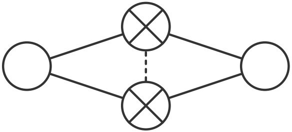
Quantum Deformations of Worldsheet Scattering

Niklas Beisert

MPI für Gravitationsphysik
Albert-Einstein-Institut, Potsdam

Integrability in Gauge and String Theory 2010

NORDITA Stockholm
30 June 2010



`arxiv:1002.1097` and
work in progress with Wellington Galleas & Takuya Matsumoto

Motivation

We know a lot about AdS/CFT integrability:

- Lax connection, spectral curve,
- Bethe ansatz, magnons, S-matrix, phase, asymptotic BE,
- Lüscher-type corrections, mirror theory, Hirota, Why-System, TBA

What do we know for sure (in the sense of: derivation, proof, proof)?

- . . . !
- . . . !?
- (. . .)!
- . . . ?
- ?

This talk towards:

- Understand algebra behind S-matrix in detail.
- How does it relate to established classical and quantum algebra?
- Let's quantum deform!

Integrability Algebras

Rational Classical
- classical scattering problems,
- untwisted affine Kac–Moody,
- Lie bialgebra structure.

$\leftarrow$

Rational Quantum
- XXX-like models,
- (double) Yangian algebra,
- partial deformation of UEA.

$\uparrow$ $\uparrow$

Trigonometric Classical
- classical scattering problems,
- any affine Kac–Moody,
- Lie bialgebra structure'.

$\leftarrow$

Quantum-Deformed
- XXZ-like models, q parameter.
- quantum affine algebra,
- full deformation of UEA.

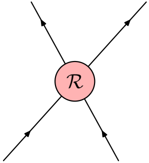
Y Trigonometric and Quantum-Deformed? (ignore elliptic/XYZ cases)
- Lose manifest Lie symmetry. Gain structural symmetry.
- Rational quantum algebra contains all.
  Everything else as (singular) limits: $q \to 1$, contractions.
- Relevant to Pohlmeyer reduced models & their quantisation. e.g. [Hollowood Miramontes]
  AdS/CFT: lost/manifest $\mathfrak{su}(2)$'s. How to match? [Grigoriev Tseytlin] [Mikhailov Schäfer-Nameki]

I. Worldsheet S-Matrix

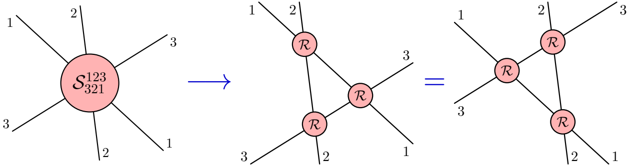
Magnon Scattering

Reminder: spin chain/worldsheet scattering picture

- infinitely extended worldsheet,
- $16 = 4 \times 4$ flavours of magnons,
- extended $\mathfrak{su}(2|2) \times \mathfrak{su}(2|2)$ residual symmetry,
- some dispersion relation,
- 2-particle scattering matrix,



- integrability: factorised multi-particle scattering, YBE.



Extended $\mathfrak{sl}(2|2)$ Bialgebra

Extended $\mathfrak{sl}(2|2)$ algebra (nevermind reality, $\mathfrak{su}(2|2)$) [NB hep-th/0511082][Gomez Hernández][Plefka Spill Torrielli]

$$\mathfrak{sl}(2)^2\text{: } \mathrm{R}^{ab}, \mathrm{L}^{\alpha\beta}, \quad \text{susy: } \mathrm{Q}^{\alpha b}, \mathrm{S}^{\alpha b}, \quad \text{centre: } \mathrm{T}, \mathrm{U}.$$

Algebra: like $\mathfrak{sl}(2|2)$, but susy relations extended

$$\begin{aligned}
\{\mathrm{Q}^{\alpha b}, \mathrm{Q}^{\gamma d}\} &= 2g\alpha\varepsilon^{\alpha\gamma}\varepsilon^{bd}(1 - \mathrm{U}^{+2}),\\
\{\mathrm{Q}^{\alpha b}, \mathrm{S}^{\gamma d}\} &= \varepsilon^{bd}\mathrm{L}^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\mathrm{R}^{bd} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}\varepsilon^{bd}\mathrm{T},\\
\{\mathrm{S}^{\alpha b}, \mathrm{S}^{\gamma d}\} &= 2g\alpha^{-1}\varepsilon^{\alpha\gamma}\varepsilon^{bd}(1 - \mathrm{U}^{-2}).
\end{aligned}$$

- Two central charges T, U;
- one coupling constant g;
- one normalisation α (Q vs. S).

