Perturbative Quantum Gravity from Gauge Theory

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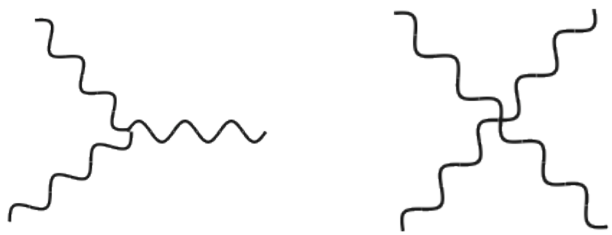
`0805.3993 [hep-th]`

Zvi Bern, John Joseph Carrasco, HJ

Gauge theory and Gravity

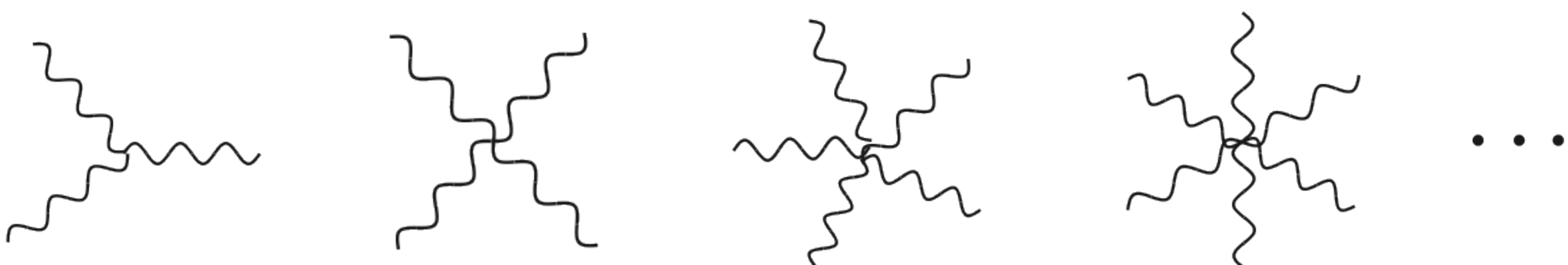
Pure Yang-Mills $\mathcal{L}_{\rm YM} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu}\,,$ A^a_μ



~ 10 terms

Pure Einstein gravity $\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R,$ $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

infinitely many vertices:



$\sim 10^2$ $\sim 10^3$ terms

the two theories are very different!

Amplitudes from standard techniques

Lagrangian, Feyman rules, etc.

Tree-level YM
difficult – but solved
Berends-Giele recursion ('88)

- factorial growth of # of diagrams
- gauge choice redundancy

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Tree-level Gravity
beyond control
using Feynman rules

additionally:

- unlimited vertex expansion
- extremely complicated vertex factors

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"impossible" beyond
4-5 points even at one loop

additionally:
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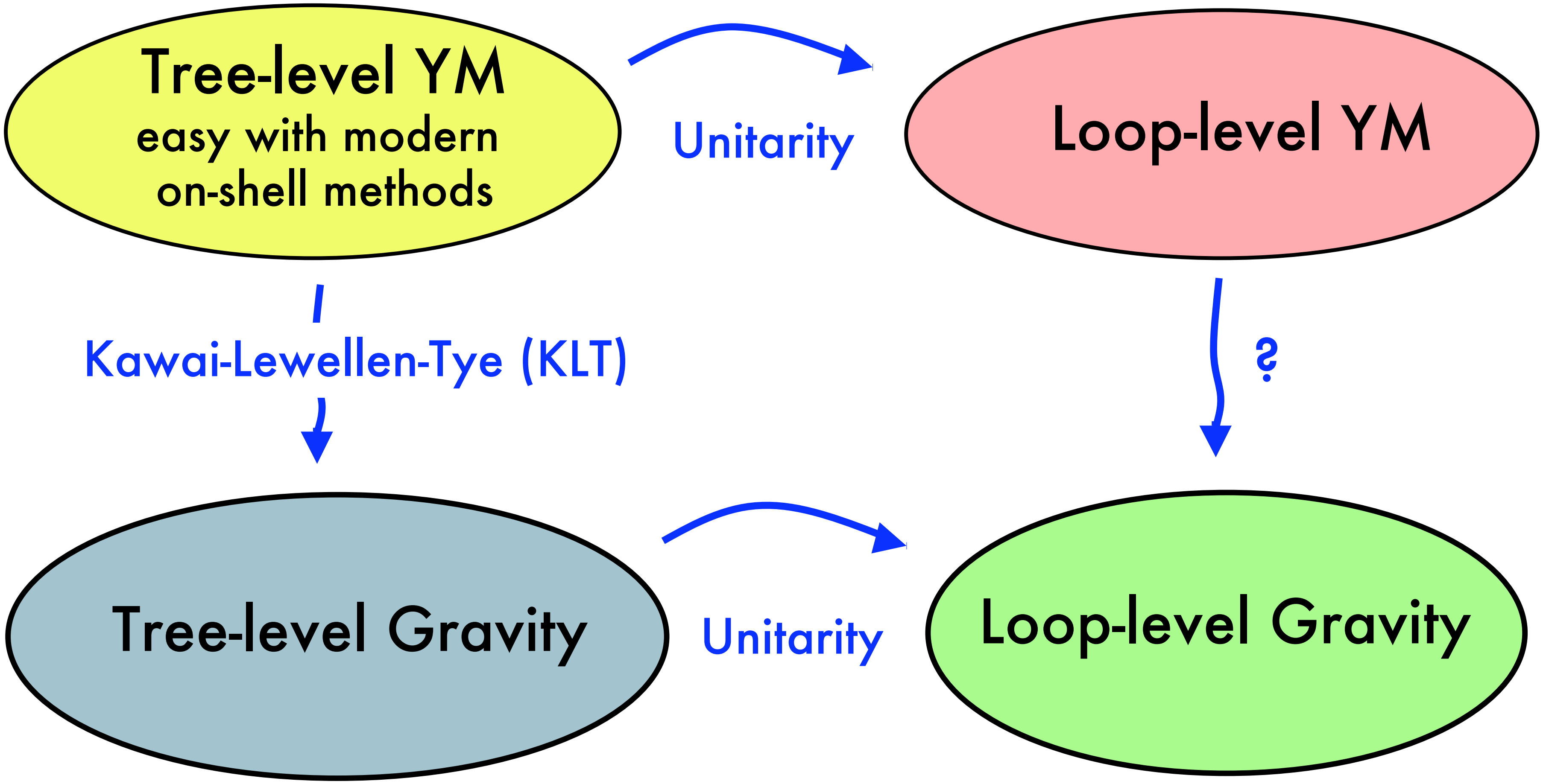
Loop-level Gravity
"insurmountable"

additionally:
- Batalin-Vilkovisky formalism

“Modern” amplitude calculations



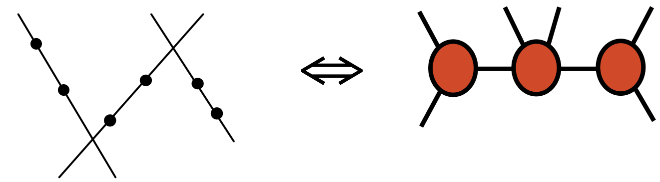
modern methods dramatically simplifies calculations,
but life is even better…

Ampl’s have hidden structures and beauty

Hidden symmetries and structures are present in gauge and gravity amplitudes:

$$\frac{\delta^{(8)}(Q)}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \cdots \langle n-1\, n\rangle \langle n\,1\rangle}$$

- Park-Taylor MHV formula (‘86)



- Witten’s twistor string theory (tree-level)
- Dual superconformal symmetry and Yangian ($\mathcal{N}$=4 SYM)

  Drummond, Henn, Korchemsky, Smirnov, Sokatchev, Plefka, etc.
- Polygon Wilson loop duality ($\mathcal{N}$=4 SYM)

  Alday, Maldacena; Drummond, Henn, Korchemsky, Sokatchev; Brandhuber, Heslop, Spence, Travaglini, etc.
- Grassmannian integrals (‘09) ($\mathcal{N}$=4 SYM)

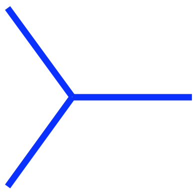
  Arkani-Hamed, Cachazo, Cheung, Kaplan; Mason, Skinner; Spradlin, Volovich, etc.

Beauty is now obvious at tree-level and the planar sector (integrability), beyond this the structure is less well studied

$\rightarrow$ discoveries awaits us....

