A & B model approaches to surface operators and Toda theories

Based on: Can Kozçaz, Sara Pasquetti & NW, 1004.2025;
Ricardo Schiappa & NW, 0911.5337; NW 0907.2189

Main other references:

Gaiotto, 0904.2715; Alday, Gaiotto and Tachikawa, 0906.3219

Dijkgraaf and Vafa, 0909.2453

Alday, Gaiotto, Gukov, Tachikawa and Verlinde, 0909.0945

Dimofte, Gukov and Hollands, 1006.0977

Alday and Tachikawa, 1005.4469

General comments

There is a very close connection between supersymmetric gauge theories (in four dimensions with extended supersymmetry) and integrable systems in various dimensions.

Several such connections have been described in other talks (primarily for $\mathcal{N} = 4$ theories). In this talk we will discuss $\mathcal{N} = 2$ gauge theories. These turn out to be related to various two-dimensional integrable (conformal) systems (Toda field theories, WZNW models, ...) in quite intricate ways. In addition, there are connections to topological strings (both A and B models) and matrix models.

In general one can go in two directions: Either use gauge theory to learn about integrable systems, or use integrable systems to learn about gauge theories.

Outline:

- Overview:
  - The A_r AGT relation
  - The A_r DV quiver matrix models
- Surface operators:
  - B-model approach: topological recursion
  - AGT approach: gauge theory and instanton counting
  - A-model approach: toric branes and topological vertex

The A_r AGT relation

For a $4d$ $\mathcal{N} = 2$ SU$(r + 1)$ gauge theory we can get a conformal theory from the following matter content: either $2r + 2$ fundamentals or one adjoint.

These theories belong to a larger class of theories, denoted $\mathcal{T}_{(n,g)}(A_r)$ (Gaiotto 2009). The theories in this larger class can be viewed as the arising from the six-dimensional A_r (2,0) theory (Witten 1995) compactified on $C \times \mathbb{R}^4$ where C is a genus g Riemann surface with n punctures.

In an $\mathcal{N} = 2$ gauge theory a fundamental object is the prepotential. This quantity is most efficiently determined using the instanton counting method of Nekrasov. In this approach one introduces a deformation of the theory with two parameters ϵ_1 and ϵ_2. The power of this deformation is that it ensures that the LMNS integrals over instanton moduli space localise to points and can hence be explicitly determined.

The fundamental object in Nekrasov's approach is the partition function $Z(a_i, \mu_j, \epsilon_1, \epsilon_2, q_k)$ where the a_i parameterise the Coulomb branch, the μ_j are the masses of the matter fields and $q_k = e^{2\pi i \tau_k}$. The partition function factorises into two parts as

$$Z = Z_{1-\text{loop}} Z_{\text{instanton}}$$

The $\mathcal{N} = 2$ prepotential $\mathcal{F}_0$ is recovered from Z when $\epsilon_1 = -\epsilon_2 = \hbar$ via the following formula (in the limit $\hbar \to 0$)

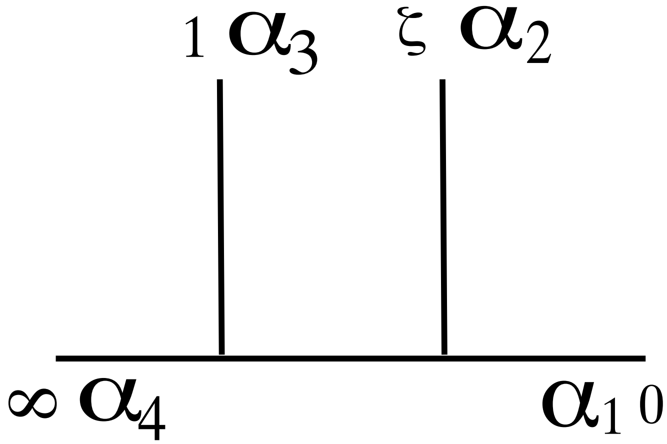
$$Z = e^{-\frac{1}{\hbar^2}\mathcal{F}_0 + \mathcal{O}(\hbar^0)}$$

