QUANTUM PHASES OF A SUPERSYMMETRIC MODEL FOR LATTICE FERMIONS

Liza Huijse – University of Amsterdam

Sept 9, 2010

INSTANS conference – Nordita, Stockholm

COLLABORATORS AND REFERENCES

- **K. Schoutens**, UvA
- **P. Fendley**, J. Halverson, UVa
- J. Vala, N. Moran, DAQIST Maynooth

P. Fendley, K. Schoutens, J. de Boer, PRL (2003)

P. Fendley, K. Schoutens, PRL (2005)

L. Huijse, J. Halverson, P. Fendley, K. Schoutens, PRL (2008)

L. Huijse, K. Schoutens, ICMP proceedings (2010)

L. Huijse, K. Schoutens, ATMP (2010)

INTRODUCTION

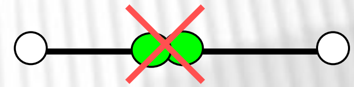
Challenge: study strongly interacting itinerant fermions at intermediate densities

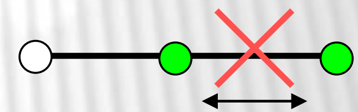
A supersymmetric model

- Analytic tools
- Interesting features:
  - integrability and quantum criticality in 1D,
  - superfrustration and possibly topological order in 2D

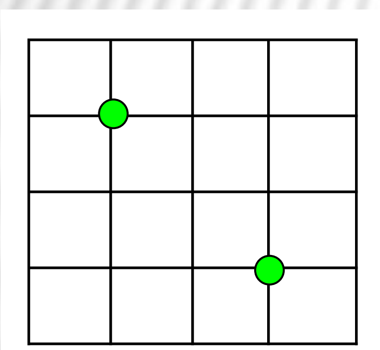
THE MODEL

- **Hardcore spinless fermions**
  - spinless fermions
  - hardcore $V_1 \rightarrow \infty$
  - hopping t





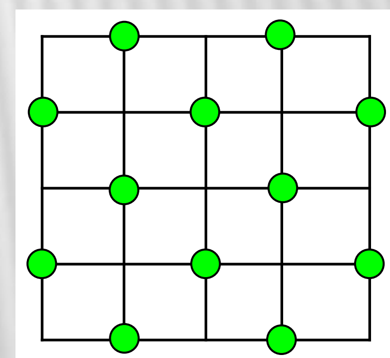
$\boldsymbol{\mu}$



Fermi liquid

Stripe phase

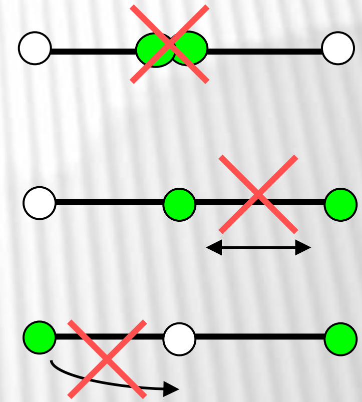
[Henley , et. al. ‘01]

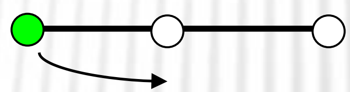


Insulator

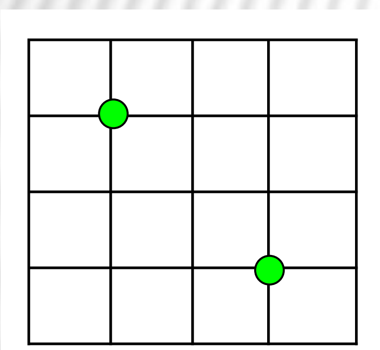
THE MODEL

- **Hardcore spinless fermions**
  - spinless fermions
  - hardcore $V_1 \rightarrow \infty$
  - hopping t





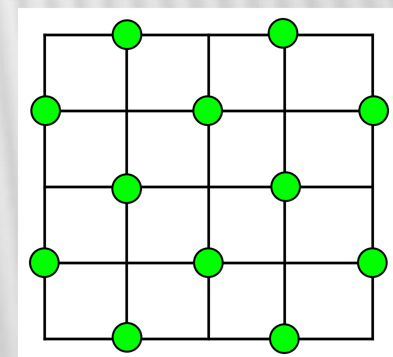
μ



Fermi liquid

Supersymmetry

[Fendley, et. al. '03]



Insulator

OUTLINE

- Supersymmetry
- The model
- 1D chain: quantum critical (SCFT)
- 2D lattices I: superfrustration
- 2D lattices II: supertopological phases?

SUPERSYMMETRIC QM

Algebraic structure

Susy charges Q^+, $Q^- = (Q^+)^\dagger$ and fermion number N_f:

$$(Q^+)^2 = 0, \quad (Q^-)^2 = 0, \quad [N_f, Q^\pm] = \pm Q^\pm$$

Hamiltonian defined as

$$H = \{Q^+, Q^-\}$$

satisfies

$$[H, Q^+] = [H, Q^-] = 0, \quad [H, N_f] = 0$$

SPECTRUM

- $E \geq 0$ for all states
- $E > 0$ states are paired into doublets of the susy algebra

$$\{|\psi\rangle, Q^{+}|\psi\rangle\} \quad Q^{-}|\psi\rangle = 0$$

- $E = 0$ iff a state is a singlet under the susy algebra

$$Q^{+}|\psi\rangle = Q^{-}|\psi\rangle = 0$$

- if $E = 0$ ground state exist, supersymmetry is unbroken.

WITTEN INDEX

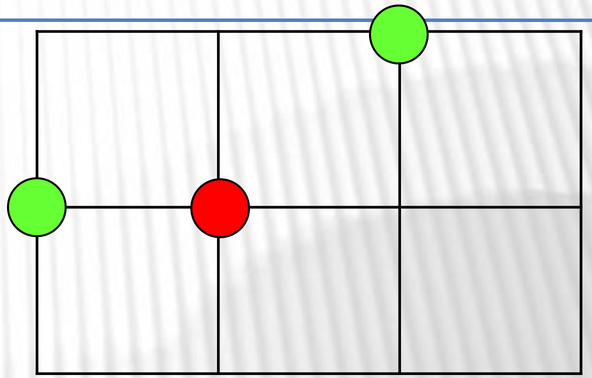
$$W = \mathrm{Tr}(-1)^{N_f}$$

- E>0 doublets $\{|\psi\rangle, Q^{+}|\psi\rangle\}$ with $N_f = f$, $N_f = f+1$ cancel in W
- only E=0 groundstates contribute

$\Rightarrow$ |W| is lower bound on # of ground states

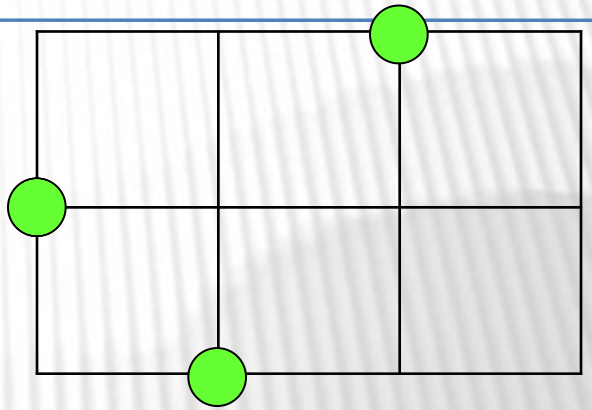
THE MODEL

SUSY LATTICE MODEL



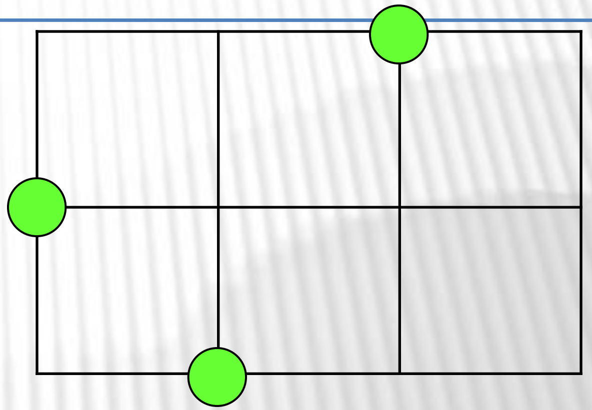
configurations:
lattice fermions with nearest neighbor exclusion

SUSY LATTICE MODEL



configurations:
lattice fermions with nearest neighbor exclusion

SUSY LATTICE MODEL



configurations:
lattice fermions with nearest neighbor exclusion

nilpotent supercharges, respecting exclusion rule:

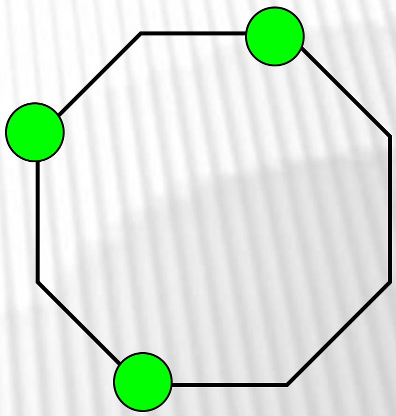
$$Q^+ = \sum_i c_i^\dagger \prod_{j \text{ next to } i} (1 - n_j), \quad Q^- = (Q^+)^\dagger, \quad n_j = c_j^\dagger c_j$$

Hamiltonian: kinetic (hopping) plus potential terms

$$H = \left\{ \mathrm{Q}^+, \mathrm{Q}^- \right\} = H_{kin} + H_{pot}$$

[Fendley - Schoutens - de Boer ‘03]

SUSY MODEL IN 1D



supercharges

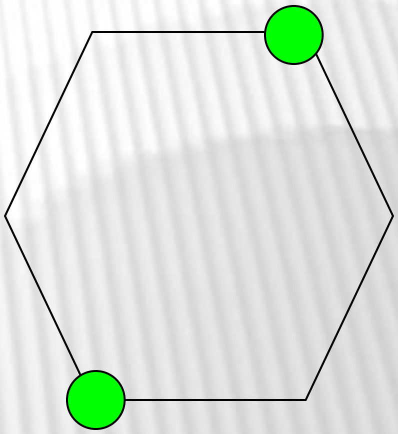
$$Q^+ = \sum_i (1 - n_{i-1}) c_i^\dagger (1 - n_{i+1}), \quad Q^- = (Q^+)^\dagger$$

Hamiltonian:

$$H = \sum_i [(1 - n_{i-1}) c_i^\dagger c_{i+1} (1 - n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1} n_{i+1} - 2N_f + L$$

L=6 MODEL: WITTEN INDEX

$$W = \mathrm{Tr}(-1)^{N_f}$$



$N_f = 0$: 1 state

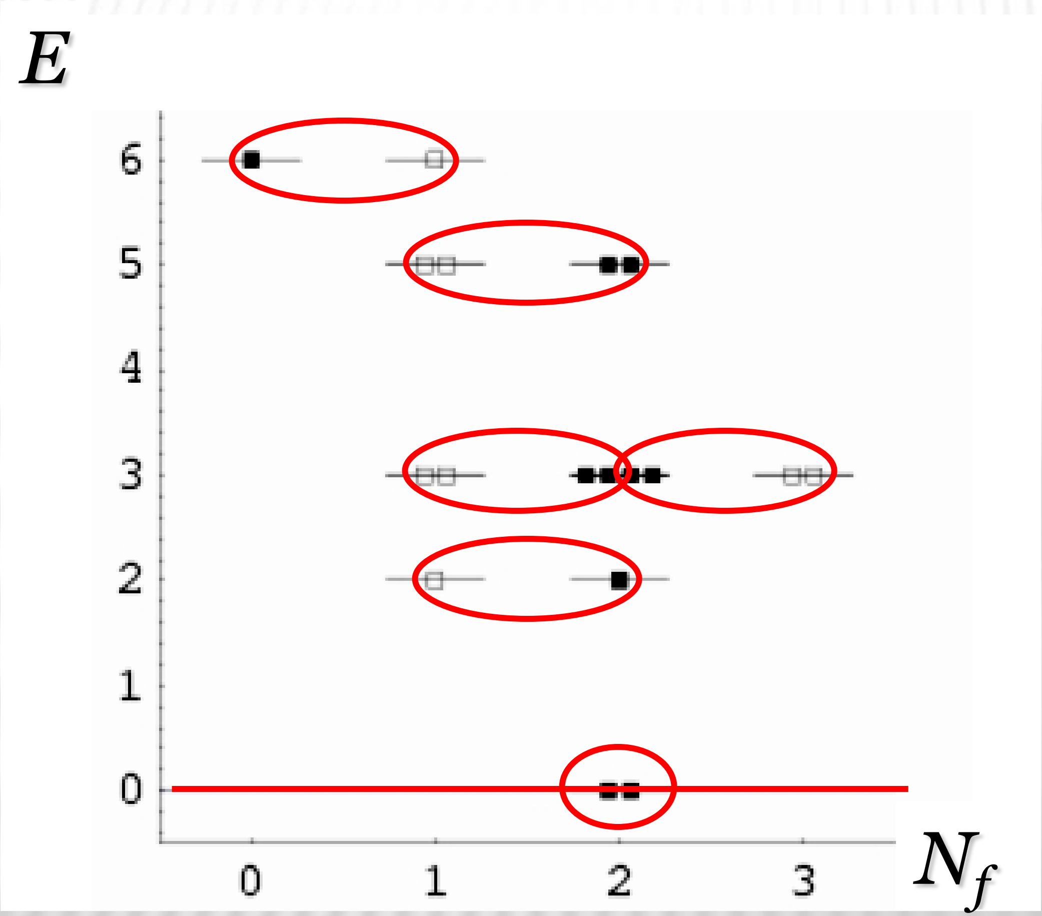
$N_f = 1$: 6 states

$N_f = 2$: 9 states

$N_f = 3$: 2 states

$$\Rightarrow W = 1 - 6 + 9 - 2 = 2$$

SPECTRUM FOR L=6 SITES



SUPERSYMMETRIC MODELS

Large variety in observed properties

- Choose underlying lattice
- Generalize hard-core constraint
- Introduce site-dependent coefficients