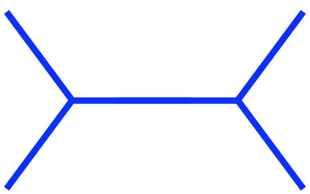
Novel structure cleans up diagrams

Duality: color $\leftrightarrow$ kinematics



$$= \mathcal{V}^{abc}(k_1, k_2, k_3) = f^{abc} V(k_1, k_2, k_3)$$

Duality: $f^{abc} \leftrightarrow V(k_1, k_2, k_3)$



$$= \frac{n_s c_s}{s} \qquad\qquad c_s = f^{abe} f^{ecd}$$

Duality: $c_s \leftrightarrow n_s$

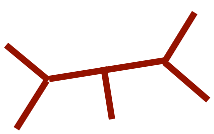
Outline

- Introduction & motivation ✓
- Duality between color and kinematics (tree-level)
  - Jacobi identity for kinematics
  - Novel relations between tree amplitudes
  - Duality begs for gravity amplitudes (~KLT)
- Does duality hold at quantum level?
  - Unitarity method and maximal cuts
  - Duality present in every cut!
  - Evidence of duality in explicit loop amplitudes
- Is duality non-perturbative?
  - Lagrangian formulation
- Conclusion

General strategy

- Claim: Exist a hidden duality between color and kinematics for generic gauge theories:
  - swapping color structures and kinematic structures leaves the amplitude unaffected.
  - color and kinematics have identical algebraic properties

- Making the duality manifest:
  - Must reorganize the tree (and loop) amplitudes such that color and kinematics are on equal footing
  - Achieved by cleaning up the Feynman diagrams:
    - Only cubic vertices are allowed $\sim f^{abc}$
    - Impose the Jacobi identity



- Explore the consequences: Gravity as a bonus, and more...

Gauge theory color decomposition

- Usual decomposition

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

- Alternative decomposition, 4pt example

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

- Map

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c)$$

color structures

$$A_4^{\text{tree}}(1,2,3,4) \equiv \frac{n_s}{s} + \frac{n_t}{t},$$

$$A_4^{\text{tree}}(1,3,4,2) \equiv -\frac{n_u}{u} - \frac{n_s}{s}$$

$$A_4^{\text{tree}}(1,4,2,3) \equiv -\frac{n_t}{t} + \frac{n_u}{u}$$

kinematic structures

color factors

$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$

$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

kinematic numerators

$$n_s, n_t, n_u$$

absorbs 4-pt contact terms
– but gauge dependent!

A Jacobi-like 4pt identity

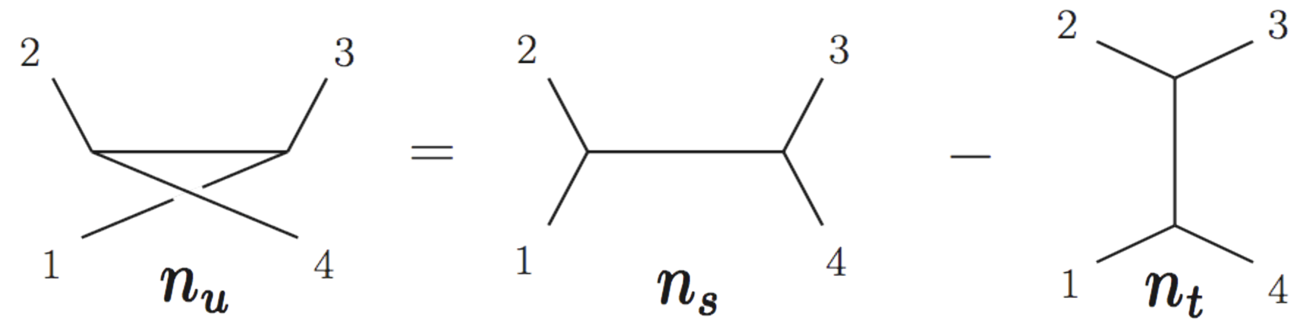
$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

color factors

$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$
$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$
$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

- Jacobi identity for color and for kinematics

$$c_u = c_s - c_t \qquad \Leftrightarrow \qquad n_u = n_s - n_t$$



- duality between color and kinematics

- Kinematic numerators gauge dependent - but 4pt identity is gauge invariant

$$-n'_s + n'_t + n'_u = -n_s + n_t + n_u + \Delta(k_j, \varepsilon_j)(s+t+u) = 0$$

~ gauge parameter

Similar duality at higher points

- Decomposing 5pt amplitude in terms of 15 cubic diagrams

kinematic numerator

color factor

propagators

$$\mathcal{A}_5^{\text{tree}} = g^3\Big(\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} + \frac{n_6c_6}{s_{14}s_{25}} + \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} + \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}} + \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\Big),$$

$$s_{ij} = (k_i + k_j)^2$$

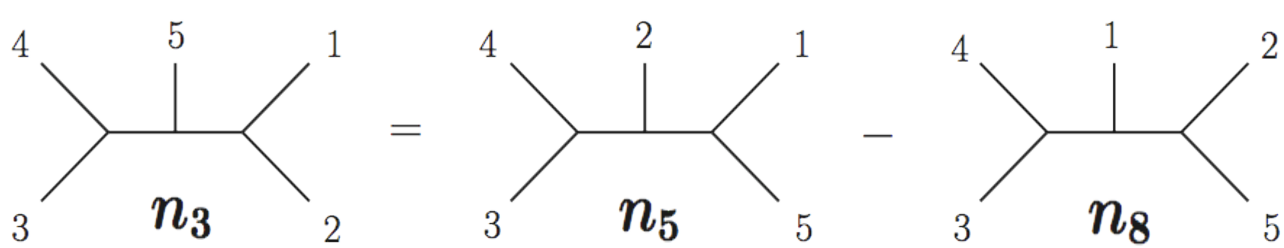
- Equivalent to partial amplitudes

$$A_5^{\text{tree}}(1,2,3,4,5) \equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}$$

etc...

- Duality between color and kinematics hold

$$n_3 - n_5 + n_8 = 0 \quad \Leftrightarrow \quad c_3 - c_5 + c_8 = 0$$



$$c_3 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_5c}\tilde{f}^{ca_1a_2}$$

$$c_5 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_2c}\tilde{f}^{ca_1a_5}$$

$$c_8 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_1c}\tilde{f}^{ca_2a_5}$$

but is no longer gauge invariant...

Generalized gauge transformation

Bern, Carrasco, HJ

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{c_i n_i}{\prod_\alpha p_\alpha^2}$$

$(2n\text{-}5)!!$ cubic diagrams

Define "generalized gauge transformation" on amplitude as

$$n_i \to n_i + \Delta_i \qquad \text{such that} \qquad \sum_i \frac{c_i \Delta_i}{\prod_\alpha p_\alpha^2} = 0$$

Amplitudes invariant under this transformation, but not duality

$$n_i + n_j + n_k \neq 0 \quad \not\Leftrightarrow \quad c_i + c_j + c_k = 0$$

Conjecture: transformation always exists such we can make n_i satisfy the Jacobi identity – making duality manifest.

Amplitude relations

- Assuming the duality can be made manifest at 5pts:
  - 15 different n_i 15
  - 9 Jacobi identities -9
  - fix 2 n_i using two partial ampliudes -2
  - remaining 4 n_i $\Leftrightarrow$ residual gauge freedom -4

Gives curios amplitude relations:

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}}$$

$$A_5^{\text{tree}}(1,2,4,3,5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5) + s_{45}(s_{12}+s_{24})A_5^{\text{tree}}(1,2,3,4,5)}{s_{24}s_{35}}$$

- Any 5pt tree is a linear combination of two basis amplitudes

$$A_5(.\,.\,.\,.\,.) = \alpha\, A_5(1,2,3,4,5) + \beta\, A_5(1,4,3,2,5)$$

Amplitude relations for any number of legs

Bern, Carrasco, HJ

- General relations for gauge theory partial amplitudes

$$A_n^{\text{tree}}(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\{\sigma\}_j\in \text{POP}(\{\alpha\},\{\beta\})} A_n^{\text{tree}}(1,2,3,\{\sigma\}_j) \prod_{k=4}^{m} \frac{\mathcal{F}(3,\{\sigma\}_j,1|k)}{s_{2,4,\ldots,k}}$$

where

$$\{\alpha\} \equiv \{4,5,\ldots,m-1,m\}, \qquad \{\beta\} \equiv \{m+1,m+2,\ldots,n-1,n\}$$

and

$$\mathcal{F}(3,\sigma_1,\sigma_2,\ldots,\sigma_{n-3},1|k) \equiv \mathcal{F}(\{\rho\}|k) = \begin{Bmatrix} \sum_{l=t_k}^{n-1} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} < t_k \\ -\sum_{l=1}^{t_k} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} > t_k \end{Bmatrix} + \begin{Bmatrix} s_{2,4,\ldots,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1} > t_k > t_{k+1} \\ 0 & \text{else} \end{Bmatrix}$$

and

$$\mathcal{G}(i,j) = \begin{Bmatrix} s_{i,j} & \text{if } i<j \text{ or } j=1,3 \\ 0 & \text{else} \end{Bmatrix}$$

and t_k is the position of leg k in the set $\{\rho\}$

$$A_n(\sigma_1,\sigma_2,..,\sigma_n) = \alpha_1 A_n(1,2,..,n) + \alpha_2 A_n(2,1,..,n) + \ldots + \alpha_{(n-3)!} A_n(3,2,..,n)$$