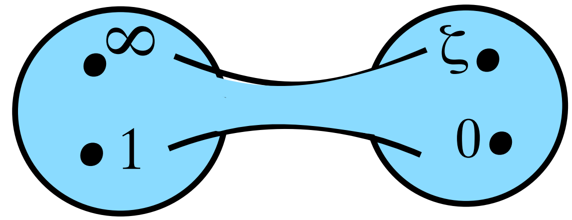
The A_r Toda field theories are defined by the action

$$S = \int \mathrm{d}^2 x \sqrt{g} \left[\frac{1}{8\pi} g^{ad} \langle \partial_a \phi, \partial_d \phi \rangle + \sum_{i=1}^{r} e^{b\langle e_i, \phi \rangle} + \frac{\langle Q\rho, \phi \rangle}{4\pi} R \right]$$

where the e_i are the simple roots of the A_r Lie algebra, $\langle \cdot, \cdot \rangle$ denotes the scalar product on the root space, ρ is the Weyl vector, $\phi = \sum_i \phi_i e_i$ and $Q = (b + \frac{1}{b})$. The ($\mathcal{W}$) primary fields are $V_\alpha = e^{\langle \alpha, \phi \rangle}$. The Liouville theory $\equiv A_1$ Toda theory.

There exists a relation between correlation functions/conformal blocks in the Toda theories and instanton partition functions of SU($r+1$) quiver gauge theories. The A_1 relation was found by Alday, Gaiotto and Tachikawa. For instance, the four-point function is related to the instanton partition function in the SU(2) theory with $N_f = 4$:





$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(\zeta) V_{\alpha_1}(0)\rangle \quad = \quad Z_{\mathrm{inst}}(\mu_1, \mu_2, \mu_3, \mu_4, q = \zeta)$$

In the table below we summarise the rules for relating the $2d$ conformal A_r Toda theories and $4d$ $\mathcal{N} = 2$ A_r (conformal) quiver gauge theories. (Note that not all relations are strict equalities.)

$2d$ A_r Toda theory	$4d$ A_r quiver gauge theory
Conformal block	Instanton partition function
Three-point function	1-loop partition function
Level k	Instanton number k
b	ϵ_1
$\frac{1}{b}$	ϵ_2
External α's	Masses, μ_j
Internal σ's	Coulomb moduli, a_i

Has been checked for various SU$(r+1)$ theories and proven for $\mathcal{N}=2^*$ SU(2) [Fateev-Litvinov], for SU(2) with $N_f=0,1,2$ [Hadasz, Jaskolski, Suchanek] and for SU$(r+1)$ with $N_f=2r+2$ when some of the masses take special values [Mironov-Morozov].

General Toda theory conformal blocks (which are not known) correspond to gauge theories that do not have a weakly coupled limit — so called generalised quivers.

The A_r DV quiver matrix models

Dijkgraaf and Vafa have presented an argument relating the $2d$ A_r Toda theories and the $4d$ A_r quiver gauge theories to a certain A_r quiver matrix model.

In past relations between matrix models, topological strings and $\mathcal{N} = 2$ gauge theories the matrix model only involved one parameter, g_s, corresponding to the restriction $\epsilon_1 = -\epsilon_2$. Dijkgraaf and Vafa suggested that the further refinement of the matrix model which is needed to treat the case with general $\epsilon_{1,2}$ is the so called β deformation.

In the eigenvalue basis with the β deformation, the matrix model correlation function $\langle \hat{V}_{\alpha_1}(z_1)\cdots\hat{V}_{\alpha_k}(z_k)\rangle_{\alpha_0,N}$ translates into

$$\int \prod_{iI} \mathrm{d}\lambda_i^I \prod_{(i,I)<(j,J)} (\lambda_i^I - \lambda_j^J)^{\beta A_{ij}} \prod_{i,I}(z_1 - \lambda_i^I)^{\frac{\sqrt{\beta}\alpha_1^i}{g_s}} \cdots \prod_{i,I}(z_k - \lambda_i^I)^{\frac{\sqrt{\beta}\alpha_k^i}{g_s}}$$

where A_{ij} is the A_r Cartan matrix, $\beta = -\epsilon_2/\epsilon_1$ and $g_s = \sqrt{-\epsilon_1\epsilon_2}$.

The proposal of Dijkgraaf and Vafa is as follows: to connect the matrix model correlation functions of k vertex operators $\hat{V}$ (on the sphere) to the conformal block of $k+1$ vertex operators V in the A_r Toda field theory one should take the large N limit. Furthermore, one should identify

$$\sqrt{\beta} g_s \sum_i N_i \equiv \sqrt{\beta} g_s\, N = \alpha_0 - \sum_i \alpha_i$$

The additional moduli, e.g. $g_s(N_i - N_{i+1})$, in a multi-cut solution are related to the internal momenta in the conformal block.