SUPERSYMMETRIC MODELS

Large variety in observed properties

- Choose underlying lattice
- Generalize hard-core constraint
  - Allow k particles to be nearest neighbors, but not k+1: M_k susy models

[Fendley , Nienhuis , Schoutens '03]

- Introduce site-dependent coefficients

SUPERSYMMETRIC MODELS

Large variety in observed properties

- Choose underlying lattice
- Generalize hard-core constraint
- Introduce site-dependent coefficients
  - Supercharge: $Q^+ = \sum_i \lambda_i c_i^\dagger \prod_{j \text{ n.t. } i} (1 - n_j)$
    - Example: staggered 1D chain

[Fendley , Hagendorf '10]

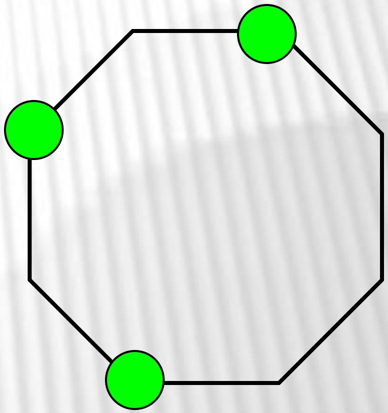
SUPERSYMMETRIC MODELS

Large variety in observed properties

- Choose underlying lattice
  - 1D: Quantum criticality
  - 2D: Superfrustration
  - 2D: 'Supertopological' phases
- Generalize hard-core constraint
- Introduce site-dependent coefficients

1D CHAIN

QUANTUM CRITICAL BEHAVIOR 1D



- periodic chain:
  2 gs for L multiple of 3, else 1 gs

- exactly solvable via Bethe Ansatz
  $\rightarrow$ continuum limit:
  $\mathcal{N}=(2,2)$ SCFT with central charge $c=1$

[Fendley, Schoutens, de Boer ’03]

$\mathcal{N}=2$ SCFT

- Virasoro algebra extended with two supercharges **G**$^{\pm}$ and a U(1) current **J**
- Supercharges have conformal dimension 3/2
- Minimal unitary series:

$$c = \frac{3k}{k+2} \quad k = 1, 2, \ldots$$

$\mathcal{N}$=2 SCFT FIRST MINIMAL MODEL

- First minimal model: k=1 ⇒ c=1
- Free boson at R=$\sqrt{3}$

$$S = \frac{2}{3\pi}\int dx\,dt\ \left[(\partial_t\Phi)^2 - (\partial_x\Phi)^2\right]$$

- Vertex operators

$$V_{m,n} = \exp\left(\imath m\Phi + \imath n\tilde{\Phi}\right) \qquad \Phi = \Phi_L + \Phi_R,\ \tilde{\Phi} = \frac{2}{3}(\Phi_L - \Phi_R)$$

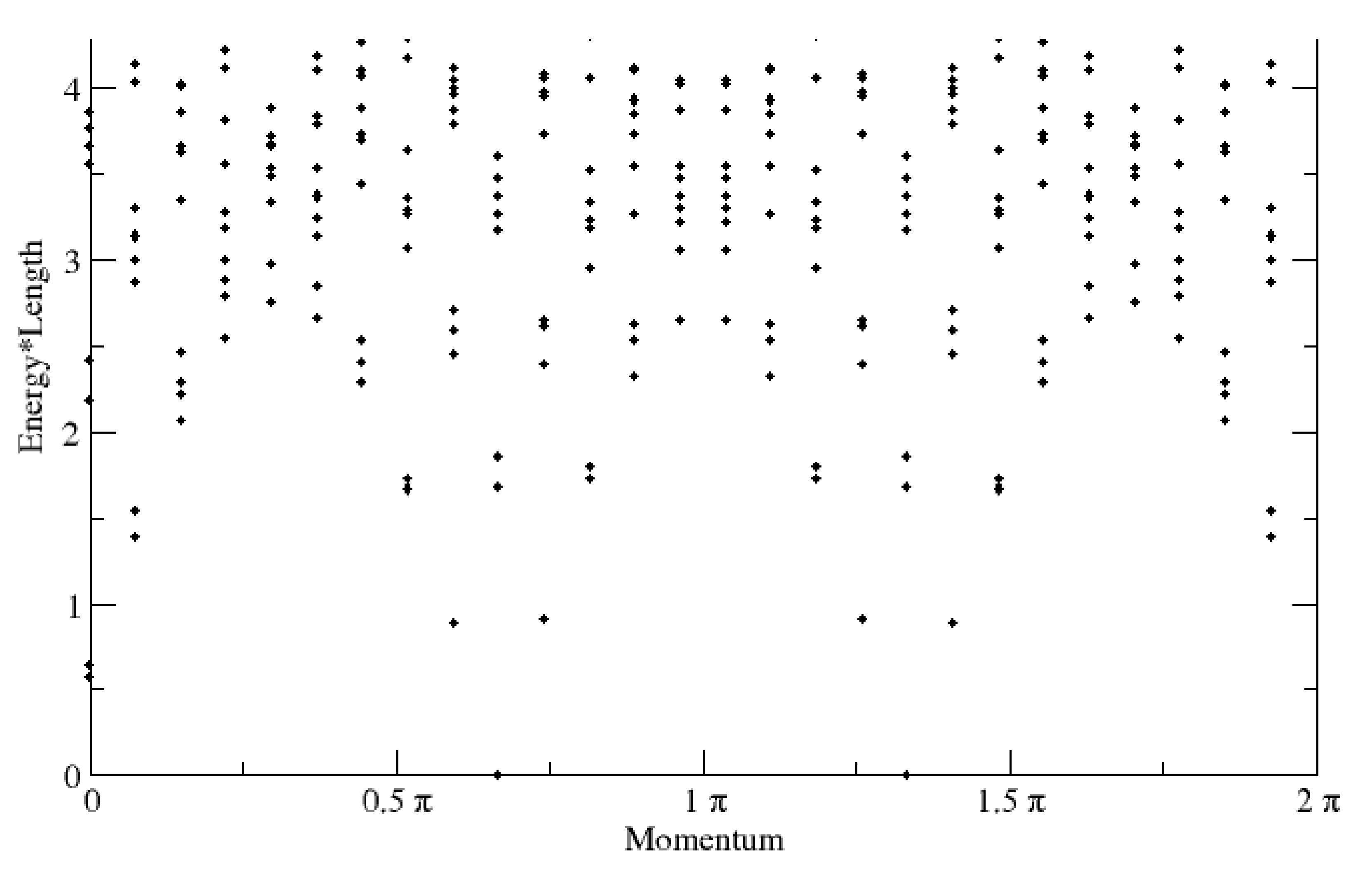
- Conformal dimensions

$$h_{L,R} = \frac{3}{8}(m \pm \frac{2}{3}n)^2$$

- Supercharges ($h_{L,R}$=3/2)

$$G_L^{\pm} = V_{\pm 1,\pm 3/2} \quad \text{and} \quad G_R^{\pm} = V_{\pm 1,\mp 3/2}$$