Coalgebra: Trivial, but susy coproducts braided by group-like U

$$\Delta(\mathrm{Q}) = \mathrm{Q} \otimes 1 + \mathrm{U}^{+1} \otimes \mathrm{Q}, \qquad \Delta(\mathrm{S}) = \mathrm{S} \otimes 1 + \mathrm{U}^{-1} \otimes \mathrm{S},$$

$$\Delta(\mathrm{U}) = \mathrm{U} \otimes \mathrm{U}.$$

Fundamental Representation

Ansatz for 4D fundamental representation on states $|\phi^a\rangle$, $|\psi^\alpha\rangle$

$$\mathrm{Q}^{\alpha b}|\phi^c\rangle = a\,\varepsilon^{bc}|\psi^\alpha\rangle, \qquad \mathrm{Q}^{\alpha b}|\psi^\gamma\rangle = -b\,\varepsilon^{\alpha\gamma}|\phi^b\rangle,$$
$$\mathrm{S}^{\alpha b}|\phi^c\rangle = -c\,\varepsilon^{bc}|\psi^\alpha\rangle, \qquad \mathrm{S}^{\alpha b}|\psi^\gamma\rangle = d\,\varepsilon_{\alpha\gamma}|\phi^b\rangle.$$

6 parameters (a, b, c, d, T, U), 4 constraints

$$ad - bc = 1, \quad ab = g\alpha(1 - U^2), \quad cd = g\alpha^{-1}(1 - U^{-2}), \quad ad + bc = T.$$

2-parameter family of representations: 1. $x^{(\pm)}$ and 2. γ (normalization).

Higher Representations: [NB nlin.SI/0610017]

- constructible from tensor products of fundamentals using coproduct Δ;
- long representations: charges T and U free (off-shell),
- short representations: charges T and U related (on-shell),
- analogous to standard $\mathfrak{su}(2|2)$ representation theory.

S-Matrix

Fundamental S-matrix $\mathcal{R} : \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2} \to \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2}$ [NB hep-th/0511082]

$$\begin{aligned}
\mathcal{R}|\phi^a\phi^b\rangle &= \tfrac{1}{2}(A+B)|\phi^b\phi^a\rangle + \tfrac{1}{2}(A-B)|\phi^a\phi^b\rangle - \tfrac{1}{2}C\varepsilon^{ab}\varepsilon_{\gamma\delta}|\psi^\gamma\psi^\delta\rangle,\\
\mathcal{R}|\psi^\alpha\psi^\beta\rangle &= -\tfrac{1}{2}(D+E)|\psi^\beta\psi^\alpha\rangle - \tfrac{1}{2}(D-E)|\psi^\alpha\psi^\beta\rangle + \tfrac{1}{2}F\varepsilon^{\alpha\beta}\varepsilon_{cd}|\phi^c\phi^d\rangle,\\
\mathcal{R}|\phi^a\psi^\beta\rangle &= G|\phi^a\psi^\beta\rangle + H|\psi^\beta\phi^a\rangle,\\
\mathcal{R}|\psi^\alpha\phi^b\rangle &= K|\psi^\alpha\phi^b\rangle + L|\phi^b\psi^\alpha\rangle.
\end{aligned}$$

Coefficient functions $A, \ldots, L$ uniquely determined by cocommutativity

$$\Delta_{\mathrm{op}}(\mathrm{J}) = \mathcal{R}^{-1}\Delta(\mathrm{J})\mathcal{R}.$$

S-matrix automatically satisfies YBE $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

Higher Representations: [Chen Dorey Okamura] [Arutyunov Frolov]

- S-matrices not uniquely determined by cocommutativity,
- YBE required, but non-linear.

Yangian

Integrability usually related to infinite-dimensional algebra: Yangian [Drinfel'd 1985]

- based on (half) loop algebra J_n, $n \geq 0$,
- quantum algebra: deformation of UEA.