Basis size: $(n-3)!$ Compare to Kleiss-Kuijf relations $(n-2)!$

Recent proofs: Bjerrum-Bohr, Damgaard, Vanhove; Feng, Huang, Jia

Example relations

4 points:

$$A_4^{\text{tree}}(1,2,\{4\},3) = \frac{A_4^{\text{tree}}(1,2,3,4)s_{14}}{s_{24}}$$

$s_{ij..} = (k_i + k_j + \ldots)^2$

5 points:

$$A_5^{\text{tree}}(1,2,\{4\},3,\{5\}) = \frac{A_5^{\text{tree}}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\text{tree}}(1,2,3,5,4)s_{14}}{s_{24}},$$

$$A_5^{\text{tree}}(1,2,\{4,5\},3) = \frac{-A_5^{\text{tree}}(1,2,3,4,5)s_{34}s_{15} - A_5^{\text{tree}}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}}$$

Relations have quite simple structure

String worldsheet monodromy

Monodromy relations on the open string worldsheet is shown to capture both the Kleiss-Kuijf relations and the relations implied by the duality

Bjerrum-Bohr, Damgaard, Vanhove (2009)

$$A(1,3,2,4) = -\mathrm{Re}\left[e^{-2i\alpha'\pi k_2\cdot k_3}A(1,2,3,4) + e^{-2i\alpha' k_2\cdot(k_1+k_3)}A(2,1,3,4)\right]$$ "Kleiss-Kuijf"

$$0 = \mathrm{Im}\left[e^{-2i\alpha'\pi k_2\cdot k_3}A(1,2,3,4) + e^{-2i\alpha' k_2\cdot(k_1+k_3)}A(2,1,3,4)\right]$$ new relations

Original relations recovered in the field theory limit:

$$A(1,3,2,4) = \frac{\sin(2i\alpha'\pi k_1\cdot k_4)}{\sin(2i\alpha'\pi k_2\cdot k_4)}A(1,2,3,4)$$

$$\alpha' \to 0\ :\ A(1,3,2,4) = \frac{k_1\cdot k_4}{k_2\cdot k_4}A(1,2,3,4)$$

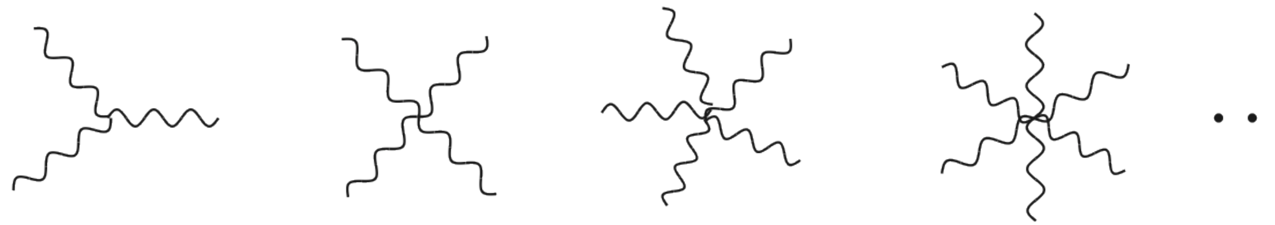
Provides partial proof of duality conjecture

Gravity as a bonus

KLT Relations

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R,$$

Feynman rules:



Kawai-Lewellen-Tye relations

Originally string theory tree level identity:

closed string ~ (left open string) × (right open string)

Field theory limit $\Rightarrow$ gravity theory ~ (gauge theory) × (gauge theory)

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)\,\widetilde{A}_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)\,\widetilde{A}_5^{\text{tree}}(2,1,4,3,5)$$
$$+is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\,\widetilde{A}_5^{\text{tree}}(3,1,4,2,5)$$

gravity states are products of gauge theory states:

$$|1\rangle_{\text{grav}} = |1\rangle_{\text{gauge}} \otimes |1\rangle_{\text{gauge}}$$

Jacobi identity + KLT

KLT:

Bern, Carrasco, HJ

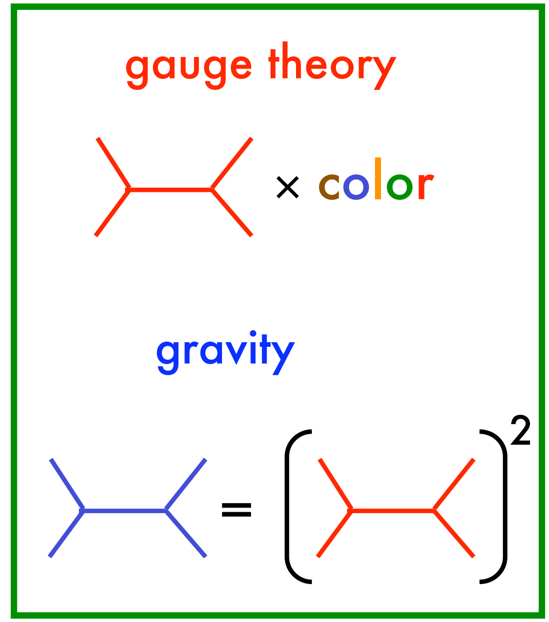
$$\mathcal{M}_4^{\text{tree}} = s_{12} A_4^{\text{tree}}(1,2,3,4) \widetilde{A}_4^{\text{tree}}(1,2,4,3) = \frac{n_s \widetilde{n}_s}{s} + \frac{n_t \widetilde{n}_t}{t} + \frac{n_u \widetilde{n}_u}{u}$$

follows after using: $n_u = n_s - n_t$

$$\mathcal{A}_4^{\text{tree}} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

$\updownarrow$ duality manifest

$$\mathcal{M}_4^{\text{tree}} = \frac{n_s \widetilde{n}_s}{s} + \frac{n_t \widetilde{n}_t}{t} + \frac{n_u \widetilde{n}_u}{u}$$

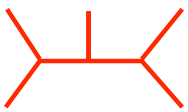


Unlike KLT this gravity formula is for local objects n_i and is manifestly crossing (Bose) symmetric

Gravity = square of YM

- At 5 points

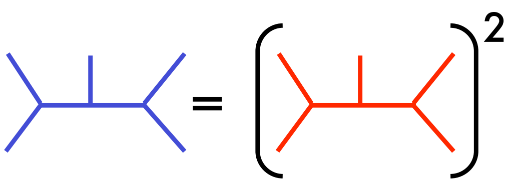
gauge theory



× color

$$\mathcal{A}_5^{\text{tree}} = g^3\Big(\frac{n_1 c_1}{s_{12}s_{45}} + \frac{n_2 c_2}{s_{23}s_{51}} + \frac{n_3 c_3}{s_{34}s_{12}} + \frac{n_4 c_4}{s_{45}s_{23}} + \frac{n_5 c_5}{s_{51}s_{34}} + \frac{n_6 c_6}{s_{14}s_{25}} + \frac{n_7 c_7}{s_{32}s_{14}} + \frac{n_8 c_8}{s_{25}s_{43}} + \frac{n_9 c_9}{s_{13}s_{25}} + \frac{n_{10} c_{10}}{s_{42}s_{13}} + \frac{n_{11} c_{11}}{s_{51}s_{42}} + \frac{n_{12} c_{12}}{s_{12}s_{35}} + \frac{n_{13} c_{13}}{s_{35}s_{24}} + \frac{n_{14} c_{14}}{s_{14}s_{35}} + \frac{n_{15} c_{15}}{s_{13}s_{45}}\Big),$$

gravity



$$\mathcal{M}_5^{\text{tree}} = i\left(\frac{\kappa}{2}\right)^3\Big(\frac{n_1 \tilde{n}_1}{s_{12}s_{45}} + \frac{n_2 \tilde{n}_2}{s_{23}s_{51}} + \frac{n_3 \tilde{n}_3}{s_{34}s_{12}} + \frac{n_4 \tilde{n}_4}{s_{45}s_{23}} + \frac{n_5 \tilde{n}_5}{s_{51}s_{34}} + \frac{n_6 \tilde{n}_6}{s_{14}s_{25}} + \frac{n_7 \tilde{n}_7}{s_{32}s_{14}} + \frac{n_8 \tilde{n}_8}{s_{25}s_{43}} + \frac{n_9 \tilde{n}_9}{s_{13}s_{25}} + \frac{n_{10} \tilde{n}_{10}}{s_{42}s_{13}} + \frac{n_{11} \tilde{n}_{11}}{s_{51}s_{42}} + \frac{n_{12} \tilde{n}_{12}}{s_{12}s_{35}} + \frac{n_{13} \tilde{n}_{13}}{s_{35}s_{24}} + \frac{n_{14} \tilde{n}_{14}}{s_{14}s_{35}} + \frac{n_{15} \tilde{n}_{15}}{s_{13}s_{45}}\Big),$$