SPECTRUM FOR 1D CHAIN, L=27, N_F=9



$\mathcal{N}=2$ SCFT DESCRIPTION FOR THE CHAIN

Different chain lengths and boundary conditions correspond to different sectors

- (anti) periodic bc $\leftrightarrow$ Ramond (NS) sector

$$(-1)^{m+2n} = \begin{cases} +1 & \text{NS} \\ -1 & \text{R} \end{cases}$$

- Chain length

  $L = 3l \leftrightarrow m \in \mathbb{Z}$

  $L = 3l \pm 1 \leftrightarrow m \in \mathbb{Z} \pm 1/3$

$\mathcal{N}$=2 SCFT DESCRIPTION FOR THE CHAIN

Identify lattice operators with operators in SCFT

- Energy

$$E_{\text{lat}} = E_{\text{CFT}}\, v_F/L = (h_L + h_R - c/12)v_F/L$$

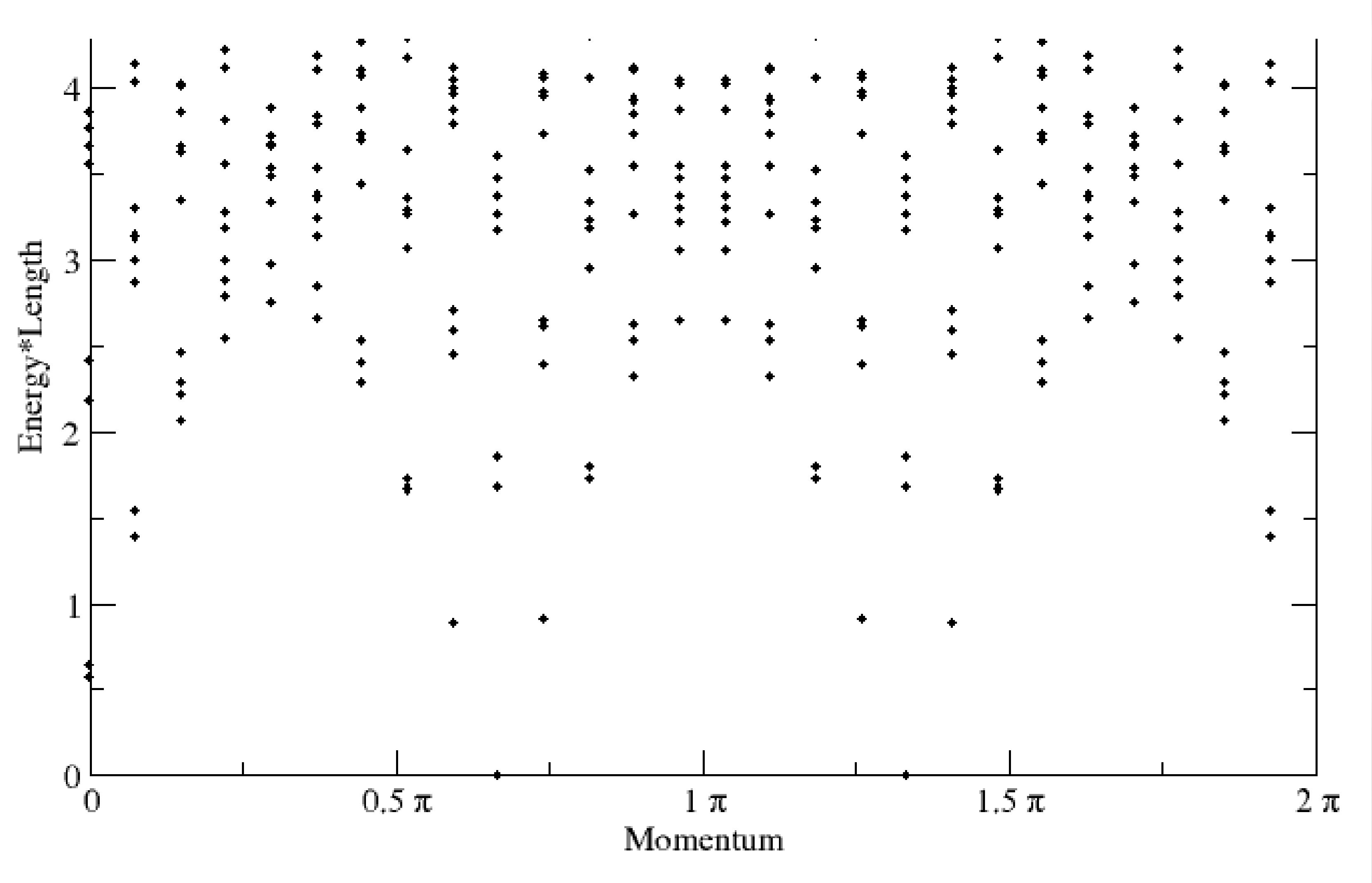
- Charge

$$N_f - N_{f_{\text{GS}}} = m$$

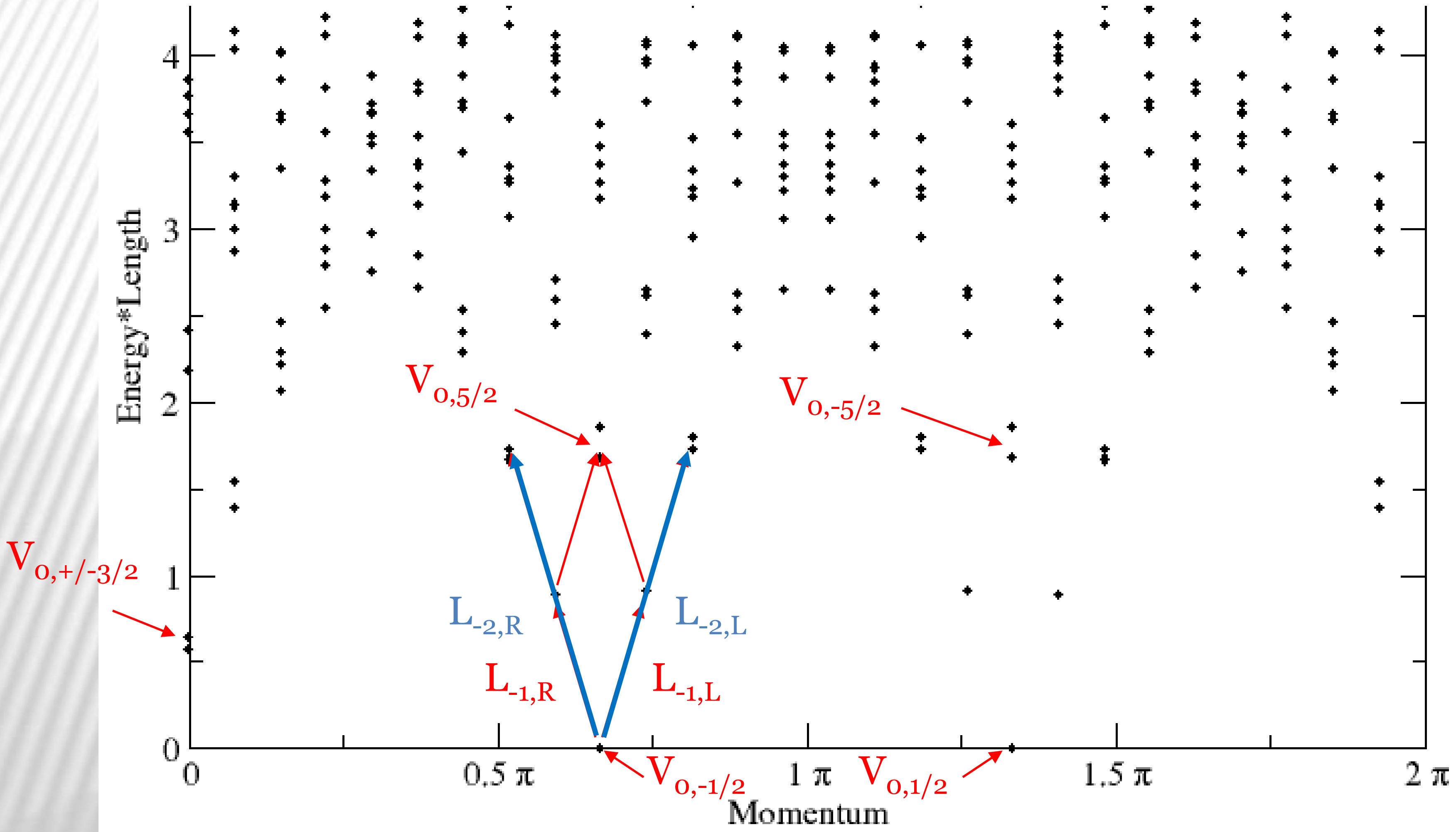
- Momentum

$$P_{\text{lat}} = 2\pi n/3 + 2\pi(h_R - h_L)/L + f_{GS}\pi \quad \text{mod}\, 2\pi$$

SPECTRUM FOR 1D CHAIN, L=27, N_F=9



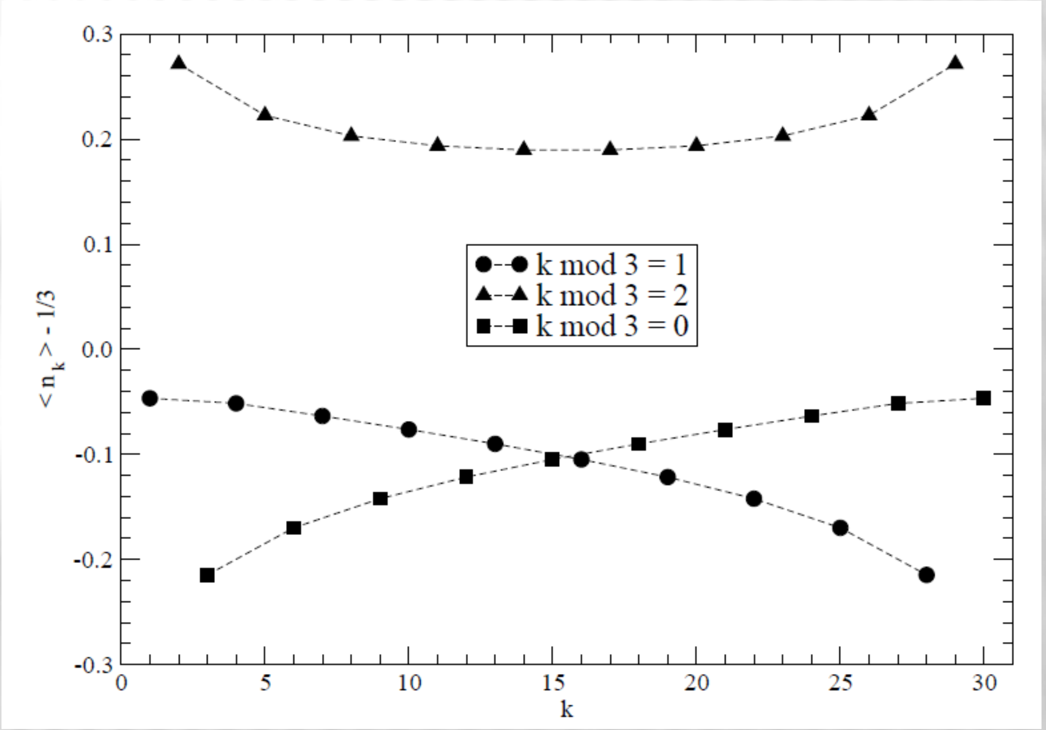
SPECTRUM FOR 1D CHAIN, L=27, N_F=9



DENSITY FLUCTUATIONS

Beccaria, De Angelis
Clear $\mathbb{Z}_3$ substructure for fermion density $<n_k>$ in chain with L Mod 3= 0 and open bc

Scaling dimension of 1/3 was extracted



[Beccaria, De Angelis '05]

DENSITY FLUCTUATIONS

Open bc: L and R movers couple

Operators: $V_m = e^{(\imath m \Phi / \sqrt{3})}$

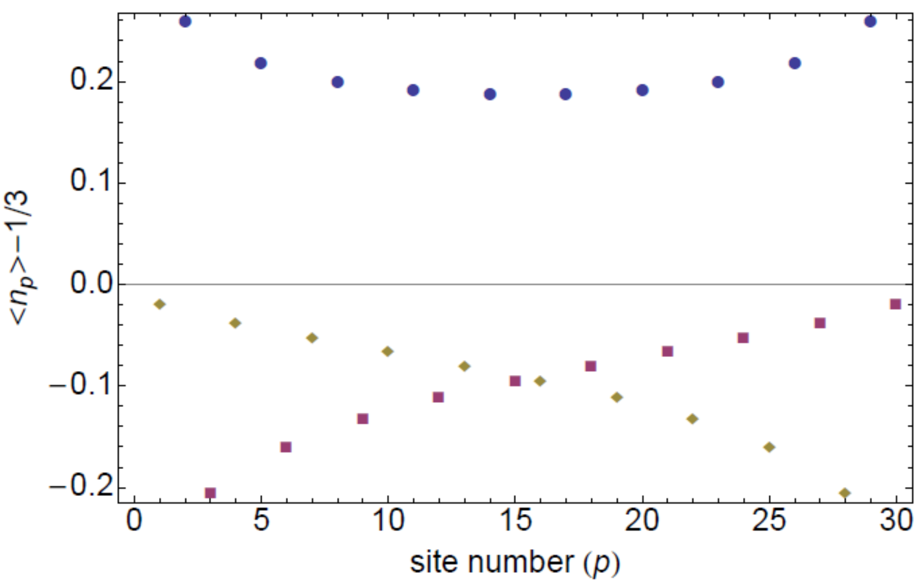
We identify:

$$\begin{aligned} 3\langle n_k \rangle &= \langle R | V_0 + A_1 (V_1 + V_{-1}) | R \rangle_{\text{strip}} \\ 3\langle n_{k\pm 1} \rangle &= \langle R | V_0 + A_1 (e^{\mp 2\pi\imath/3} V_1 + e^{\pm 2\pi\imath/3} V_{-1}) | R \rangle_{\text{strip}} \end{aligned}$$

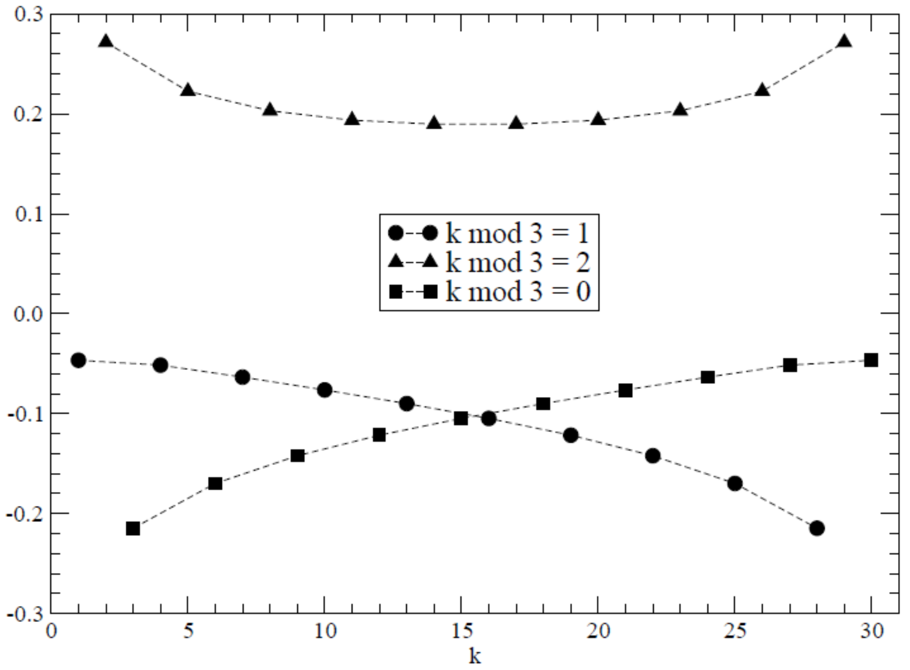
DENSITY FLUCTUATIONS

SCFT Result: scaling dimension 1/3 and density fluctuations near boundary

SCFT



Beccaria et. al.