Start with extended $\mathfrak{sl}(2|2)$ Lie algebra generated by J^A (level-zero)

$$[\mathrm{J}^A, \mathrm{J}^B\} = f^{AB}_C \mathrm{J}^C, \qquad \Delta(\mathrm{J}^A) = \mathrm{J}^A \otimes 1 + \mathrm{U}^{[A]} \otimes \mathrm{J}^A.$$

Introduce level-one generators $\widehat{\mathrm{J}}^A$. Adjoint/coproduct/Serre: [NB 0704.0400]

$$[\mathrm{J}^A, \widehat{\mathrm{J}}^B\} = f^{AB}_C \widehat{\mathrm{J}}^C,$$

$$\Delta(\widehat{\mathrm{J}}^A) = \widehat{\mathrm{J}}^A \otimes 1 + \mathrm{U}^{[A]} \otimes \widehat{\mathrm{J}}^A + \hbar f^A_{BC} \mathrm{J}^B \mathrm{U}^{[C]} \otimes \mathrm{J}^C,$$

$$\big[[\mathrm{J}^A, \widehat{\mathrm{J}}^B\}, \widehat{\mathrm{J}}^C\big\} + 2 \text{ cyclic} = \tfrac{1}{6}\hbar^2 a^{ABC}_{DEF}\{\mathrm{J}^D, \mathrm{J}^E, \mathrm{J}^F].$$

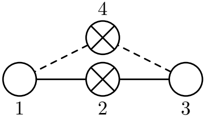
- Evaluation representation: $\widehat{\mathrm{J}}^A \simeq u\, \mathrm{J}^A$ ($x^\pm$ and u related).
- All R-matrices uniquely determined (apparently). [de Leeuw 0804.1047] [Arutyunov de Leeuw Torrielli]
- Understanding not complete: Role of U? Further generators? ...

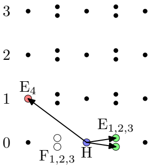
Chevalley–Serre Presentation

Only levels 0 and 1 used. Higher levels follow from algebra.
Can even reduce the basis of essential generators further.

raising	Cartan	lowering
$E_1 \sim L^{11}$,	$H_1 \sim L^{12}$,	$F_1 \sim L^{22}$,
$E_2 \sim Q^{22}$,	$H_2 \sim -\frac{1}{2}H_1 - \frac{1}{2}H_3 - C$,	$F_2 \sim S^{11}$,
$E_3 \sim R^{11}$,	$H_3 \sim R^{12}$,	$F_3 \sim R^{22}$,
$E_4 \sim \widehat{S}^{22}$,	no H_4,	no F_4.

$$\left(\begin{array}{cc|cc} H & E_1 & & \\ F_1 & H & E_2 & \\ \hline & F_2 & H & E_3 \\ E_4 & & F_3 & H \end{array}\right)$$





Algebra Relations

Mixed brackets (A_{jk} symmetric Cartan matrix)

$$[\mathrm{H}_j, \mathrm{E}_k] = +A_{jk}\mathrm{E}_k, \quad [\mathrm{H}_j, \mathrm{F}_k] = -A_{jk}\mathrm{F}_k, \quad [\mathrm{E}_j, \mathrm{F}_k\} = \pm\delta_{jk}\mathrm{H}_k.$$

Serre relations (commuting generators & Jacobi identities)

$$0 = [\mathrm{E}_1, \mathrm{E}_3] = \{\mathrm{E}_2, \mathrm{E}_2\} = \big[[\mathrm{E}_2, \mathrm{E}_1], \mathrm{E}_1\big] = \big[[\mathrm{E}_2, \mathrm{E}_3], \mathrm{E}_3\big],$$

$$0 = [\mathrm{F}_1, \mathrm{F}_3] = \{\mathrm{F}_2, \mathrm{F}_2\} = \big[[\mathrm{F}_2, \mathrm{F}_1], \mathrm{F}_1\big] = \big[[\mathrm{F}_2, \mathrm{F}_3], \mathrm{F}_3\big].$$

Extended susy relations

$$\big\{[\mathrm{E}_2, \mathrm{E}_1], [\mathrm{E}_2, \mathrm{E}_3]\big\} = g\alpha^{+1}(1 - \mathrm{U}^{+2}),$$

$$\big\{[\mathrm{F}_2, \mathrm{F}_1], [\mathrm{F}_2, \mathrm{F}_3]\big\} = g\alpha^{-1}(1 - \mathrm{U}^{-2}).$$

Coalgebra (with $[k] = \delta_{k,2}$)

$$\Delta(\mathrm{E}_k) = \mathrm{E}_k \otimes 1 + \mathrm{U}^{+[k]} \otimes \mathrm{E}_k, \quad \Delta(\mathrm{F}_k) = \mathrm{F}_k \otimes 1 + \mathrm{U}^{-[k]} \otimes \mathrm{F}_k.$$

Relations for E_4 similar to E_2, but not all established. E.g. $\Delta(\mathrm{E}_4)$ messy.