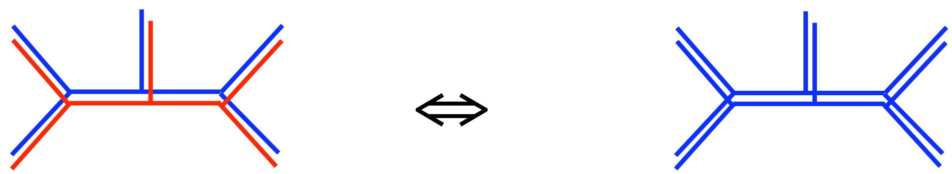
Remarkably only one family of numerators (either n_i or $\tilde{n}_i$) need to satisfy the Jacobi identities.

Tree-level gravity to all orders

- Conjecture to all orders (checked trough 8 points)

Bern, Carrasco, HJ

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{n_i c_i}{\prod_\alpha p_\alpha^2} \quad \Leftrightarrow \quad \mathcal{M}_n^{\text{tree}} = \sum_i \frac{n_i \tilde{n}_i}{\prod_\alpha p_\alpha^2}$$



double copy of YM

Proof: Bern, Dennen, Huang, Kiermaier

Connection to Heterotic string by Tye and Zhang

$$\mathcal{A}^{\text{het}}\Big|_{\alpha'\to 0} = \sum_i \frac{n_{\mathrm{L},i}\,\tilde{n}_{\mathrm{R},i}}{\prod_\beta p_\beta^2}$$

Left sector $n_{\mathrm{L},i}$ $\Leftrightarrow$ modes in spacetime $R^{(1,D-1)}$

Right sector $\tilde{n}_{\mathrm{R},i}$ $\Leftrightarrow$ modes in spacetime $R^{(1,D-1)} \times T^{N_c}$

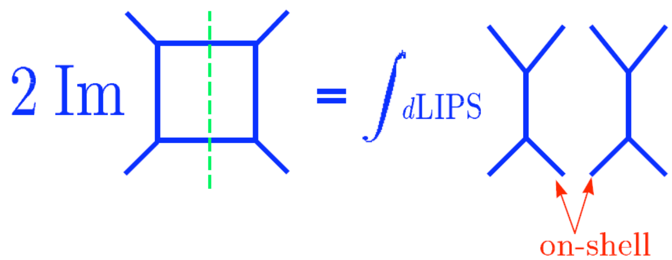
Classical → Quantum

Unitarity

Optical theorem:

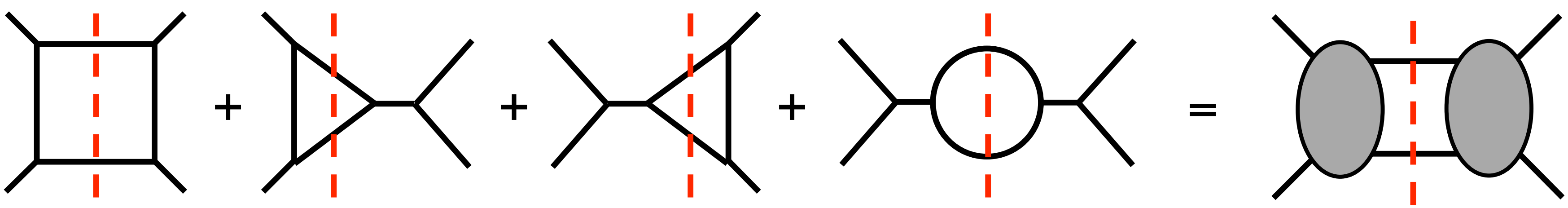
$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$$

$$2\,\mathrm{Im}\,T = T^\dagger T$$



The unitarity method reconstructs the amplitudes avoiding dispersion relations

Bern, Dixon, Dunbar, Kosower (1994)



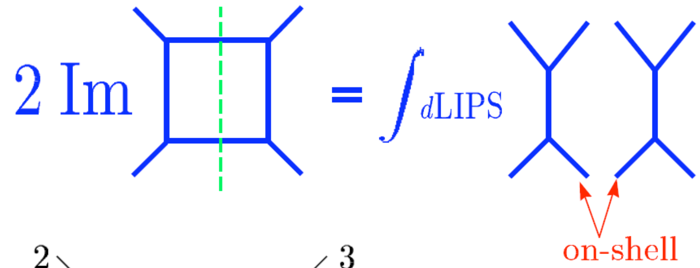
Compute a cut: put loop legs on-shell in amplitude = sew trees amplitudes

checking every cut channel will fix the loop integrals

Unitarity method developments

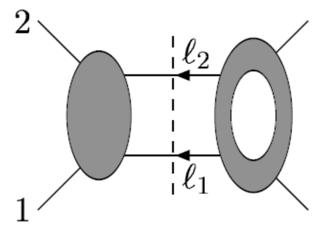
TIME

optical theorem



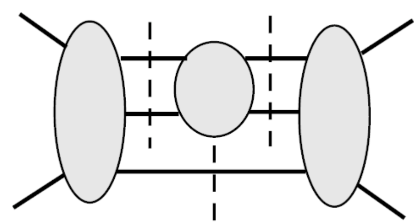
unitarity method

Bern, Dixon, Dunbar and Kosower (1994)



generalized unitarity

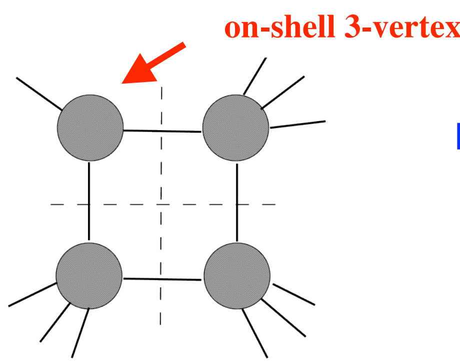
Bern, Dixon and Kosower



quadruple cut
& leading singularity

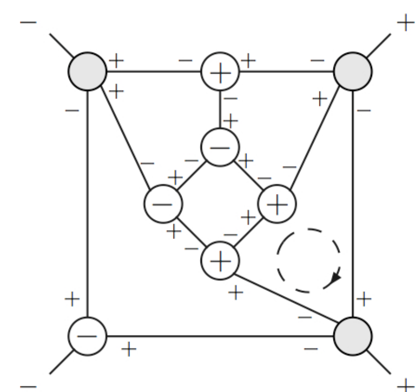
Britto, Cachazo, Feng;
Buchbinder, Cachazo (2004)

Cachazo and Skinner
Cachazo, Spradlin, Volovich
(2008)



maximal cut

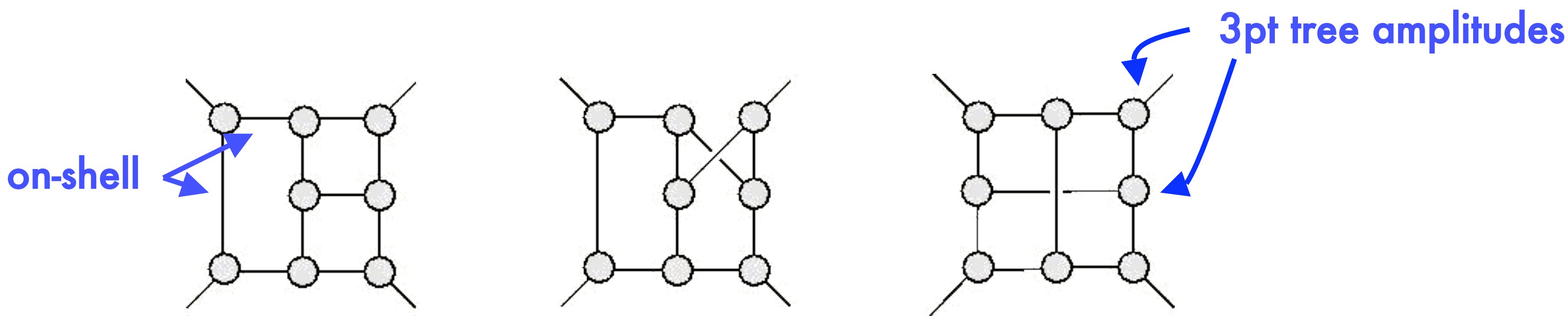
Bern, Carrasco, HJ
and Kosower (2007)