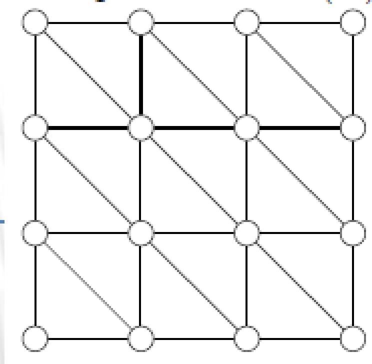


2D LATTICES

POWERFUL TOOLS

- Witten index
  - gives lower bound to number of ground states: $W=\#GS_B-\#GS_F$
- Cohomology of Q
  - GS are in 1-1 correspondence with cohomology elements
    - gives total number of gs
    - gives fermion number of gs
    - often gives relation between gs and geometric object

WITTEN INDEX



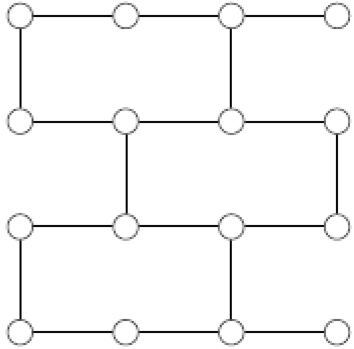
Triangular lattice

$N \times M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

[van Eerten, ‘05]

WITTEN INDEX



Honeycomb lattice

$N \times M$ sites with periodic BC

	2	4	6	8	10	12	14	16	18
2	-1	-1	2	-1	-1	2	-1	-1	2
4	3	7	18	47	123	322	843	2207	5778
6	-1	-1	32	-73	44	356	-1387	2087	2435
8	3	7	18	55	123	322	843	2215	5778
10	-1	-1	152	-321	-171	7412	-26496	10079	393767
12	3	7	156	1511	6648	29224	150069	1039991	6208815
14	-1	-1	338	727	-5671	1850	183560	-279497	-4542907
16	3	7	1362	12183	31803	379810	5970107	55449303	327070578

[van Eerten, ‘05]

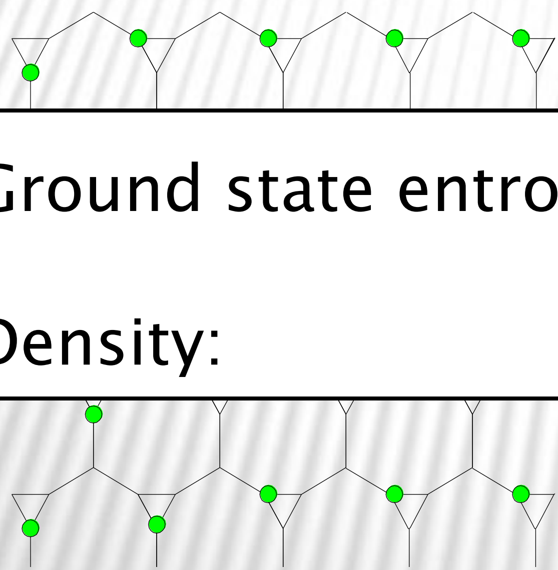
COHOMOLOGY

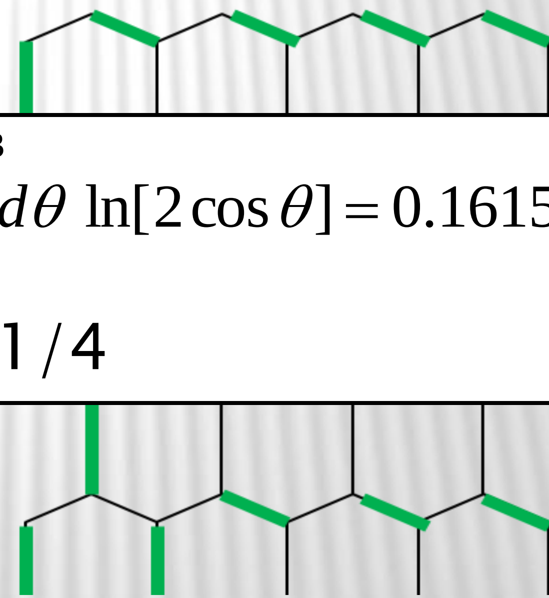
Martini lattice

Number of ground states

$\longleftrightarrow$

number of dimer coverings of honeycomb lattice



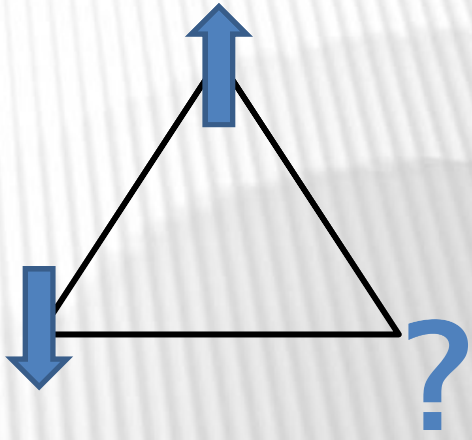


Ground state entropy: $\frac{S_{\rm gs}}{L} = \frac{1}{\pi}\int_0^{\pi/3} d\theta \, \ln[2\cos\theta] = 0.16153\ldots$

Density: $N_f/L = 1/4$

[Fendley, et. al. '05]

SUPERFRUSTRATION



Frustration

Competing terms in hamiltonian

→ multiple ground states

Supersymmetry

Subtle competition between kinetic and potential terms

→ for 2D, 3D lattices exponential ground state degeneracy

→ extensive ground state entropy

SUPERFRUSTRATION

‘3-rule’

- repulsive interactions favor 3-site interparticle distance
- chemical potential favors higher densities

Combined with kinetic terms

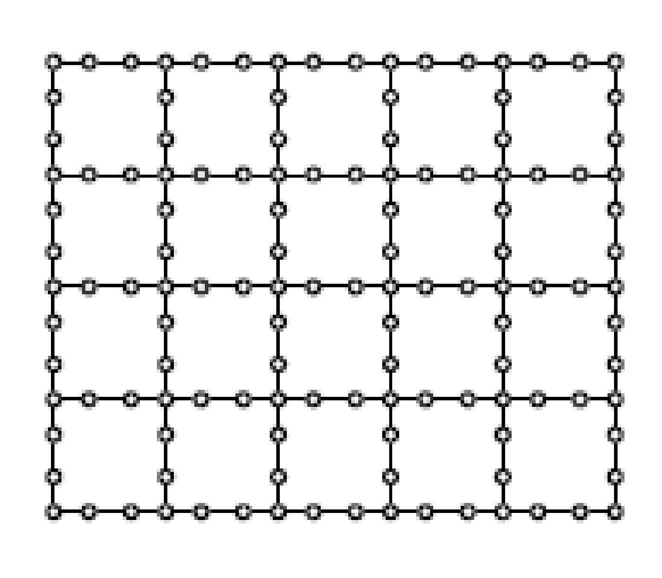
→ quantum charge frustration at intermediate densities

[Fendley, Schoutens, ‘05]

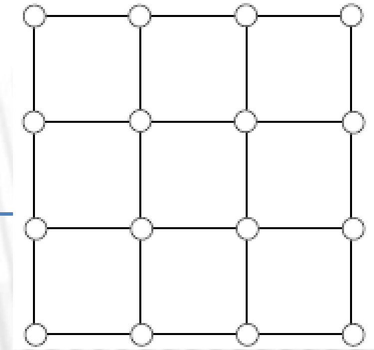
REDUCED FRUSTRATION

Λ_3 lattice nicely accommodates 3-rule

- 1 gs @ $N_f/L = 1/5$
  all particles on
  the corners
- 1 gs @ $N_f/L = 2/5$
  all particles resonating
  on the bond-sites



SQUARE LATTICE



Witten index

$N \times M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3	1	−1	1	3
3	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1	4	1	1
4	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7	1	3	1	7
5	1	1	1	1	−9	1	1	1	1	11	1	1	1	1	−9	1	1	1	1	11
6	1	−1	4	3	1	14	1	3	4	−1	1	18	1	−1	4	3	1	14	1	3
7	1	1	1	1	1	1	1	1	1	1	1	1	1	−27	1	1	1	1	1	1
8	1	3	1	7	1	3	1	7	1	43	1	7	1	3	1	7	1	3	1	47
9	1	1	4	1	1	4	1	1	40	1	1	4	1	1	4	1	1	76	1	1
10	1	−1	1	3	11	−1	1	43	1	9	1	3	1	69	11	43	1	−1	1	13
11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	1	3	4	7	1	18	1	7	4	3	1	166	1	3	4	7	1	126	1	7
13	1	1	1	1	1	1	1	1	1	1	1	1	−51	1	1	1	1	1	1	1
14	1	−1	1	3	1	−1	−27	3	1	69	1	3	1	55	1	451	1	−1	1	73
15	1	1	4	1	−9	4	1	1	4	11	1	4	1	1	174	1	1	4	1	11

[Fendley - Schoutens - van Eerten '05]

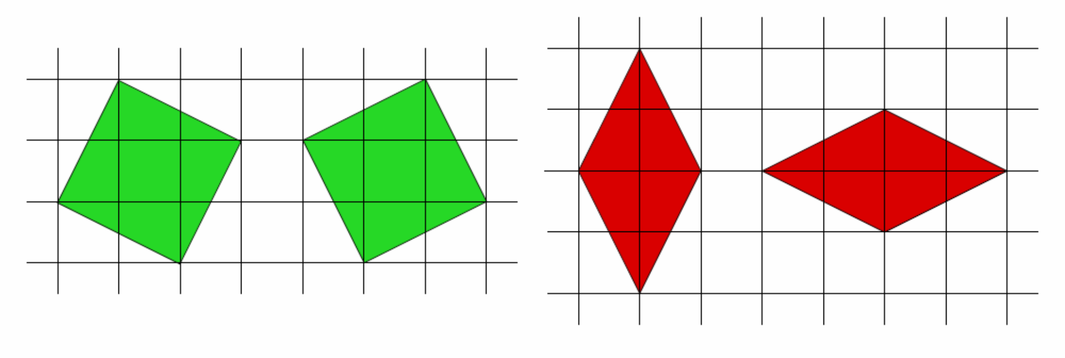
SQUARE LATTICE: WITTEN INDEX

Witten index related to rhombus tilings of the lattice

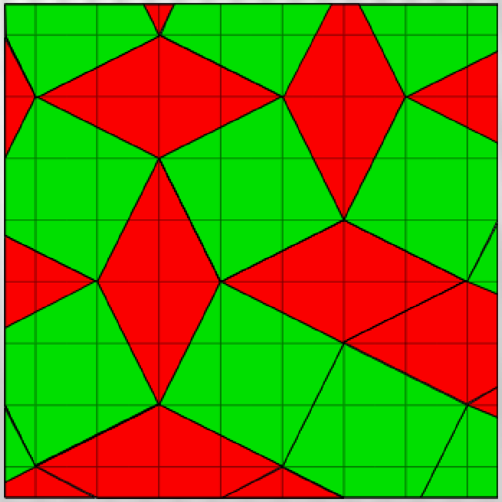
Theorem **[Jonsson 2005]**

$$W_{\vec{u},\vec{v}} = t_{even} - t_{odd} - (-1)^{d_-}\theta_{d_-}\theta_{d_+}$$

with $d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2)$, $\theta_{3p} = 2 \quad \theta_{3p\pm1} = -1$