An example: The matrix model expression corresponding to a four–point conformal block in the A_1 theory is the integral

$$\int \prod_I \mathrm{d}\lambda^I \prod_{I<J} |\lambda^I - \lambda^J|^{2\beta} \prod_I (z-\lambda^I)^{-2\alpha_3/\epsilon_1} (\lambda^I)^{-2\alpha_1/\epsilon_1} (1-\lambda^I)^{-2\alpha_2/\epsilon_1}$$

When α_3 is equal to $-\epsilon_1/2$ $(= -b/2)$ it has been shown by Kaneko that the above integral satisfies the hypergeometric equation with solution ${}_2F_1(A, B; C; z)$ where

$$A = -N, \quad B = \frac{1}{\beta}(-2\frac{\alpha_1}{\epsilon_1} - 2\frac{\alpha_2}{\epsilon_1} + 2) + N - 1, \quad C = \frac{1}{\beta}(-2\frac{\alpha_1}{\epsilon_1} + 1)$$

Using $\beta = -\frac{\epsilon_2}{\epsilon_1}$, $\epsilon_2 N - \alpha_0 + \alpha_1 + \alpha_2 - \epsilon_1/2 = 0$, and $\epsilon_1 = b$, $\epsilon_2 = 1/b$ one finds complete agreement with the Liouville result.

One can also consider the A_r case, as well as the cases with k insertions of $V_{-b/2}$. (Also for a conjectural q-deformed matrix model related to $5d$ quiver theories and to q-deformed Virasoro.)

Summary I

Three ways to calculate the same quantities:

- Using $4d$ (generalised) $\mathcal{N} = 2$ quiver gauge theories
- Using $2d$ conformal Toda field theories
- Using $0d$ quiver matrix models

These three classes of theories are all classified by Lie algebras (here we only considered the $A_r \cong \mathrm{SU}(r+1)$ case). In all three models a two-dimensional Riemann surface plays a crucial role. In the Toda field theories the Riemann surface is the manifold on which the theory is defined; in the $\mathcal{N} = 2$ quiver gauge theories the Riemann surface is (essentially) the Seiberg-Witten curve, and in the quiver matrix models the Riemann surface is the one appearing from the loop equations in the large N limit.

Surface operators

Insertions of additional operators has also been studied, e.g. Wilson and 't Hooft loops [Drukker, Gomis, Okuda & Teschner], [AGGTV]. Surface operators have also been studied:

A surface operator in a $4d$ gauge theory is a certain object localised on a two-dimensional submanifold, just like Wilson and 't Hooft loops are localised on one-dimensional submanifolds.

Surface operators are classified by ways of breaking the gauge group. Number of parameters = numbers of unbroken U(1)'s. $U(1)^r$: Full surface operator; U(1): Simple surface operator.

Alday, Gaiotto, Gukov, Tachikawa and Verlinde conjectured that an insertion of a simple surface operator in an SU(2) quiver gauge theory should correspond to an insertion of the degenerate operator $V_{-b/2}$ in the dual Liouville theory.

Recently Alday & Tachikawa argued that SU(2) theories in the presence of a full surface operator can be described by certain correlation functions in a SL(2) WZNW model. Note: For SU(2) a full surface operator is the same as a simple surface operator.

In the semi-classical limit, $\alpha_i \to \frac{\alpha_i}{\hbar}$ with $\hbar \to 0$, one has e.g.

$$Z_{\mathsf{null}}(z) \equiv \frac{\langle V_{\alpha_1}(\infty) V_{\alpha_2}(1) V_{-b/2}(z) V_{\alpha_3+\frac{b}{2}}(0)\rangle}{\langle V_{\alpha_1}(\infty) V_{\alpha_2}(1) V_{\alpha_3}(0)\rangle} = e^{\frac{b}{\hbar} G(z) + \cdots}$$

The degenerate operator $V_{-b/2}$ satisfies $(L_{-1}^2 + b^2 L_{-2}) V_{-b/2} = 0$.

B-model approach: topological recursion

The above result can be shown [AGGTV] to imply that

$$G(z) = \int^{z} \lambda_{\mathsf{SW}}$$

where λ_{SW} is the Seiberg-Witten differential.

We argue that higher order terms beyond the semi-classical limit can be calculated using the B-model topological recursion method [Eynard & Orantin; Bouchard, Klemm, Mariño & Pasquetti] leading to the 'Baker-Akhiezer' function:

$$Z_{\mathsf{null}}(z) = \exp\left[\sum_{g,n}\left(\frac{\hbar}{b}\right)^{2g-2+n}\frac{1}{n!}\int^{z}\cdots\int^{z}\mathcal{W}_{n}^{(g)}(z,\cdots z)\right]$$

This result agrees with the Liouville result (when $b+\frac{1}{b}=0$).

For multiple (e.g. two) insertions one has

$$e^{-\frac{b}{\hbar}G(z_1)+\frac{b}{\hbar}G(z_2)+G(z_1,z_2)+\mathcal{O}(\hbar)}$$

Again we checked the agreement with the Liouville expressions.

