_Q:

$$\log Y_{M|vw} = \log(1+Y_{M-1|vw})(1+Y_{M+1|vw})\star s$$
$$-\log(1+Y_{M+1})\star s + \delta_{M1}\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\, s \qquad \star \mathrm{K}_M$$

$$\mathrm{K}_M - s\star(\mathrm{K}_{M+1}+\mathrm{K}_{M-1}) = \delta \mathrm{K}_M \quad (M\geq 2), \qquad \mathrm{K}_1 - s\star \mathrm{K}_2 = \delta \mathrm{K}_1\,,$$

This allows one to express the sum $\log\left(1+\frac{1}{Y_{M'|vw}}\right)\star K_{vwx}^{M'Q}$ and leads to the hybrid form of the TBA equations

Finite-size correction to BYE from the mirror TBA

Frolov, Suzuki and G.A. '10

1. Linearize TBA equations

$$Y = Y^o(1+\mathscr{Y}),$$

2. Linearize the exact Bethe equation $Y_{1*}(u_k) = -1$

$$\underbrace{\pi(2n_k+1) = J\,p_k + i\sum_{j=1}^{2}\log S_{\mathfrak{sl}(2)}(u_j,u_k)}_{\text{BYE}} + \mathcal{R}_k$$

where

$$\begin{aligned}\delta\mathcal{R}_k &= \sum_{M=1}^{\infty}\int_{-\infty}^{\infty} du\, Y_M^o(u)\,\frac{1}{\pi}\,\frac{u-u_k}{(M+1)^2+(u-u_k)^2} \\ &\quad + \int_{-\infty}^{\infty} du\, A_{1|vw}(u)\,\mathscr{Y}_{1|vw}(u)\,\frac{1}{2\sinh\frac{\pi}{2}(u-u_k)} \\ &+ \sum_{M=1}^{\infty}\int_{-\infty}^{\infty} du\, Y_{M+1}^o(u)\,\frac{1}{4\pi i}\Big[-i\,\frac{4(u-u_k)}{M^2+(u-u_k)^2} \\ &\quad + \psi\big(\tfrac{M}{4}+\tfrac{i}{4}(u-u_k)\big) - \psi\big(\tfrac{M}{4}-\tfrac{i}{4}(u-u_k)\big) \\ &\quad - \psi\big(\tfrac{M+2}{4}+\tfrac{i}{4}(u-u_k)\big) + \psi\big(\tfrac{M+2}{4}-\tfrac{i}{4}(u-u_k)\big)\Big]\end{aligned}$$

and $A_{M|vw} = \frac{Y^o_{M|vw}}{1+Y^o_{M|vw}}$. Looks puzzling, but...

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\delta\mathcal{R}_k \quad = \quad & \sum_{M=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_M^o(u)\,\frac{1}{\pi}\,\frac{u-u_k}{(M+1)^2+(u-u_k)^2} \\
& + \int_{-\infty}^{\infty}\mathrm{d}u\, A_{1|vw}(u)\mathscr{Y}_{1|vw}(u)\,\frac{1}{2\sinh\frac{\pi}{2}(u-u_k)} \\
+ \quad & \sum_{M=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{M+1}^o(u)\,\frac{1}{4\pi i}\Big[-i\,\frac{4(u-u_k)}{M^2+(u-u_k)^2} \\
& + \psi\big(\tfrac{M}{4}+\tfrac{i}{4}(u-u_k)\big) - \psi\big(\tfrac{M}{4}-\tfrac{i}{4}(u-u_k)\big) \\
& - \psi\big(\tfrac{M+2}{4}+\tfrac{i}{4}(u-u_k)\big) + \psi\big(\tfrac{M+2}{4}-\tfrac{i}{4}(u-u_k)\big)\Big]
\end{aligned}$$

and $A_{M|vw} = \frac{Y^o_{M|vw}}{1+Y^o_{M|vw}}$. Looks puzzling, but...