periodicities $\vec{u},\vec{v}$



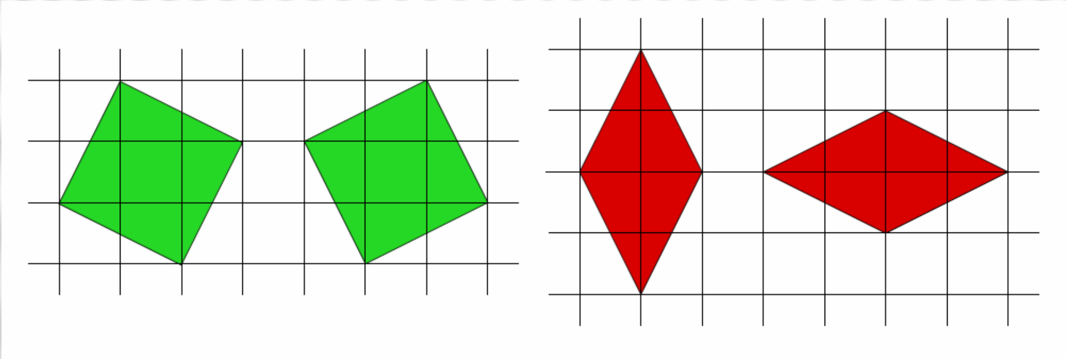
SQUARE LATTICE: GROUND STATES

Total # of ground states related to rhombus tilings

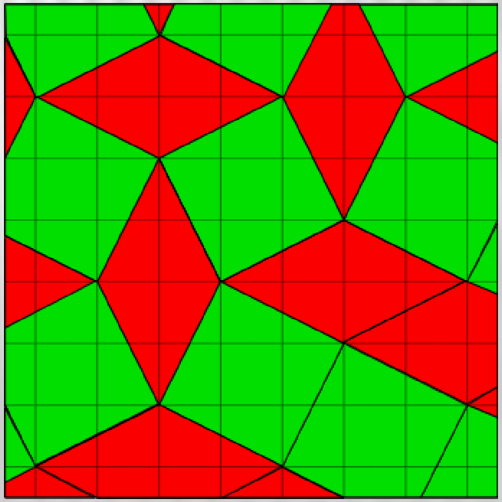
Conjecture [Fendley]

$$\#\,\mathrm{GS} = t_{even} + t_{odd} + \Delta$$

with $|\Delta| = |(-1)^{d_-}\theta_{d_-}\theta_{d_+}|$



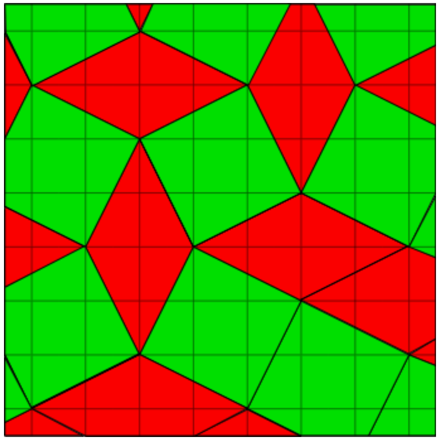
periodicities $\vec{u}, \vec{v}$



SQUARE LATTICE: GROUND STATES

Total # of ground states related to rhombus tilings

Theorem **[LH - Schoutens 2010]**



$$N_i = t_i + \Delta_i$$

for $\vec{u} = (m, -m)$ and $v_1 + v_2 = 3p$

with $\Delta_i = \begin{cases} -(-1)^{(\theta_m+1)p}\theta_{d_-}\theta_{d_+} & \text{if } i = [2m/3]p \\ 0 & \text{otherwise,} \end{cases}$

$$d_\pm = \gcd(u_1 \pm u_2, v_1 \pm v_2) \quad \theta_{3p} = 2 \quad \theta_{3p\pm 1} = -1$$

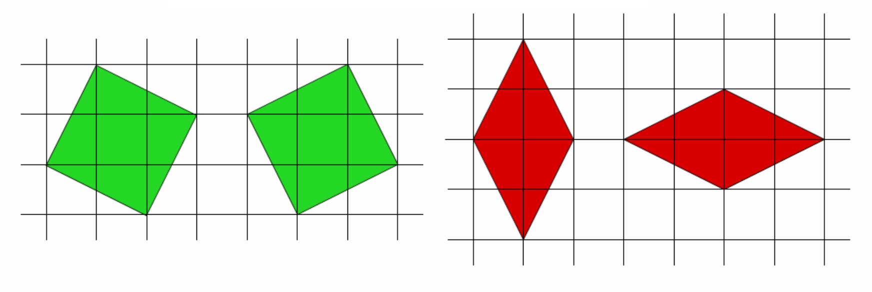
SQUARE LATTICE: GROUND STATES

- # gs grows exponentially with the linear size of the system

$$\# \text{ GS} \sim \frac{4^{p+q}}{\sqrt{pq}} \quad \text{for periodicities} \quad \begin{aligned} \vec{u} &= (3p, -3p) \\ \vec{v} &= (3q, 3q) \end{aligned}$$

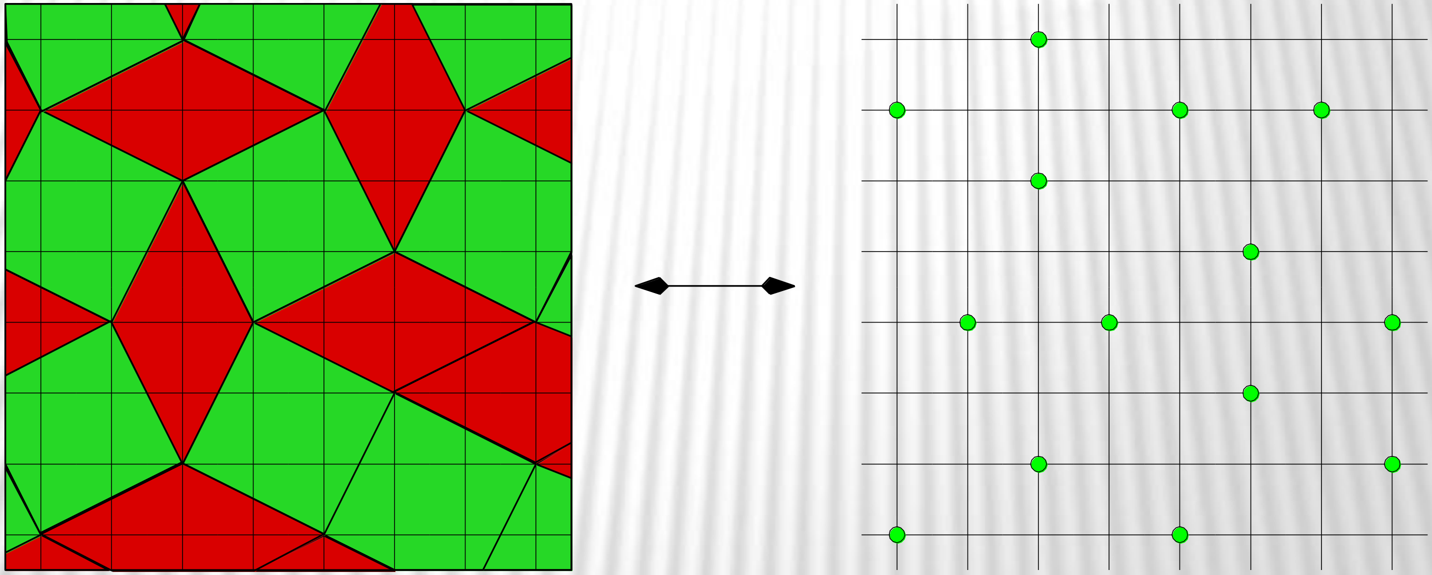
- zero energy ground states found at intermediate density:

$$\frac{f}{L} \in \left[\frac{1}{5}, \frac{1}{4}\right] \cap \mathbb{Q}$$



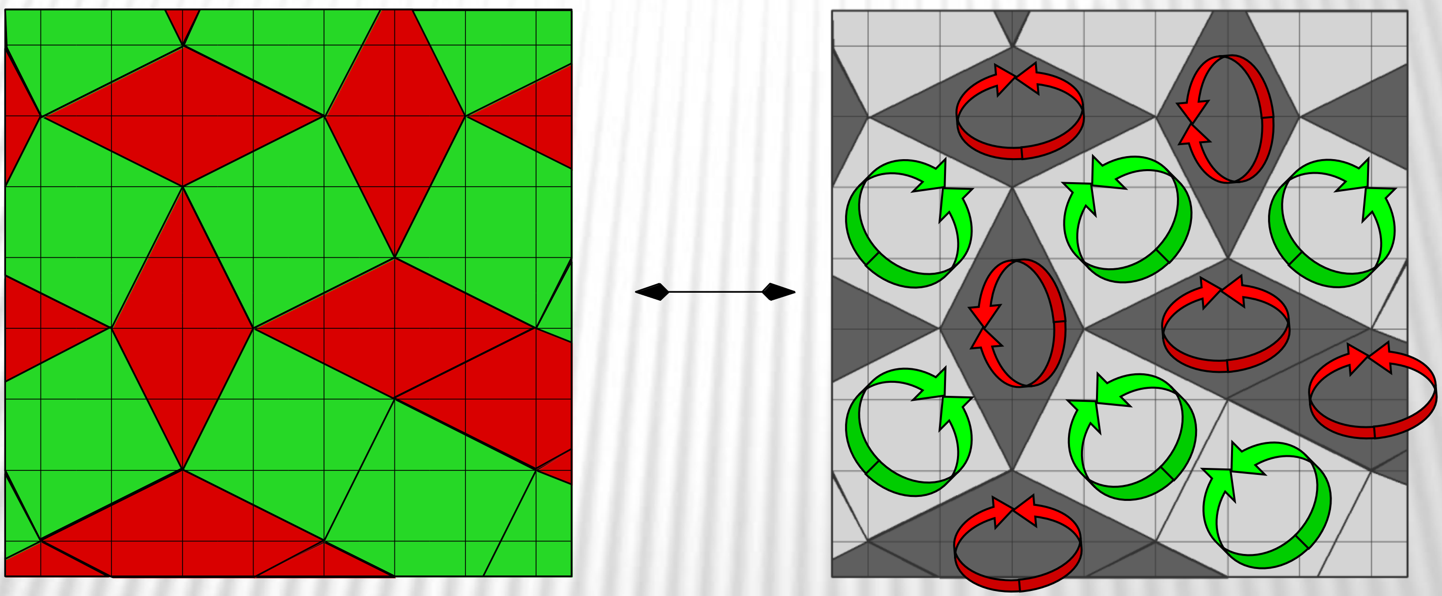
SQUARE LATTICE: GROUND STATES

Tilings as ground states?



SQUARE LATTICE: GROUND STATES

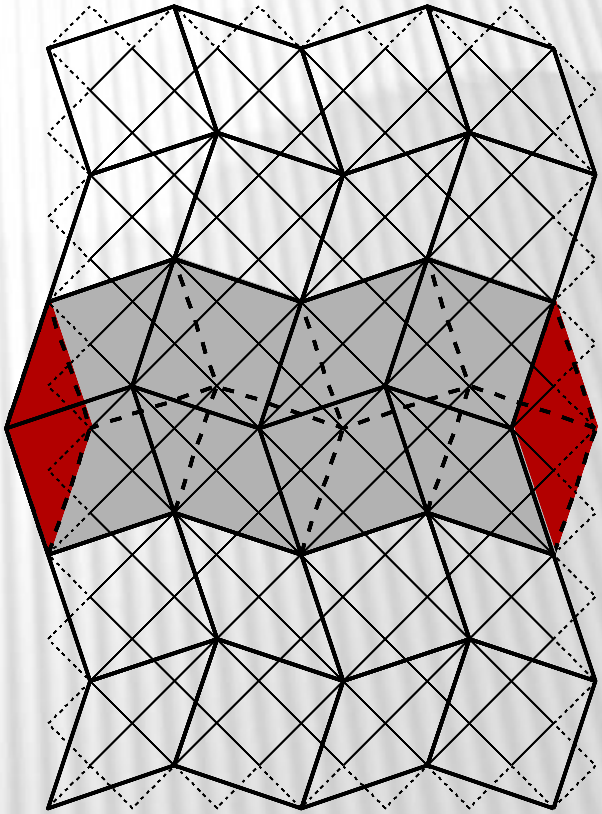
Tilings as ground states?