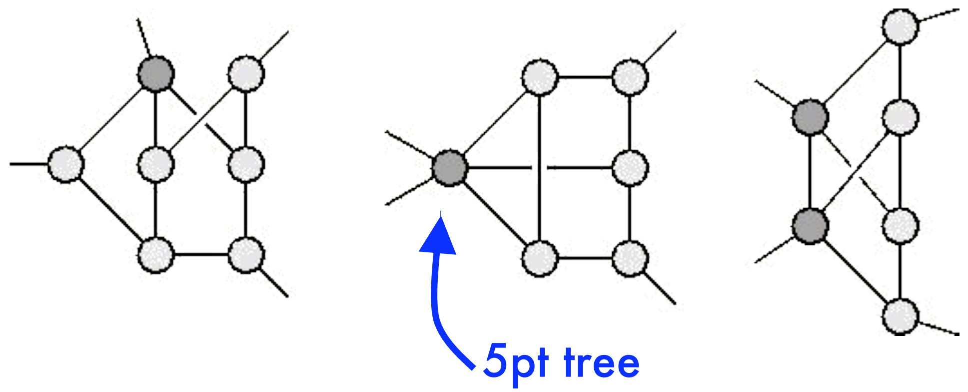
Maximal cuts - a systematic approach for any theory

Bern, Carrasco, HJ
and Kosower (2007)

- put maximum number of propagator on-shell → simplifies calculation



- systematically release cut conditions → great control of missing terms



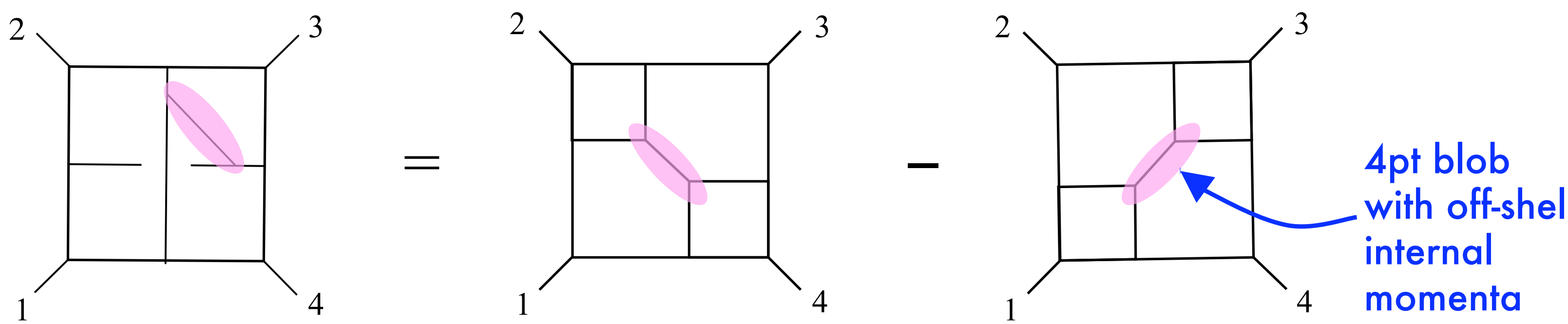
Reconstructs the amplitude piece-by-piece (or term-by-term)

Duality at loop level

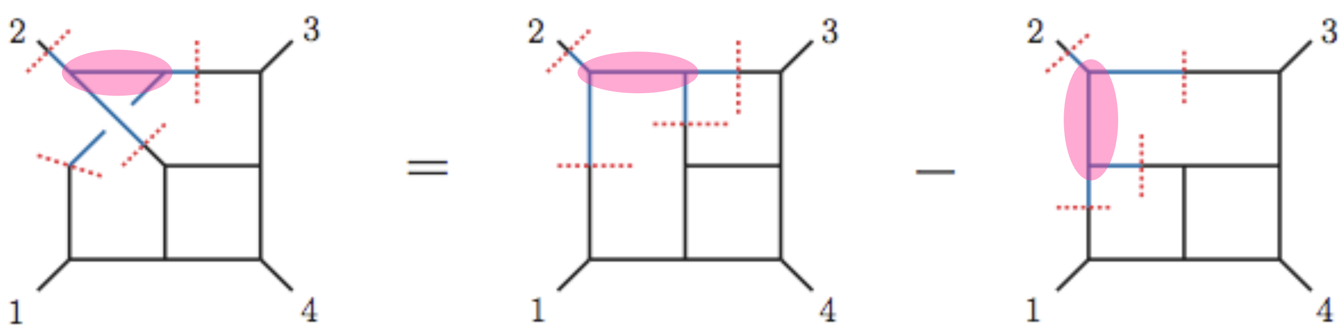
“on-shell” duality first observed in 4-Loop cuts

Studying near-maximal cut at 4-loops reveals that the diagrams (numerators) entering the cut are not independent

4 loops
$\mathcal{N}$=4 SYM



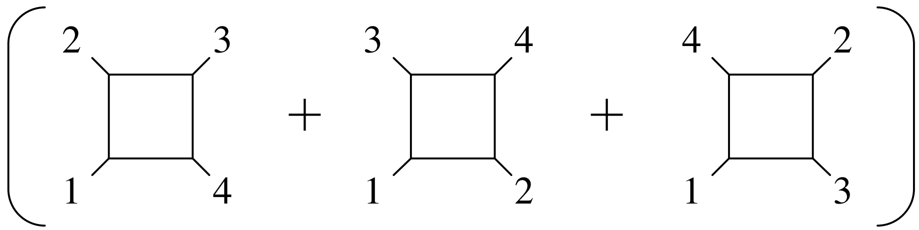
3 loops
$\mathcal{N}$=4 SYM



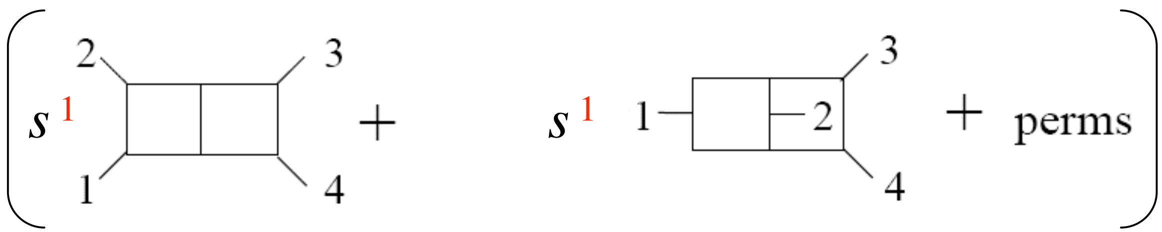
The cut duality follows directly from the tree duality as long as every loop has at least one on-shell leg

“off-shell” duality in $\mathcal{N}=4$ SYM 4-pt ampl.

For particularly simple loop amplitudes one can show that the off-shell duality follows from the tree-level one.