II. Quantum Deformation

Quantum-Deformed Extended $\mathfrak{sl}(2|2)$

Have all ingredients to address quantum deformations. $\left[{}^{\text{NB}}_{\text{Koroteev}}\right]$

Consider first extended $\mathfrak{sl}(2|2)$ with

- deformation parameter q (and coupling constant g, normalisation α),
- generators E_k, F_k, $\mathrm{K}_k = q^{\mathrm{H}_k}$ (Cartan charges exponentiated)
- central charges U, $\mathrm{V} = q^{\mathrm{T}}$ (note $\mathrm{V}^2 = \mathrm{K}_1^{-1}\mathrm{K}_2^{-2}\mathrm{K}_3^{-1}$).

Mixed relations:

$$\mathrm{K}_j\mathrm{E}_k = q^{A_{jk}}\mathrm{E}_k\mathrm{K}_j, \quad \mathrm{F}_k\mathrm{K}_j = q^{A_{jk}}\mathrm{K}_j\mathrm{F}_k, \quad [\mathrm{E}_j, \mathrm{F}_k\} = \pm\delta_{jk}\frac{\mathrm{K}_k - \mathrm{K}_k^{-1}}{q - q^{-1}}\,.$$

Serre and extended susy relations, similar for F's: (with $j = 1, 3$)

$$0 = [\mathrm{E}_1, \mathrm{E}_3] = \{\mathrm{E}_2, \mathrm{E}_2\} = \big[\mathrm{E}_j, [\mathrm{E}_j, \mathrm{E}_2]\big] - (q - 2 + q^{-1})\mathrm{E}_j\mathrm{E}_2\mathrm{E}_j.$$

$$\big\{[\mathrm{E}_2, \mathrm{E}_1], [\mathrm{E}_2, \mathrm{E}_3]\big\} - (q - 2 + q^{-1})\mathrm{E}_2\mathrm{E}_1\mathrm{E}_3\mathrm{E}_2 = g\alpha(1 - \mathrm{U}^2\mathrm{V}^2).$$

Coalgebra compatible with algebra:

$$\Delta(\mathrm{E}_k) = \mathrm{E}_k \otimes 1 + \mathrm{K}_k^{-1}\mathrm{U}^{+[k]} \otimes \mathrm{E}_k, \quad \Delta(\mathrm{F}_k) = \mathrm{F}_k \otimes \mathrm{K}_k + \mathrm{U}^{-[k]} \otimes \mathrm{F}_k.$$

Fundamental Representation

4D fundamental representation

$$
\mathrm{E}_1 \simeq \left(\begin{array}{cc|cc} 0&1&0&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&0&0&0 \end{array}\right), \qquad
\mathrm{E}_2 \simeq \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&a&0\\ \hline 0&0&0&0\\ b&0&0&0 \end{array}\right), \qquad
\mathrm{E}_3 \simeq \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&1\\ 0&0&0&0 \end{array}\right),
$$

$$
\mathrm{F}_1 \simeq \left(\begin{array}{cc|cc} 0&0&0&0\\ 1&0&0&0\\ \hline 0&0&0&0\\ 0&0&0&0 \end{array}\right), \qquad
\mathrm{F}_2 \simeq -\left(\begin{array}{cc|cc} 0&0&0&c\\ 0&0&0&0\\ \hline 0&d&0&0\\ 0&0&0&0 \end{array}\right), \qquad
\mathrm{F}_3 \simeq \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&0&1&0 \end{array}\right).
$$

6 parameters (a, b, c, d, U, V), 4 constraints

- 2-parameter family ($x^{(\pm)}$, γ),
- deformation of undeformed case.