SQUARE LATTICE: EDGE STATES

- for 'diagonal' open boundary conditions there is a unique gs; expect that 'vanished' torus gs's form band of edge modes

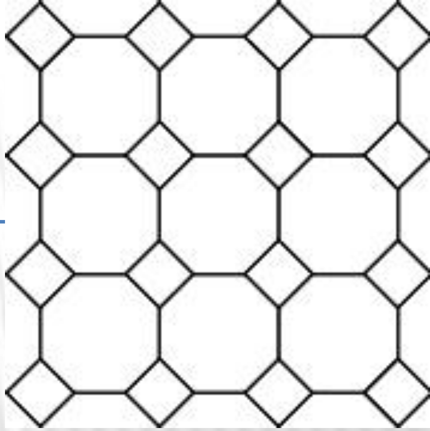
- explicit evidence for critical modes from ED studies of various ladder geometries



[LH - Halverson - Fendley - Schoutens '08]

SUPERTOPOLOGICAL PHASE?

OCTAGON–SQUARE LATTICE



- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette
- $N \times M$ plaquettes with closed bc: 2^M+2^N-1 gs
- gapless defects by adding diagonal link on plaquette

WORK IN PROGRESS

Understand character of GS using insights from cohomology

- Unique gs: ‘filled Landau level’
- Defects: interpret extra link as flux through plaquette

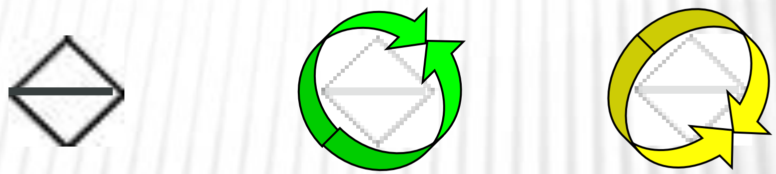
With J. Vala and N. Moran

Numerical studies: 3x3 plaquettes

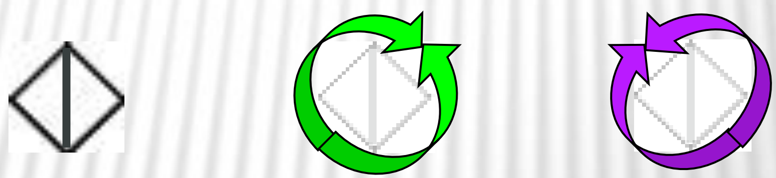
SINGLE PLAQUETTE

plaquette
(1 gs)

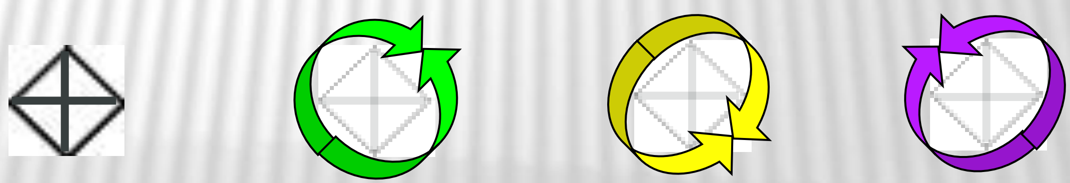
H-defect
(2 gs)



V-defect
(2 gs)

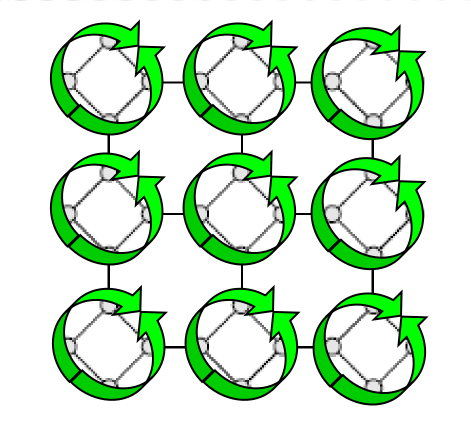


HV-defect
(3 gs)

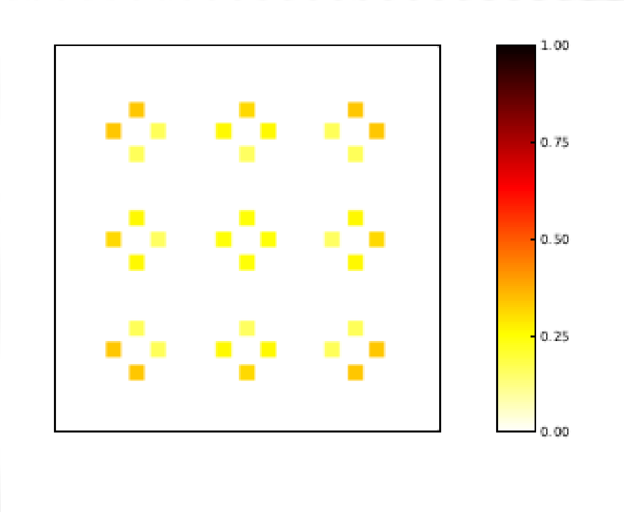


PROJECTED SINGLE PARTICLE STATES

Single particle product state with n.n. occupancies projected out (PPS)



- Fermion density per site



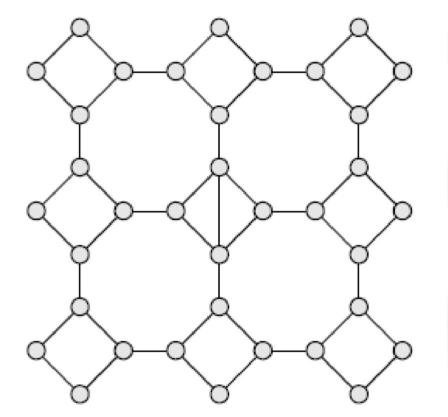
- $|\langle\Psi|\Psi_{\text{PPS}}\rangle|^2 = 0.651$
- 2 particles on 1 plaquette:

$\langle n_i n_k \rangle \approx 10^{-2}$

V–DEFECT

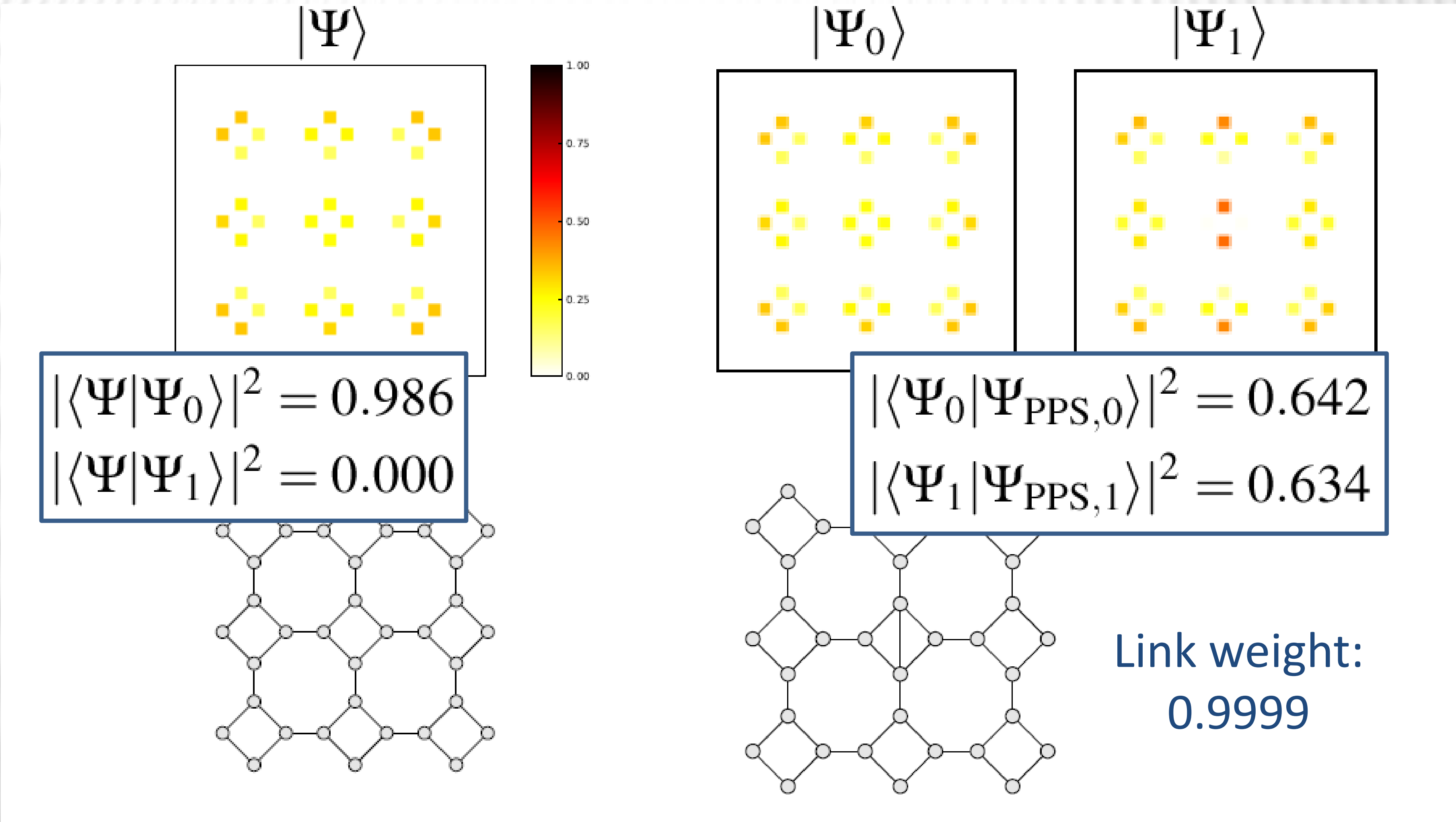
To test PPS for defect we need to lift gs degeneracy:

add small weight to extra link



IDENTIFY DEFECT STATE

Fermion density per site



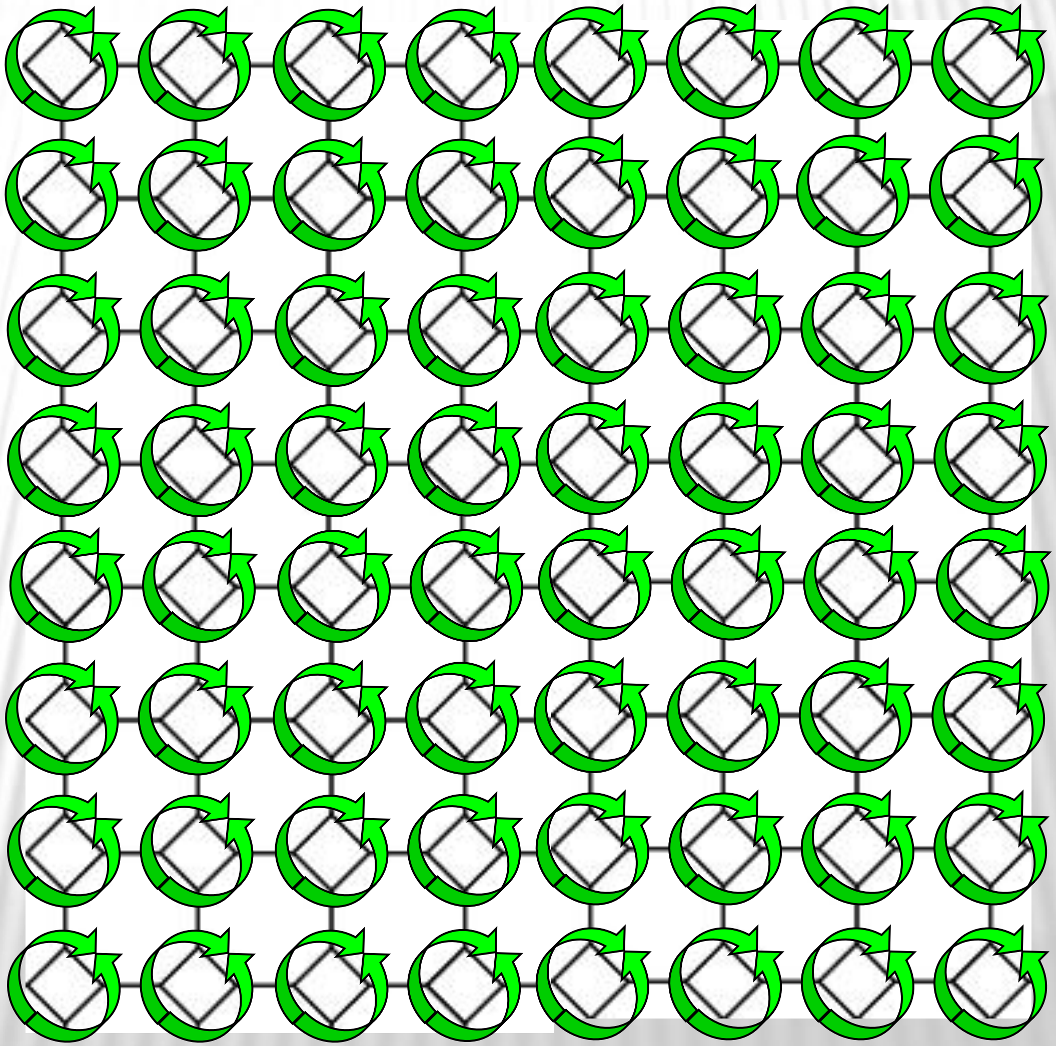
$$|\langle\Psi|\Psi_0\rangle|^2 = 0.986$$
$$|\langle\Psi|\Psi_1\rangle|^2 = 0.000$$

$$|\langle\Psi_0|\Psi_{\mathrm{PPS},0}\rangle|^2 = 0.642$$
$$|\langle\Psi_1|\Psi_{\mathrm{PPS},1}\rangle|^2 = 0.634$$