1-loop: K^{1} 

Green, Schwarz, Brink (1982)

2-loop: K^{1} 

Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

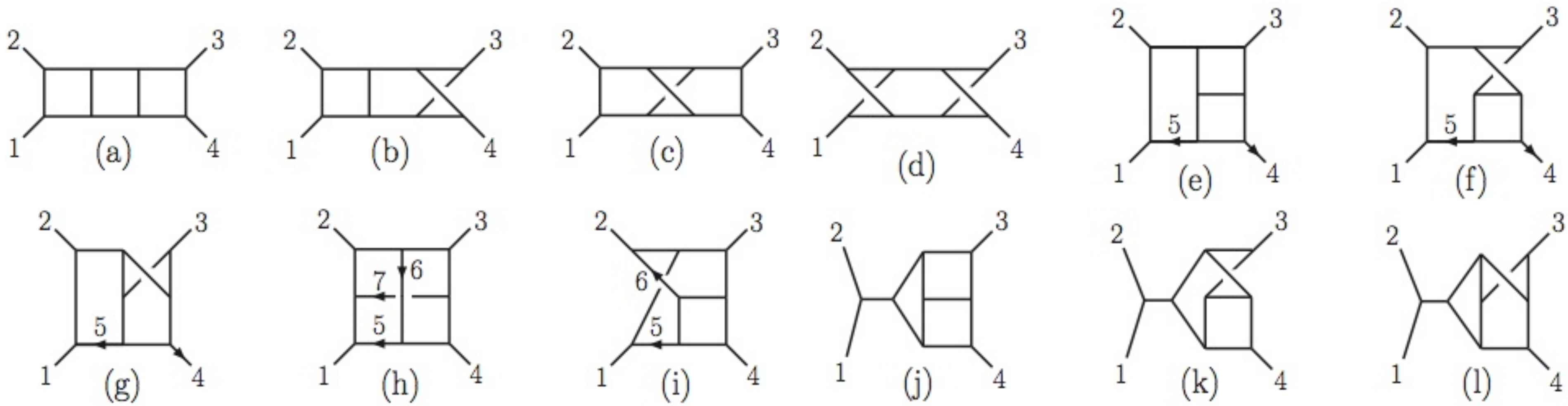
prefactor contains helicity structure: $K = st A_4^{\text{tree}}$

Duality: $\mathcal{N}=8$ sugra is obtained if $1 \to 2$ “numerator squaring”

New nontrivial evidence

3-loop $\mathcal{N}$=4 SYM admits manifest realization of duality – and $\mathcal{N}$=8 sugra is simply the square

1004.0476 [hep-th]
Bern, Carrasco, HJ



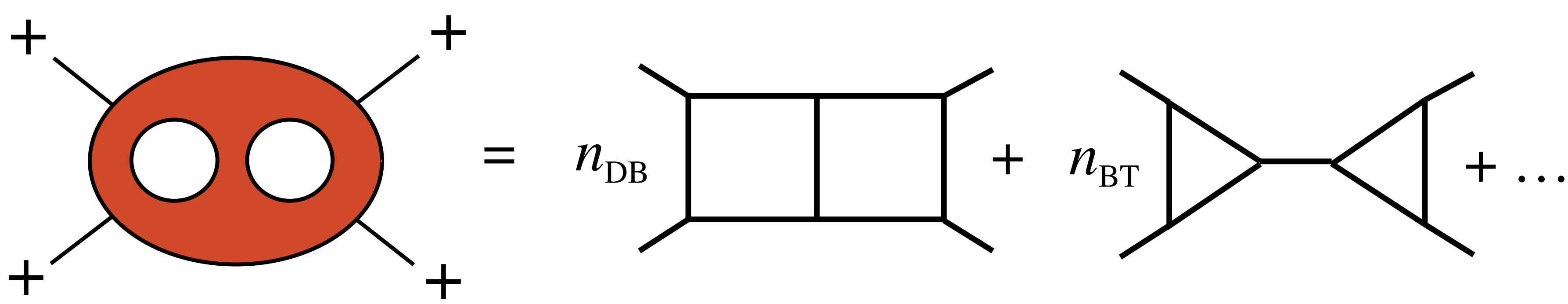
Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$\left(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2\right)/3$
(h)	$\left(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)\right.$ $\left.+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17}) + s^2\right)/3$
(i)	$\left(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)\right.$ $\left.+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46}) + u\,\tau_{25} + s^2\right)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2k_i \cdot l_j$$

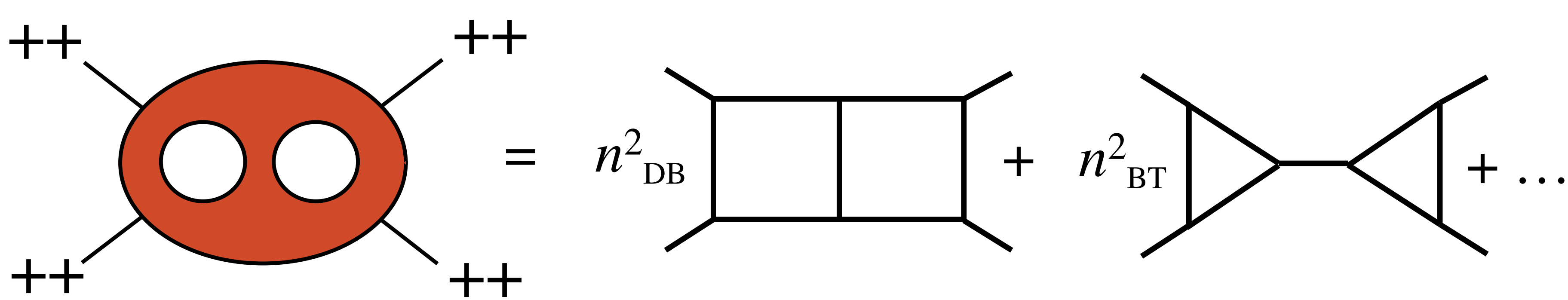
Works for non-susy theories

All-plus helicity QCD amplitude:

1004.0476 [hep-th]
Bern, Carrasco, HJ



All-plus helicity Einstein gravity amplitude:



(with dilation and axions in loops)

Lagrangian formulation

- Lagrangian formulation with manifest duality

1004.0693 [hep-th]
Bern, Dennen, Huang, Kiermaier

YM Lagrangian receives corrections at 5 points and higher

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \dots$$

corrections proportional to the Jacobi identity (thus equal to zero)

$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}\big([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho]\big)$$

Introduction of auxiliary "dynamical" fields gives local cubic Lagrangian

$$\mathcal{L}_{YM} = \tfrac{1}{2}A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - g f^{abc}(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu})A^{b\mu}A^{c\nu} \quad + \dots$$

"squaring" gives gravity Lagrangian.

$\rightarrow$ nonperturbative insight ?

Summary

- Studies of non-planar multi-loop amplitudes revealed the presence of a new hidden structure in gauge theory
- The structure is a duality between color and kinematics at tree-level, for generic gauge theories, including pure Yang-Mills
- Although the duality is not automatic (gauge invariant), enforcing it implies nontrivial relations for gauge invariant partial tree amplitudes
- The duality gives new local, manifestly crossing (Bose) symmetric relationship between gravity and gauge theory, clarifying KLT
- Nontrivial checks at two and tree loops shows that duality survives at the quantum level – natural extension of conjecture
- Lagrangian formulation, connection to string theory, give hints of future potential. <u>May be a key tool for nonplanar gauge theory</u>

Outlook

- For tree-level gauge theory the complete proof of duality is still missing
  - Show that n-pt tree amplitudes can be reorganized to make Jacobi identity manifest.
  - For which theories can this be done?
- Loop level evidence of duality needs to be completed by more nontrivial example, higher points, and more nontrivial QCD examples. Or by direct proof to all orders.
- How do we exploit the mapping between gravity and gauge theory Lagrangians? Can one match nonperturbative physics?
- What does it all mean? Is there "real physics" here?
- Is there an underlying "Lie group" for the kinematics, in analogy to the gauge group that underlies the color structure?
- What is the physical interpretation of gravity as a double copy of gauge theory? Compositeness?
- Detailed physical understanding awaits us!

Extra slides

Some comments on theories

Amplitude relations "derived" from abstract properties:

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_i \frac{n_i c_i}{\prod_\alpha p_\alpha^2}$$

$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

But which concrete theories satisfy the duality/relations?

- Hope: every (massless) gauge theory at tree level?
  - Pure ($\mathcal{N}$=0) YM evidence ✓ BCJ
  - YM + matter (single flavor) ✓ Søndergaard
  - $\mathcal{N}$=4,2,1,0 SYM (open string) ✓ Bjerrum-Bohr, Damgaard, and Vanhove
  - YM + 2 fermion flavors fails ✗ note: $A(f_1,f_2,f_1,f_2) = 0$