Fundamental S-matrix $\mathcal{R} : \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2} \to \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2}$

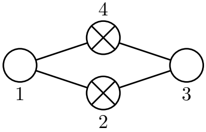
- 10 coefficient functions uniquely determined by cocommutativity,
- similar to undeformed S-matrix, just deformed,
- no manifest $\mathfrak{sl}(2)^2$ invariances,
- R-matrix for Alcaraz–Bariev model; thus quantum-deformed Hubbard.

No shocking surprises, just deformations.

Extended Quantum Affine $\mathfrak{sl}(2|2)$

Consider quantum affine extension. Affine $\mathfrak{sl}(2|2)$ Kac–Moody [NB Galleas Matsumoto]

$$A = \begin{pmatrix} +2 & -1 & & -1 \\ -1 & & +1 & \\ & +1 & -2 & +1 \\ -1 & & +1 & \end{pmatrix}$$



- Observe: node 4 is a copy of node 2; fully interchangeable!
- $\mathrm{E}_4, \mathrm{K}_4, \mathrm{F}_4$ should obey same relations as $\mathrm{E}_2, \mathrm{K}_2, \mathrm{F}_2$.
- Can use different constants $(g, \alpha) \to (g_2, \alpha_2), (g_4, \alpha_4)$.
- Can use different central charges $(\mathrm{U}, \mathrm{V}) \to (\mathrm{U}_2, \mathrm{V}_2), (\mathrm{U}_4, \mathrm{V}_4)$.

$$\big\{[\mathrm{E}_2, \mathrm{E}_1], [\mathrm{E}_2, \mathrm{E}_3]\big\} - (q - 2 + q^{-1})\mathrm{E}_2\mathrm{E}_1\mathrm{E}_3\mathrm{E}_2 = g_2\alpha_2(1 - \mathrm{U}_2^2\mathrm{V}_2^2),$$

$$\big\{[\mathrm{E}_4, \mathrm{E}_1], [\mathrm{E}_4, \mathrm{E}_3]\big\} - (q - 2 + q^{-1})\mathrm{E}_4\mathrm{E}_1\mathrm{E}_3\mathrm{E}_4 = g_4\alpha_4(1 - \mathrm{U}_4^2\mathrm{V}_4^2),$$

$$\Delta(\mathrm{E}_2) = \mathrm{E}_2 \otimes 1 + \mathrm{K}_2^{-1}\mathrm{U}_2^{+1} \otimes \mathrm{E}_2,$$

$$\Delta(\mathrm{E}_4) = \mathrm{E}_4 \otimes 1 + \mathrm{K}_4^{-1}\mathrm{U}_4^{+1} \otimes \mathrm{E}_4.$$

Mixed Commutators

What about $\{\mathrm{E}_2, \mathrm{F}_4\}$ and $\{\mathrm{E}_4, \mathrm{F}_2\}$?
Compatibility with coproduct constrains form

$$\{\mathrm{E}_4, \mathrm{F}_2\} = -\tilde{g}\tilde{\alpha}^{+1}(\mathrm{K}_2 - \mathrm{U}_2^{-1}\mathrm{U}_4\mathrm{K}_4^{-1}),$$
$$\{\mathrm{E}_2, \mathrm{F}_4\} = +\tilde{g}\tilde{\alpha}^{-1}(\mathrm{K}_4 - \mathrm{U}_4^{-1}\mathrm{U}_2\mathrm{K}_2^{-1}).$$

- another coupling $\tilde{g}$ and normalisation $\tilde{\alpha}$.

Consideration of fundamental representation suggests

$$g := g_2 = g_4 = \frac{\tilde{q} - \tilde{q}^{-1}}{2i(q - q^{-1})},$$
$$\tilde{g} = \frac{i(\tilde{q} - \tilde{q}^{-1})}{(q - q^{-1})(\tilde{q} + \tilde{q}^{-1})},$$
$$\alpha := \alpha_2 = \alpha_4\tilde{\alpha}^{-2}.$$

Fundamental Representation

Ansatz for fundamental representation ($k=2,4$)

$$\mathrm{E}_k \simeq \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & a_k & 0 \\ \hline 0 & 0 & 0 & 0 \\ b_k & 0 & 0 & 0 \end{array}\right), \qquad \mathrm{F}_k \simeq -\left(\begin{array}{cc|cc} 0 & 0 & 0 & c_k \\ 0 & 0 & 0 & 0 \\ \hline 0 & d_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