Five-loop Konishi from the mirror TBA

Equation for $\mathscr{Y}_{Q|vw}$:

$$\mathscr{Y}_{Q|vw} \star \Omega_{QM} = -Y^o_{M+1} \star s\,, \qquad s(u) = \frac{1}{4\cosh\frac{\pi u}{2}}$$

where the fluctuation operator is

$$\Omega_{QM} = \delta_{QM} - \delta_{Q,M-1} A_{M-1|vw}\, s - \delta_{Q,M+1} A_{M+1|vw}\, s\,.$$

This equation is solved by iterations. Formal solution is

$$\mathscr{Y}_{M|vw} = -Y^o_{Q+1} \star s \star \Omega^{-1}_{QM}$$

The solution $\mathscr{Y}_{1|vw}$ is plugged into $\delta\mathcal{R}_k$

Five-loop Konishi from the mirror TBA

Comparison of the numerical results

M	TBA	Lüscher	TBA-Lüscher	$\frac{\text{TBA-Lüscher}}{\text{Lüscher}}$
2	-0.0108303	-0.0108304	$5.21829 \cdot 10^{-8}$	$4.81819 \cdot 10^{-6}$
3	-0.000118621	-0.000118665	$4.34822 \cdot 10^{-8}$	0.000366429
4	$-4.63638 \cdot 10^{-6}$	$-4.6417 \cdot 10^{-6}$	$5.32116 \cdot 10^{-9}$	0.00114638
5	$-3.78671 \cdot 10^{-7}$	$-3.79407 \cdot 10^{-7}$	$7.35899 \cdot 10^{-10}$	0.0019396
6	$-4.89345 \cdot 10^{-8}$	$-4.94077 \cdot 10^{-8}$	$4.73166 \cdot 10^{-10}$	0.00957676

The contributions to $\delta\mathcal{R}_1$ and $-\Phi^{(8)}(u_1)$ produced by individual Y_M's for $M = 2, \ldots, 6$

The perfect numerical agreement

$$\delta\mathcal{R}_k + \Phi_k^{(8)} = 0$$

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Recently the agreement has been confirmed analytically (cf. Arpad's talk on Friday)

Balog, Hegedus '10

Four magnons at weak coupling

- Each of $Y_{M|vw}$ has four real zeroes ($Y_{1|vw}$ has six) already at $g \sim 0$

  Such an analytic structure leads to extra driving terms[1] in the TBA equations already at weak coupling. Two-particle states are a bit special in this respect

- As, in the two-particle case, we find a universal relation between length and charge

$$L = J + 2$$

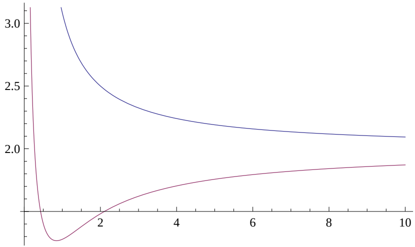
  Recall that semi-classically $L = J$

[1] That is to those which are not induced by $\log(1 + Y_1)$

Putting more magnons...

Four magnons at weak coupling

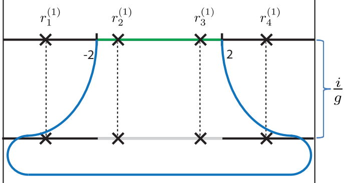
Example: $J = 2, N = 4, n_1 = n_2 = 1$
Four Bethe roots $u_1 = -u_3,\ u_2 = -u_4$



Roots $Y_{M|vw}(r_i^{(M)}) = 0$

$$\log Y_{M|vw}(u) = \sum_{i=1}^{2+2\delta_{M1}} \log S(r_i^{(M)} - \frac{i}{g} - u) + \sum_{i=1}^{2} \log S(r_i^{(M+2)} - \frac{i}{g} - u)$$
$$+ \log(1 + Y_{M-1|vw})(1 + Y_{M+1|vw}) \star s + \delta_{M1} \log \frac{1 - Y_-}{1 - Y_+} \hat{\star}\, s$$
$$- \log(1 + Y_{M+1}) \star s$$

Integration contour for $Y_\pm$-functions



$$Y_+(r_i^{(1)} - \tfrac{i}{g}) = \infty$$

Summary

The mirror TBA: from string hypothesis to the exact spectrum

1. The TBA equations for the vacuum follow from the string hypothesis
2. The TBA equations for excited states are determined by the TBA equations for the vacuum in conjunction with the contour deformation trick
3. With the integration contour tightened back to the real mirror line, the TBA equations for different states have different driving terms, even in the two-particle case!
4. The form of the equations might depend on the coupling constant. For any state there could be critical values of g, crossing which the TBA equations must be modified. Prove their (non)-existence is an important open problem!
5. Relation length – charge exhibits universality (a conjecture explicitly checked for 2-,4-, and 6-particle states)

$$L = J + 2$$