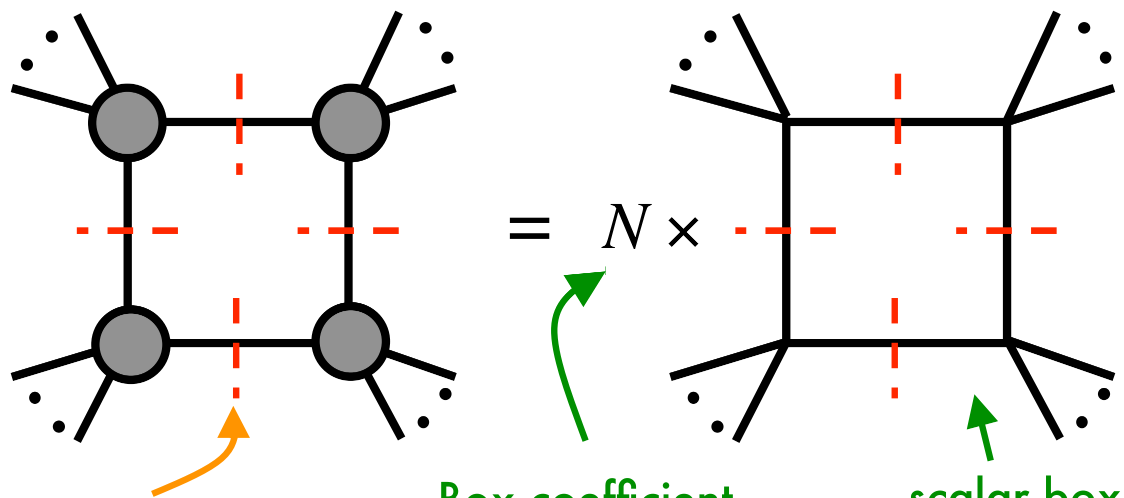
Naively any theory with more than one coupling constant fails!
– problems for spontaneously broken theories?
+ hints that formalism implies unification (→gravity implications)

Strong connection to string theory hints that formalism have more general validity than naive considerations suggests

Quadruple cut - the simplest cut

quadruple cut

Britto, Cachazo, Feng

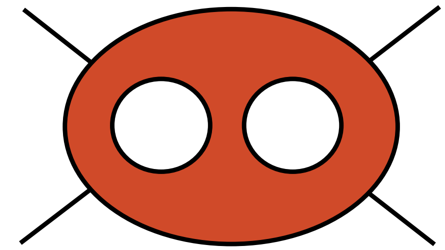


$$N = A^{\text{tree}}_{(1)} A^{\text{tree}}_{(2)} A^{\text{tree}}_{(3)} A^{\text{tree}}_{(4)}$$

On-shell conditions completely freezes momenta in *D*=4, requires complex momenta

Loop-Level Amplitudes

Some milestones for loop-level calculations:



- Feynman diagrams; ghosts, renormalization, dim. reg. etc. ('30-'70s)
- Unitarity - optical theorem ('60s)
- Passarino-Veltman reduction ('79), (1-loop integral basis: Box, triangle, bubble)
- String inspired methods, unitarity cuts, generalized unitarity ('90s)

  Bern, Dixon, Dunbar, Kosower
- Quaduple cuts ('04) Britto, Cachazo,Feng
- Multiloop generalizations: hepta/octa cuts, leading singularities, maximal cuts

  Buchbinder, Cachazo; Cachazo Skinner; Cachazo, Spradlin, Volovich; Bern, Carrasco, HJ, Kosower

Gauge theory amplitude properties

- Tree level, adjoint representation

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

- Well-known partial amplitude properties

$$A_n^{\text{tree}}(1,2,\ldots,n) = A_n^{\text{tree}}(2,\ldots,n,1)$$ cyclic symmetry

$$A_n^{\text{tree}}(1,2,\ldots,n) = (-1)^n A_n^{\text{tree}}(n,\ldots,2,1)$$ reflection symmetry

$(n - 1)!$

$$\sum_{\sigma\in\text{cyclic}} A_n^{\text{tree}}(1,\sigma(2,3,\ldots,n)) = 0$$ "photon"-decoupling identity

$$A_n^{\text{tree}}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in\text{OP}(\{\alpha\},\{\beta^T\})} A_n^{\text{tree}}(1,\{\sigma\}_i,n)$$ Kleiss-Kuijf relations

$(n - 2)!$

- *New relations reduce independent basis to* $(n - 3)!$ Bern, Carrasco, HJ

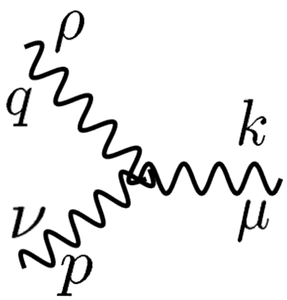
Pure Yang-Mills

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu}$$

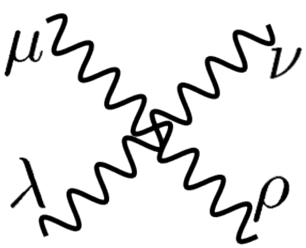
Feynman gauge propagator :

$$= \frac{-i\eta_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$$

vertices:



$$= -g f^{abc} \left[\eta_{\mu\nu}(k-p)_\rho + \eta_{\rho\mu}(q-k)_\nu + \eta_{\nu\rho}(p-q)_\mu \right]$$

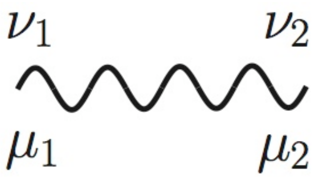


$$= \begin{array}{l} -ig^2 f^{abe} f^{ecd} (\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ -ig^2 f^{ace} f^{edb} (\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\ -ig^2 f^{ade} f^{ebc} (\eta_{\mu\nu}\eta_{\sigma\rho} - \eta_{\mu\rho}\eta_{\sigma\nu}) \end{array}$$

Pure Einstein Gravity

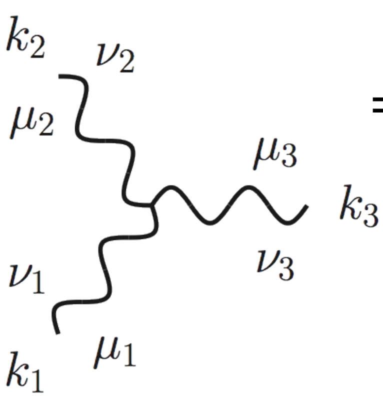
$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

propagator (de Donder gauge):



$$= \frac{1}{2}\left[\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right]\frac{i}{p^2+i\epsilon}$$

cubic vertex:



$$\begin{aligned}
= \mathrm{sym}[&-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}) - \frac{1}{2}P_6(k_{1\mu_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_3\nu_3}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3})\\
&+P_6(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}) + 2P_3(k_{1\mu_2}k_{1\nu_3}\eta_{\mu_1\nu_1}\eta_{\nu_2\mu_3}) - P_3(k_{1\nu_2}k_{2\mu_1}\eta_{\nu_1\mu_1}\eta_{\mu_3\nu_3})\\
&+P_3(k_{1\mu_3}k_{2\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + P_6(k_{1\mu_3}k_{1\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + 2P_6(k_{1\mu_2}k_{2\nu_3}\eta_{\nu_2\mu_1}\eta_{\nu_1\mu_3})\\
&+2P_3(k_{1\mu_2}k_{2\mu_1}\eta_{\nu_2\mu_3}\eta_{\nu_3\nu_1}) - 2P_3(k_1\cdot k_2\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}\eta_{\nu_3\mu_1})]
\end{aligned}$$

After symmetrization ~ 100 terms !

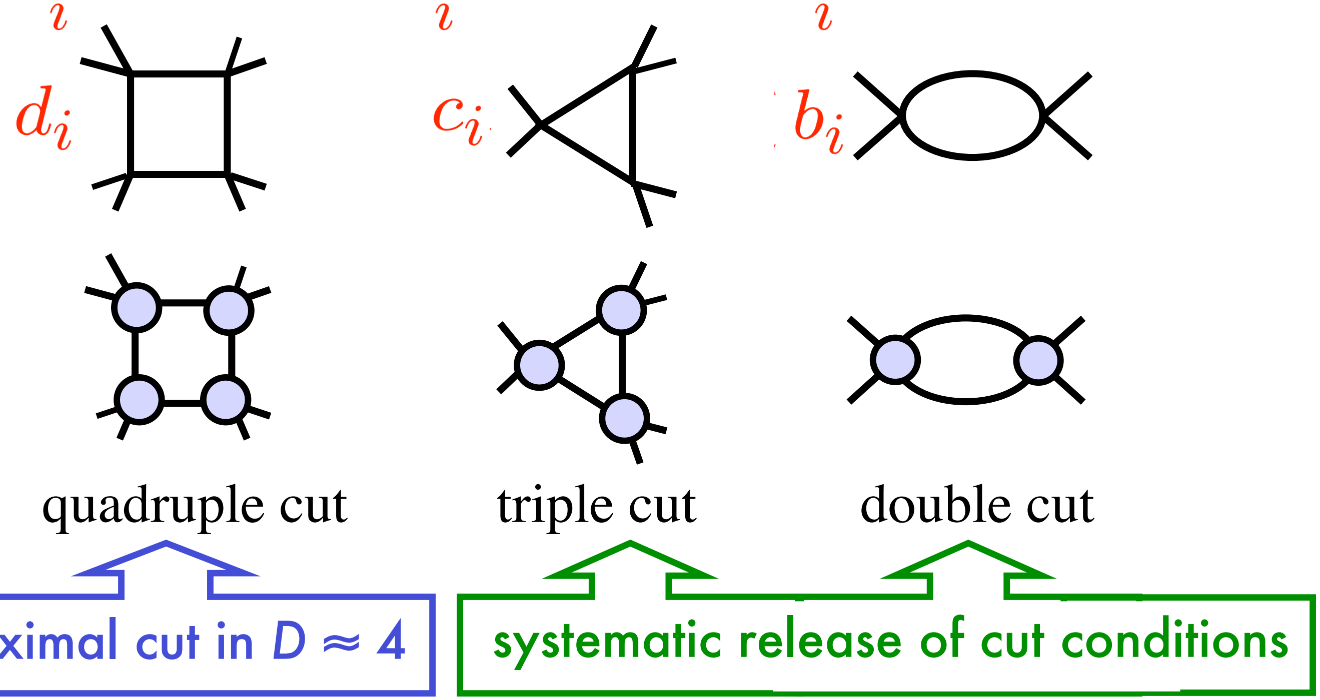
One-loop calculations

Maximal-cut method is similar to the strategy of one-loop calculations

One loop Integral basis well known:

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)} + \text{Rational}$$

$D = 4 - 2\varepsilon$



Having an integral basis is not necessary - but convenient

Details of duality conjecture - YM sector

- A gauge theory tree amplitude can be expanded in purely cubic diagrams

full amplitude $\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = g^{n-2} \sum_i \frac{n_i c_i}{(\prod_j p_j^2)_i}$

partial amplitude $A_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_j \frac{n_j}{(\prod_m p_m^2)_j}$

color factors $c_i = \tilde{f}^{abc} \tilde{f}^{cde} \ldots \tilde{f}^{xyz}$

- Jacobi identity true for both color and kinematics…

$$c_\alpha = c_\beta - c_\gamma \qquad \Leftrightarrow \qquad n_\alpha = n_\beta - n_\gamma$$

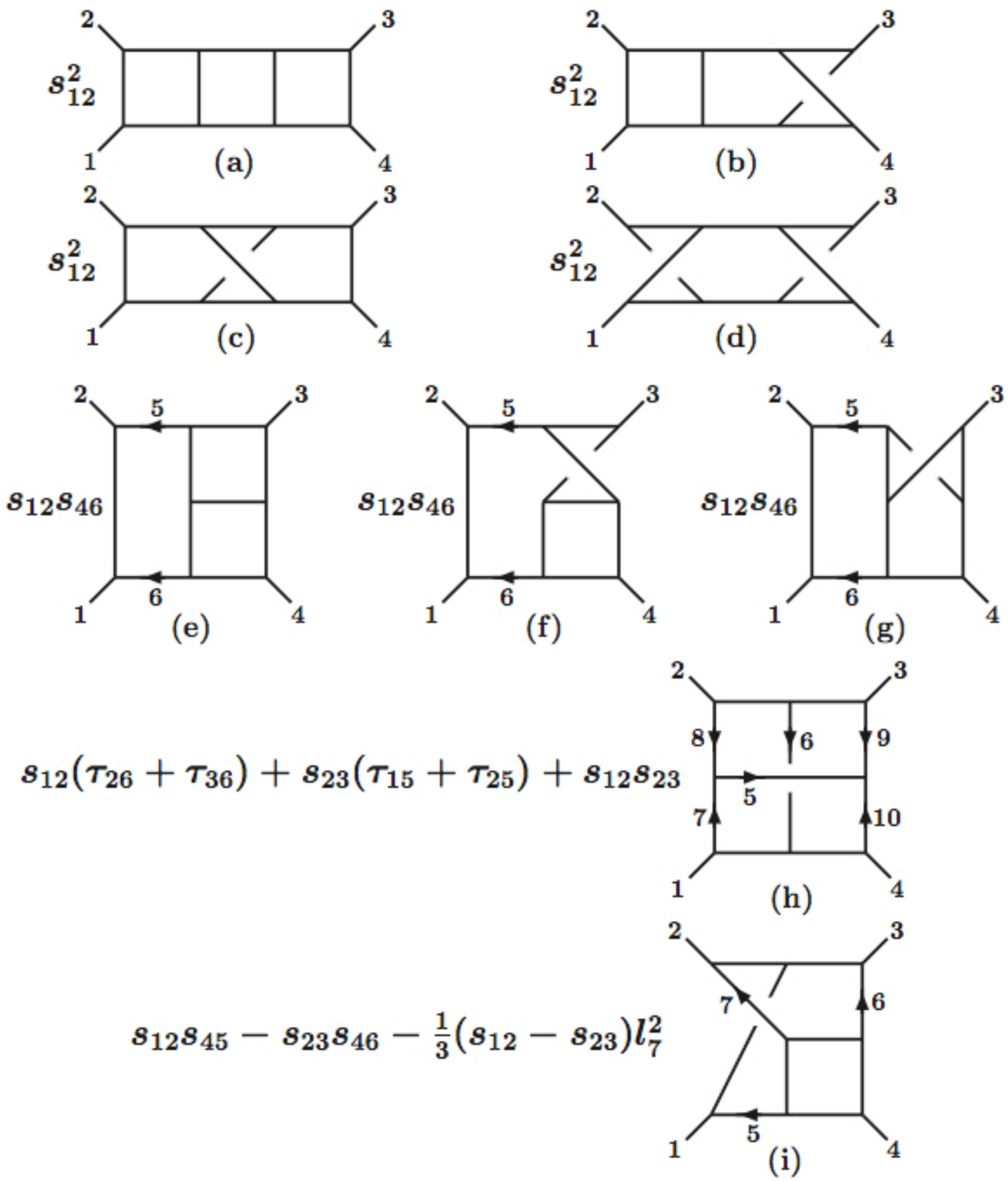
…as long as gauge invariance is enforced for $(n - 3)!$ partial amplitudes

$$A_n^{\text{tree}}(\mathcal{P}_i\{1,2,3,\ldots,n\}) = \left[\sum_j \frac{n_j}{(\prod_m p_m^2)_j}\right]_i$$

Checked through 8 pts!

$\Rightarrow$ only $(n - 3)!$ linearly independent partial amplitudes
- (down from $(n - 2)!$ for the Kleiss-Kuijf relations)

Old 3-loop $\mathcal{N}$=4 amplitude



Non-planar $\mathcal{N}$=4 more complicated

Used no established guiding principle for writing down integrals

Heuristic rules for some pieces are known: rung rule, etc.

We now have new organization principle: color-kinematics duality

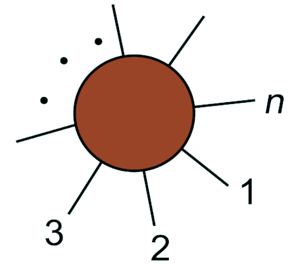
Integration is still challenging

$$s_{ij} = (k_i + k_j)^2$$
$$\tau_{ij} = 2k_i \cdot l_j$$

Bern, Carrasco, Dixon, HJ, Kosower, Roiban,
Bern, Carrasco, Dixon, HJ, Roiban

Gauge theory tree amplitudes

Some milestones for tree-level calculations:

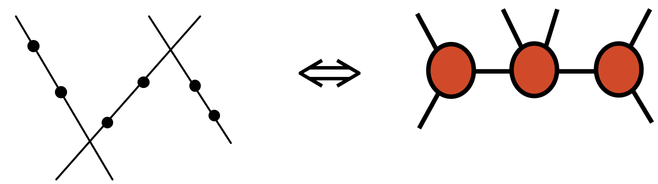


- **Feynman** diagrams ('40s)

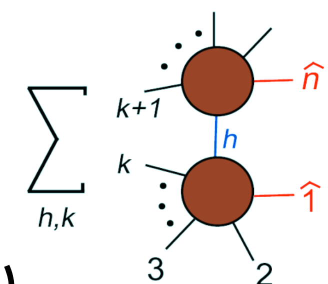
- **Park-Taylor** MHV formula ('86)

$$\frac{\delta^{(8)}(Q)}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle\, \langle 3\, 4\rangle \cdots \langle n-1\, n\rangle\, \langle n\, 1\rangle}$$

- **Berends-Giele** off-shell recursion ('88)
- **Cachazo-Svrcek-Witten (CSW)** rules ('04)



- **Britto-Cachazo-Feng-Witten (BCFW)** on-shell recursion ('04)



- **Drummond-Henn** tree formulas for Yang-Mills (& $\mathcal{N}$=4 SYM) ('09)
- Grassmannian integrals ('09) **Arkani-Hamed, Cachazo, Cheung, Kaplan; Mason, Skinner**