- 12 parameters ($a_k, b_k, c_k, d_k, U_k, V_k$), 10 constraints,
- 2-parameter family ($x^{(\pm)}$, γ),
- conjugate representations of nodes 2 and 4 (antipodes):

$$U_4 = U_2^{-1}, \quad V_4 = V_2^{-1}, \quad x_4^\pm = s(x_2^\pm) = \frac{1 - \frac{1}{2}(\tilde{q} - \tilde{q}^{-1})x_2^\pm}{x_2^\pm - \frac{1}{2}(\tilde{q} - \tilde{q}^{-1})}\,.$$

- fundamental S-matrix $\mathcal{R}$ invariant under full quantum affine!

Conclusions Extended $\mathfrak{sl}(2|2)$ Quantum Affine

Structure of algebra:

- 4 sets of raising/Cartan/lowering generators $\mathrm{E}_k, \mathrm{F}_k, \mathrm{K}_k$, $k = 1, 2, 3, 4$,
- 4 central charges $\mathrm{U}_k, \mathrm{V}_k$, $k = 2, 4$, (V_k among K_k),
- 2-parameter family of 4D fundamental representations,

Summary of parameters:

- quantum deformation parameter q,
- coupling constant $\tilde{q}$,
- generator normalizations $\alpha, \tilde{\alpha}$.

Outlook:

- Explore structure of the algebra more closely.
  E.g. similarity $\mathrm{U}_k \leftrightarrow \mathrm{V}_k$ and $q \leftrightarrow \tilde{q}$.
- Add and understand affine $\mathfrak{gl}(1)$ for extended affine $\mathfrak{gl}(2|2)$.
- Add and understand affine derivation (affine charge in $\mathrm{V}_2\mathrm{V}_4$).

III. Classical Trigonometric

Classical Limit

Y Classical Limit?

- Affine Lie algebra much simpler than quantum deformed UEA.
- Deformation only in coalgebra encoded only in r-matrix.
- Explore structure of algebra (affine derivation, affine $\mathfrak{gl}(1)$).
- Explore moduli space: $q, \tilde{q}$
- Surprises?

How? Classical limit

- of a quantum-deformed algebra: $q \to 1$, keep u in $z = q^u$;
- of AdS/CFT scattering problem: $g \to \infty$, keep $x^\pm$ finite.

Classical limit of quantum-deformed AdS/CFT scattering: [NB 1002.1097]

$$g \to \infty\,, \quad q = 1 + \frac{h}{2g} + \mathcal{O}(g^{-2})\,, \quad x^\pm \to x.$$

Curiously, one parameter h remains in classical description!

Classical Algebra

Back to full loop algebra spanned by $z^n \mathrm{J}^A$.
Use experience from undeformed/rational algebra. [NB Spill]
Susy brackets to central charge T

$$\begin{aligned}
\{\mathrm{Q}^{\alpha b}, \mathrm{Q}^{\gamma d}\} &= \varepsilon^{\alpha\gamma}\varepsilon^{bd} W_{12}(z)\,\mathrm{T},\\
\{\mathrm{Q}^{\alpha b}, \mathrm{S}^{\gamma d}\} &= -\varepsilon^{\alpha\gamma}\mathrm{R}^{bd} + \varepsilon^{bd}\mathrm{L}^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\varepsilon^{bd} W_{11}(z)\,\mathrm{T},\\
\{\mathrm{S}^{\alpha b}, \mathrm{S}^{\gamma d}\} &= -\varepsilon^{\alpha\gamma}\varepsilon^{bd} W_{21}(z)\,\mathrm{T}.
\end{aligned}$$

Introduce a $\mathfrak{gl}(1)$ transformation W

$$\begin{aligned}
[\mathrm{W}, \mathrm{Q}^{\alpha b}] &= W_{11}(z)\,\mathrm{Q}^{\alpha b} + W_{12}(z)\,\mathrm{S}^{\alpha b},\\
[\mathrm{W}, \mathrm{S}^{\alpha b}] &= W_{21}(z)\,\mathrm{Q}^{\alpha b} + W_{22}(z)\,\mathrm{S}^{\alpha b}.
\end{aligned}$$

For the trigonometric case we find the matrix [NB 1002.1097]

$$W(z) = \begin{pmatrix} +h^{-1}(z-1) & 2\alpha \\ 2\alpha^{-1} z & -h^{-1}(z-1) \end{pmatrix}.$$