Link weight:
0.9999

2D LATTICE (OPEN)

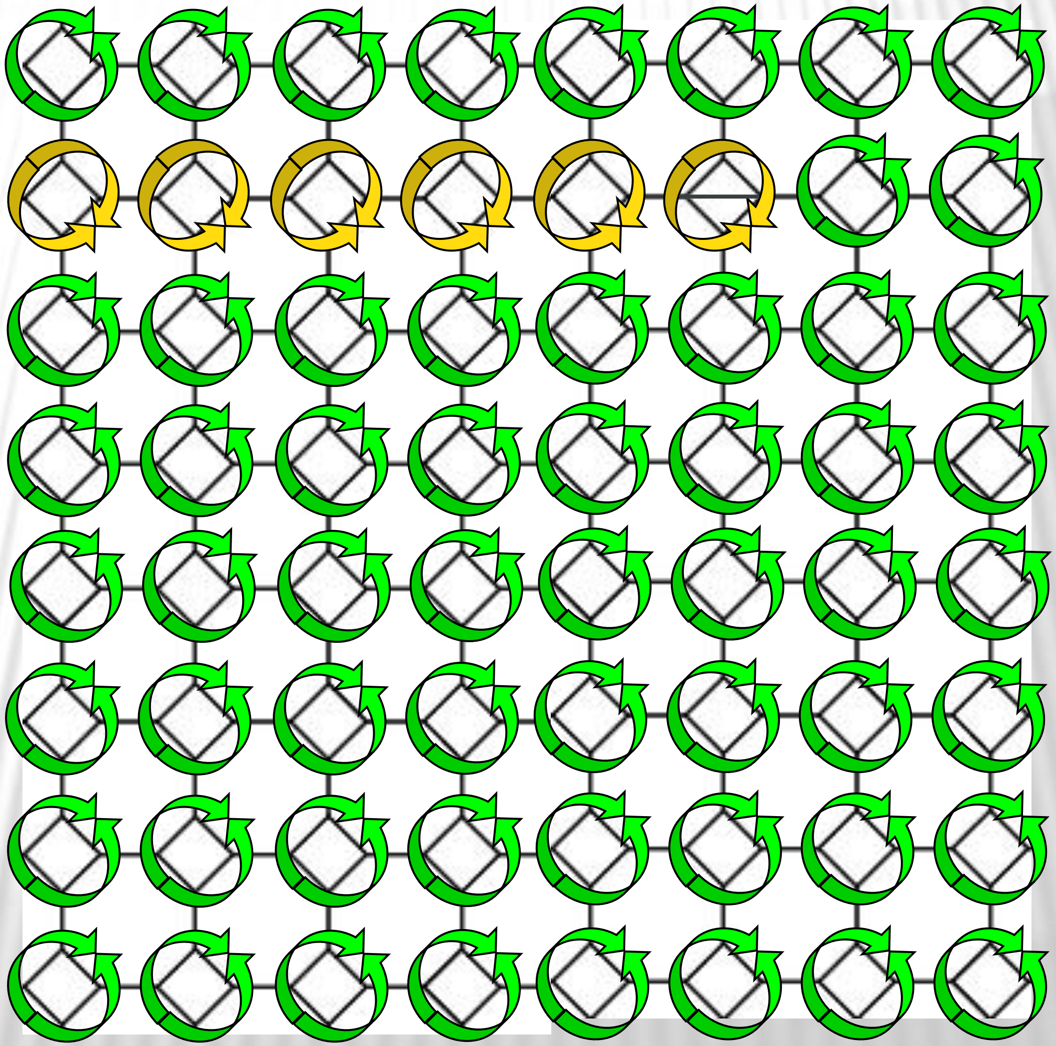
open bc
(1 gs)



“filled
Landau
level”

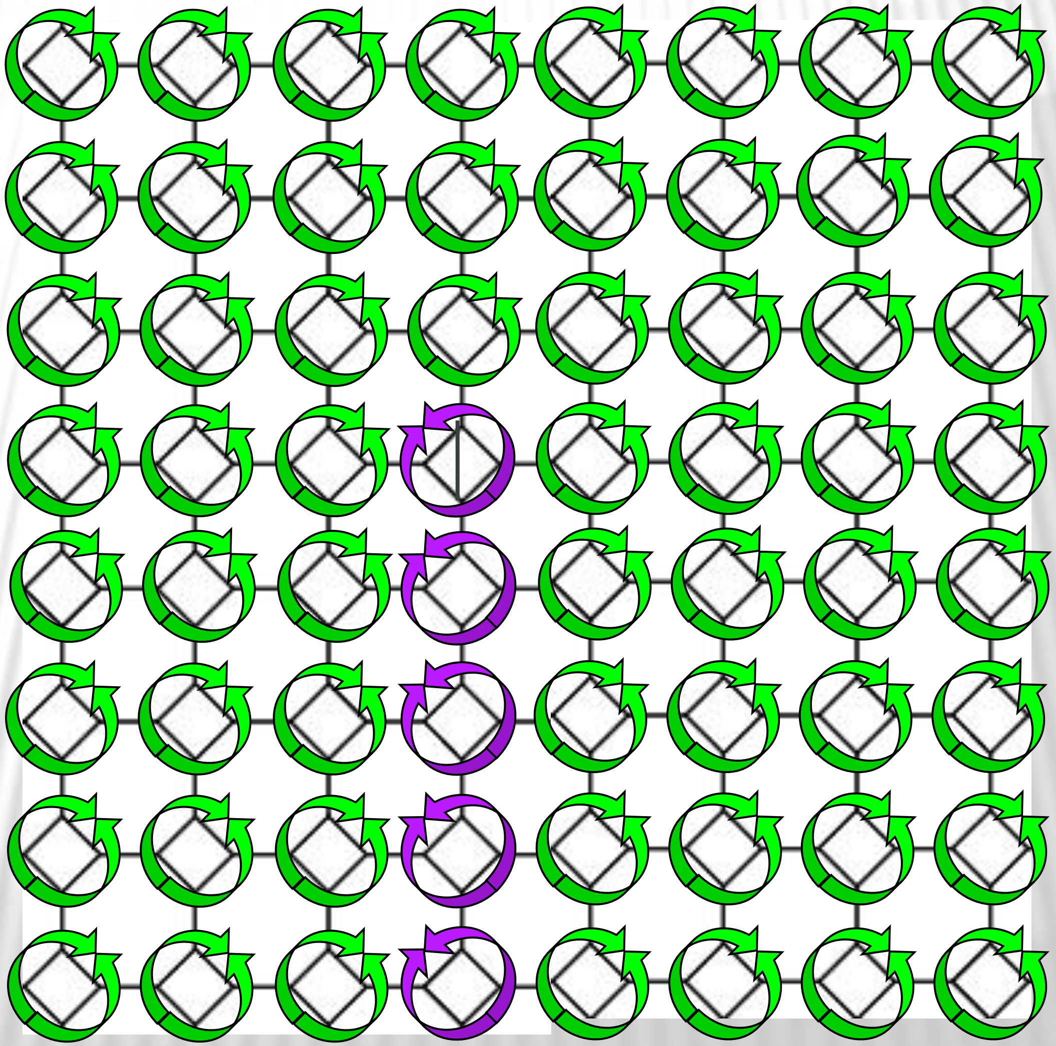
2D LATTICE (H–DEFECT)

H-defect
(2 gs)



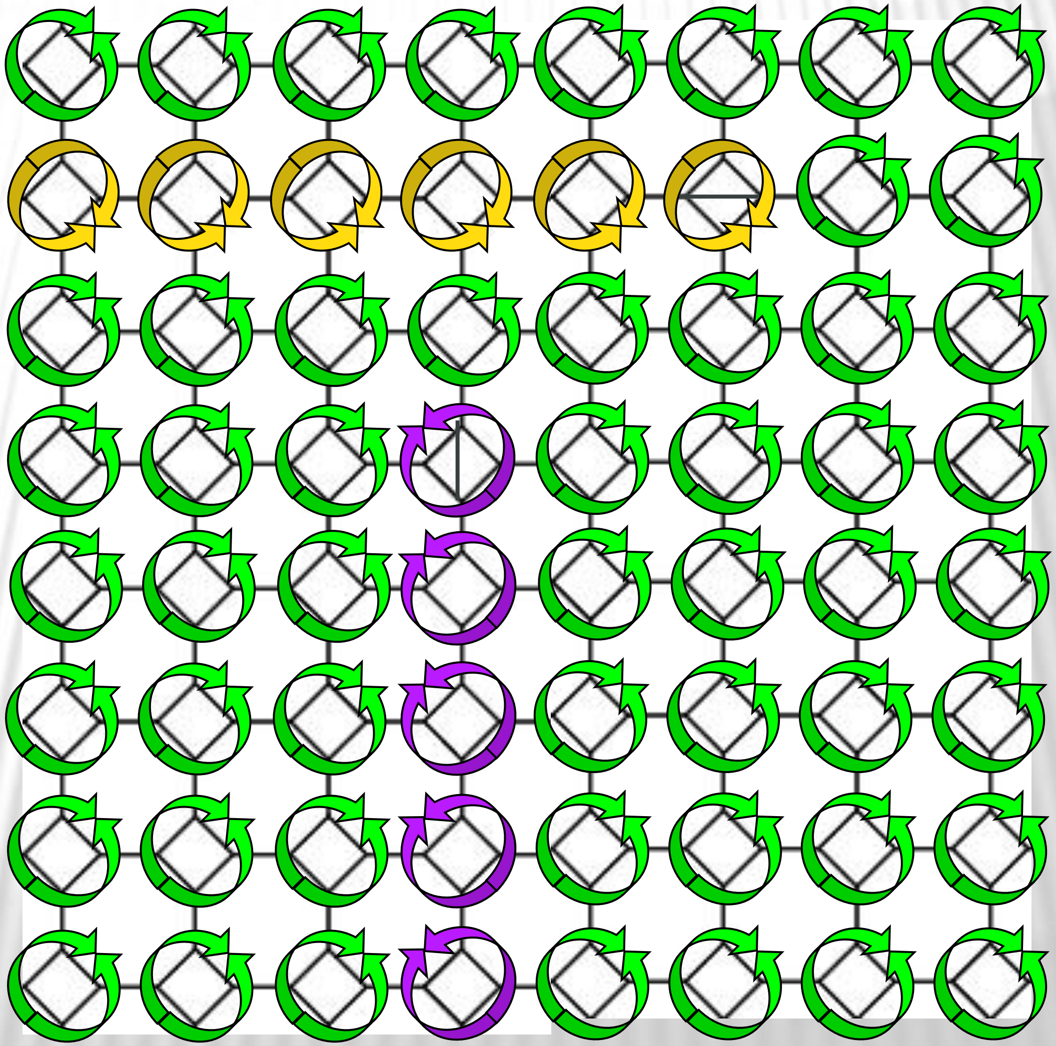
2D LATTICE (V–DEFECT)

V-defect
(2 gs)