Comment: A similar structure arises in the recent work of Nekrasov and Shatashvili where they take the limit $\epsilon_2 \to 0$ of the prepotential, leading to a function $\mathcal{W}(z)$ similar to $G(z)$ above, but the exact relation to surface operators is not completely clear.

When $\epsilon_2 \to 0$ NS argue that $\epsilon_1 = \hbar$ leads to a quantisation of the classical integrable system well known to arise when $\epsilon_{1,2} \to 0$ (the limit in which one recovers the Seiberg-Witten solution).

See also Nekrasov & Witten; Alday & Tachikawa; Teschner; and Maruyoshi & Taki for further similar work.

AGT approach: gauge theory and instanton counting

Another approach is the following. Consider the prototypical case of inserting a single $V_{-b/2}$ operator into the four-point function in the Liouville theory. The resulting expression can be viewed as a restriction of a generic five-point function, schematically:

$$\langle\alpha_1|V_{\alpha_2}|\sigma\rangle\langle\sigma|V_{\alpha_3}|\alpha_4-\tfrac{b}{2}\rangle\langle\alpha_4-\tfrac{b}{2}|V_{-b/2}|\alpha_4\rangle =$$

$$\langle\alpha_1|V_{\alpha_2}|\sigma\rangle\langle\sigma|V_{\alpha_3}|\sigma'\rangle\langle\sigma'|V_{\alpha_5}|\alpha_4\rangle\Big|_{\substack{\alpha_5=-b/2\\ \sigma'=\alpha_4-b/2}}$$

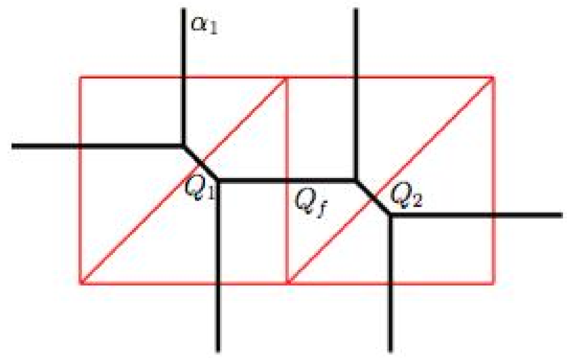
Now, via the AGT conjecture, a five point function can be related to the instanton partition function in the SU(2)×SU(2) theory with a bifundamental matter multiplet and two fundamental matter multiplets in each of the two SU(2) factors. Furthermore, the restrictions can also be translated into gauge theory language. Imposing these restrictions allows us to simplify the expression. But if the AGGTV conjecture is correct the result of these manipulations should correspond to the instanton partition function for the SU(2) gauge theory with four fundamental matter multiplets together with a surface operator insertion. The result has a form which agrees with general expectations. It has a sum over conventional $4d$ instantons labelled by a pair of Young tableaux as well as a sum over "two-dimensional instantons" due to the surface operator, labelled by an integer m.

This method works equally well for the $SU(N)$ theories, as well as for the cases with several surface operator insertions.

A-model approach: toric branes and topological vertex

Yet another approach is the following. It is well known that the Nekrasov partition function (in $d = 5$) is equal to the (A-model) topological string partition function. The topological vertex [Aganagic, Klemm, Mariño & Vafa] is a powerful method to compute the topological string partition function.

We have checked that surface operator insertions in the gauge theory (lifted to $d = 5$) correspond to insertions of toric branes in the topological string framework (as also proposed by Gukov). The effects of such insertions can be expediently calculated (using in particular the ‘vertex on a strip’ results of Iqbal and Kashani-Poor). Consider for instance



This geometry corresponds to "half" of SU(2) with $N_f = 4$. With representations in the form of columns on one of the external legs (and trivial representations on the others) we find:

$$Z_{\text{open}}(z) = \sum_{m=0}^{\infty} z^m \prod_{k=1}^{m} \frac{\left(1 - q^k Q_1\right)\left(1 - q^k Q_1 Q_f Q_2\right)}{\left(1 - q^k\right)\left(1 - q^k Q_1 Q_f\right)}$$

This is a q-hypergeometric function which in the $4d$ limit reduces to the previous results.

Summary II

We have shown that the effects of surface operator insertions in the A_r quiver gauge theories, and their dual incarnations in the A_r quiver matrix models and A_r Toda theories, can be computed using various alternative approaches such as B-model branes/topological recursion (also for multiple insertions), as well as by using the identification of surface operators with A-model toric branes. The latter approach agrees with what one obtains by combining the AGGTV and AGT conjectures

TACK!