Fundamental Representation

Ansatz for fundamental evaluation representation

$$
\begin{aligned}
\mathrm{Q}^{\alpha b}|\phi^c\rangle &= T_{11}\,\varepsilon^{bc}|\psi^\alpha\rangle, & \mathrm{Q}^{\alpha b}|\psi^\gamma\rangle &= T_{12}\,\varepsilon^{\alpha\gamma}|\phi^b\rangle,\\
\mathrm{S}^{\alpha b}|\phi^c\rangle &= T_{21}\,\varepsilon^{bc}|\psi^\alpha\rangle, & \mathrm{S}^{\alpha b}|\psi^\gamma\rangle &= T_{22}\,\varepsilon^{\alpha\gamma}|\phi^b\rangle,\\
\mathrm{T}|\phi^a\rangle &= \tfrac{1}{2}q\,|\phi^a\rangle, & \mathrm{T}|\psi^\alpha\rangle &= \tfrac{1}{2}q\,|\psi^\alpha\rangle,\\
\mathrm{W}|\phi^a\rangle &= -\tfrac{1}{2}q^{-1}|\phi^a\rangle, & \mathrm{W}|\psi^\alpha\rangle &= +\tfrac{1}{2}q^{-1}|\psi^\alpha\rangle.
\end{aligned}
$$

Solved for parameters z, q, T in terms of x, γ

$$
z(x) = \frac{i(x+ih)}{x(hx+i)}\,, \qquad q(x) = \frac{-x(hx+i)}{x^2+2ihx-1}\,,
$$

$$
T(x,\gamma) = \begin{pmatrix} \gamma & -\alpha\gamma^{-1}q \\ -i\alpha^{-1}\gamma x^{-1} & i\gamma^{-1}xzq \end{pmatrix}.
$$

Note that $TM = qWT$ with $M = \mathrm{diag}(+1,-1)$ required for compatibility.

Classical r-Matrix

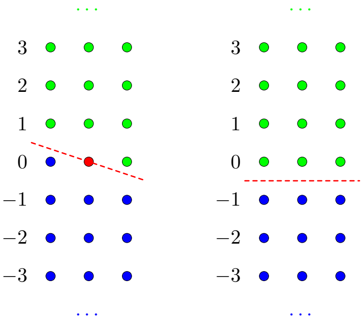
Standard form of trigonometric r-matrix yields classical limit of $\mathcal{R}$

$$r_{12} = \frac{z_1}{z_1 - z_2}\, s_{12} + \frac{z_2}{z_1 - z_2}\, s_{21}, \qquad \text{compare } r_{12}^{\text{rat}} = \frac{s_{12} + s_{21}}{u_1 - u_2}\,,$$

Here $s_{12} = \sum_a \mathrm{J}_a \otimes \mathrm{J}^a$ with $\mathrm{J}_a/\mathrm{J}^a$ in negative/positive subalgebra.
Quasi-triangular Lie bialgebra: r satisfies CYBE (algebraically)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Expansion of r_{12} in z element of negative $\otimes$ positive subalgebra. triangular decomposition (trigonometric/rational)



Affine Derivation

Affine Kac–Moody algebra consists of

- loop algebra $z^n \mathrm{J}^A$,
- affine derivation zd/dz and
- affine central charge (trivial for evaluation representations).

Y Affine Derivation?

- Evaluation representation $D(x, \gamma)$ belongs to loop algebra.
- Affine derivation shifts $x(z)$: affine representation is $\oplus_x D(x, \gamma(x))$
- Physically: 2D on-shell field. zd/dz represents "Lorentz" boost.
- Affine $\mathfrak{gl}(1)$ (W) momentum-dependent shift $\delta\gamma(x)$: Gauge transf.!

Concretely for our extended $\mathfrak{gl}(2|2)$ affine algebra: [Young 0704.2069] [Gómez Hernández] [NB 1002.1097]

- Non-trivial action on $\mathrm{Q}, \mathrm{S}, \mathrm{T}, \mathrm{W}$.
- r-Matrix not invariant under affine derivation zd/dz.
- Violation of difference form of r,
- deformed Lorentz transformations through cobrackets.