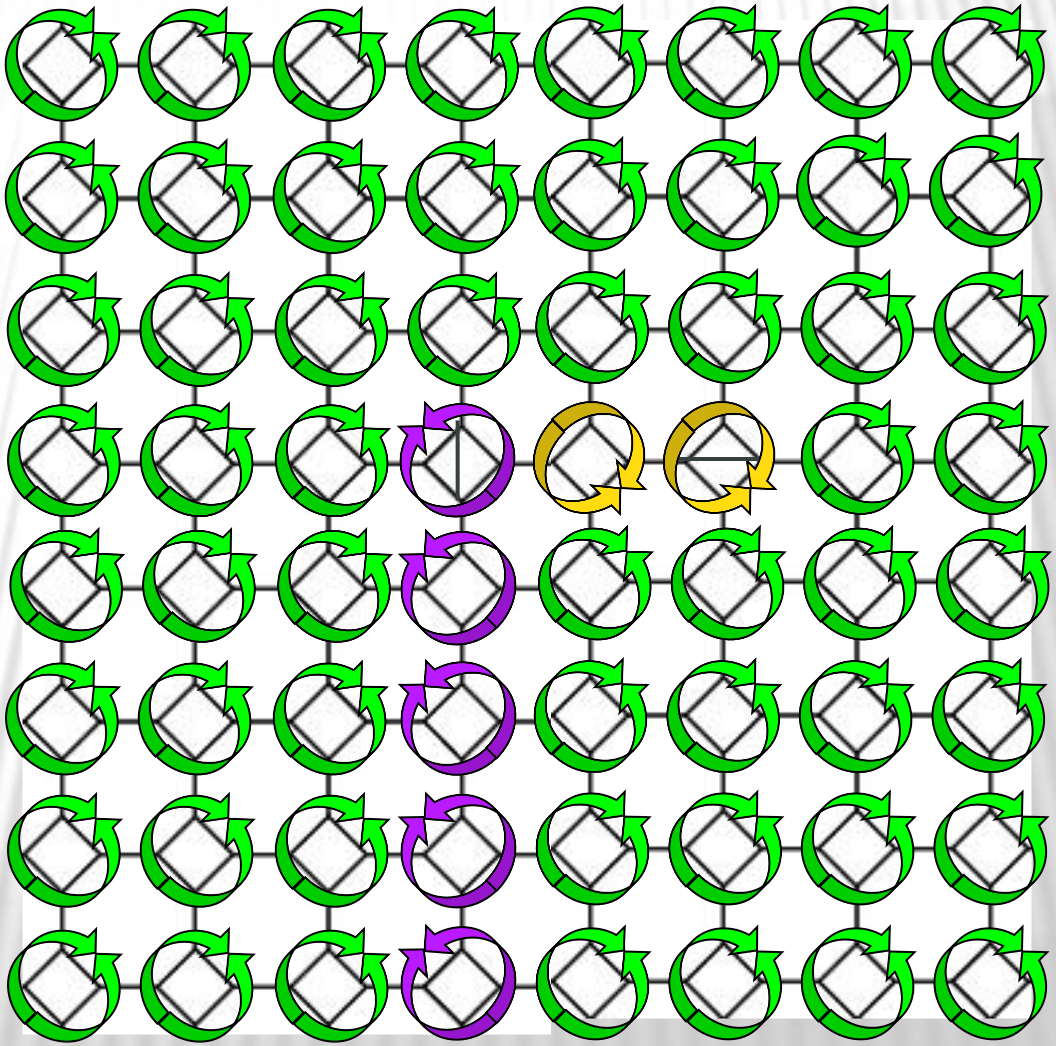
2D LATTICE (2 DEFECTS)

H-defect
plus
V-defect
(4 gs)



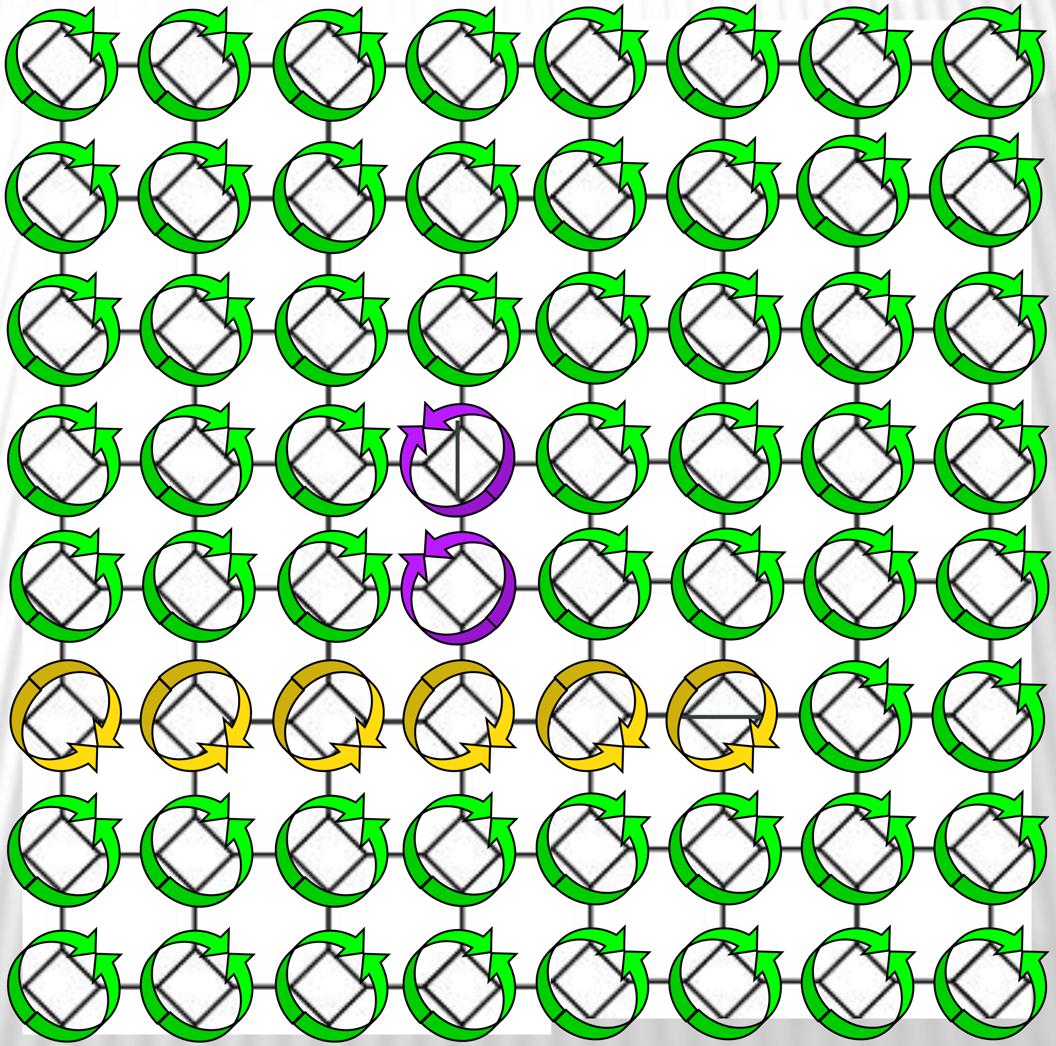
2D LATTICE (2 DEFECTS)

H-defect
plus
V-defect
(4 gs)

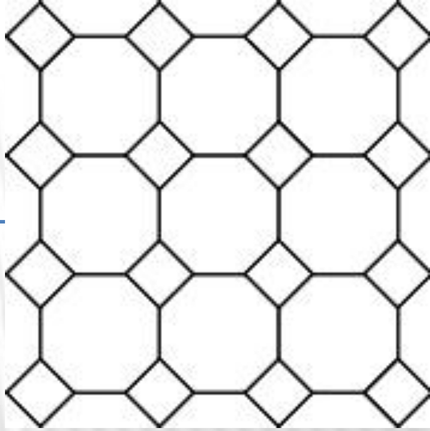


2D LATTICE (2 DEFECTS)

H-defect
plus
V-defect
(4 gs)



SUPERTOPOLOGICAL PHASE?



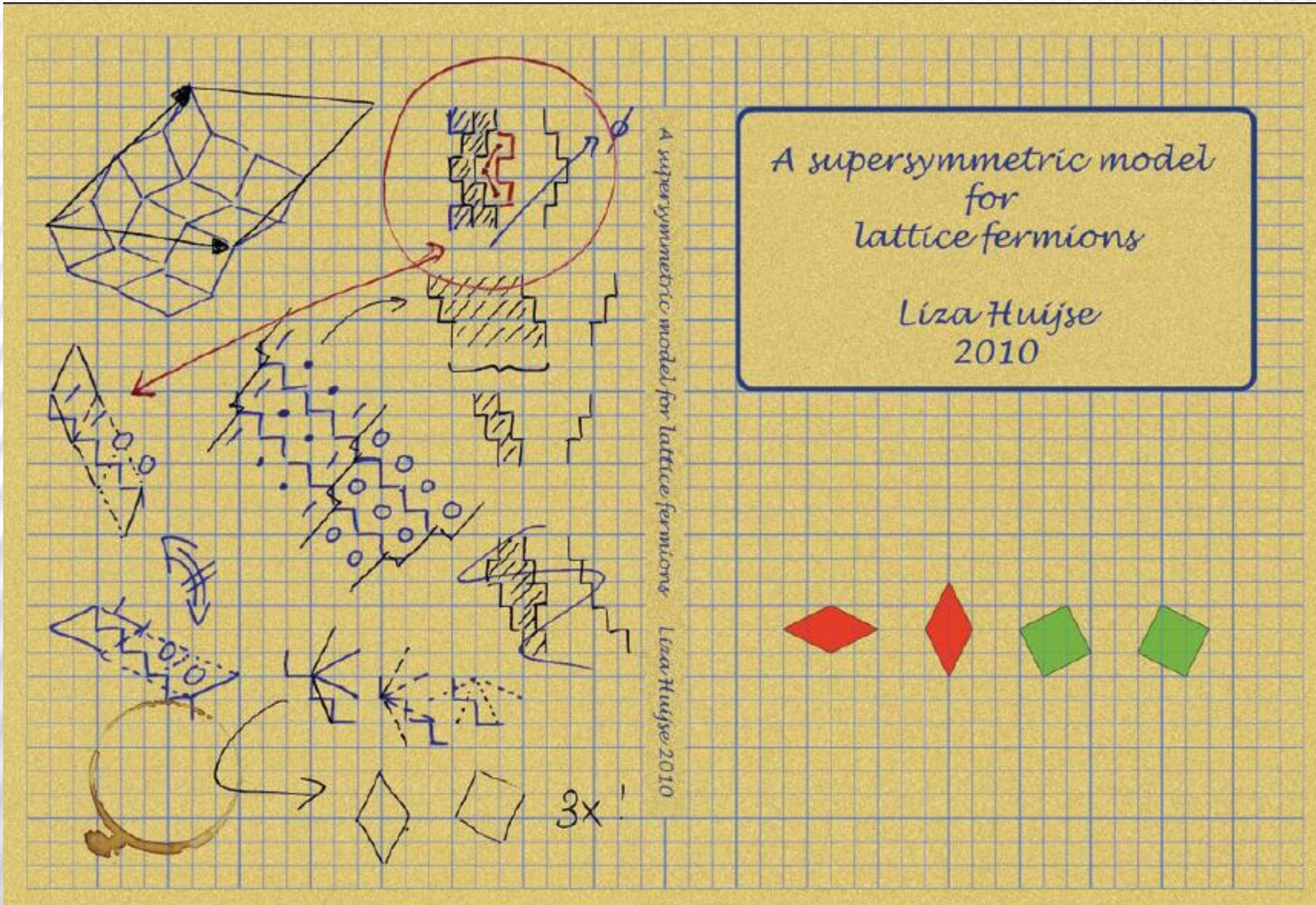
need to understand

- gap above torus gs?
- edge modes for open system?
- topological interactions and braiding of H, V and HV defects?
- …

CONCLUSION

Supersymmetry:

- Powerful analytic tools
- Subtle interplay between kinetic and interaction terms
- Variety of features: quantum criticality, superfrustration, edge modes, supertopological phase?, …


A supersymmetric model
for
lattice fermions
Liza Huijse
2010
A supersymmetric model for lattice fermions   Liza Huijse 2010
3x!

THANK YOU