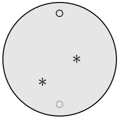
Limits

Trigonometric r-matrix $r_{12}(x_1, x_2; h)$ should have various limits:

- classical AdS/CFT worldsheet s-matrix,
- conventional trigonometric r-matrix,
- ...

How to find & understand the various limits?

- Consider the 4 singular points: $z^\circ_\pm = 0, \infty$ and $z^*_\pm$ (self-dual).
- Limit means zooming into neighbourhood of some point z.
- Configuration of singular points depends on constant h.
- Can move singular points around while zooming in.

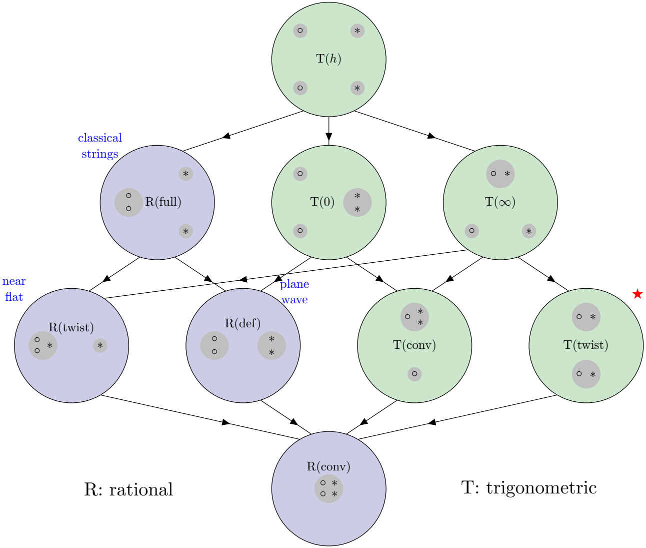


Two insights:

- Zooming in means moving everything else to a common point.
- Constant h is a Moebius invariant of $z^\circ_\pm$ and $z^*_\pm$.

Move singular points around freely; limits at various coincidences.

Cascade of Limits



R: rational

T: trigonometric

Twisted Trigonometric Case

Trigonometric r-matrix (excerpt)

$$\begin{aligned}
r|\phi^1\phi^1\rangle &= A|\phi^1\phi^1\rangle,\\
r|\phi^1\phi^2\rangle &= \tfrac{1}{2}(A+B+1)|\phi^2\phi^1\rangle + \tfrac{1}{2}(A-B)|\phi^1\phi^2\rangle - \tfrac{1}{2}C\varepsilon_{\alpha\beta}|\psi^\alpha\psi^\beta\rangle,\\
r|\phi^2\phi^1\rangle &= \tfrac{1}{2}(A-B)|\phi^2\phi^1\rangle + \tfrac{1}{2}(A+B-1)|\phi^1\phi^2\rangle + \tfrac{1}{2}C\varepsilon_{\alpha\beta}|\psi^\alpha\psi^\beta\rangle,\\
r|\phi^2\phi^2\rangle &= A|\phi^2\phi^2\rangle.
\end{aligned}$$

Coefficient functions for twisted trigonometric case

$$A = \frac{1}{4}\,\frac{y_1+y_2}{y_1-y_2}\,,\quad \tfrac{1}{2}(A-B) = \frac{1}{4}\,\frac{y_1-y_2}{y_1+y_2}\,,\quad C \simeq \frac{1}{2}\,\frac{1}{y_1+y_2}\,,\quad \ldots$$

- Surprise Coefficients match precisely with Pohlmeyer reduction. [Hoare Tseytlin]
- However, matrix structure not trigonometric but rational! Mistake?
- Contra: trigonometric structure breaks $\mathfrak{su}(2)$'s.
- Pro: YBE requires trigonometric matrix structure.
- Pro: denominator has desired Lorentz form $\sinh(\vartheta_1-\vartheta_2)$. ?!

Conclusions Classical Trigonometric

Structure of classical algebra:

- vector space of affine $\mathfrak{gl}(2|2)$,
- one-parameter deformation of conventional algebra,
- affine extensions: deformed Lorentz boost,
- standard classical r-matrix,
- new quasi-triangular Lie bialgebra.

Special limits:

- many limiting cases exist (algebraic contractions),
- all previously known $\mathfrak{gl}(2|2)$ r-matrices contained in one.

Outlook:

- Elliptic generalisation exist?
- Quantise and compare with quantum affine.