Non-abelian anyons with ultracold atoms in artificial gauge potentials

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Stockholm, September 10, 2010

In collaboration with:

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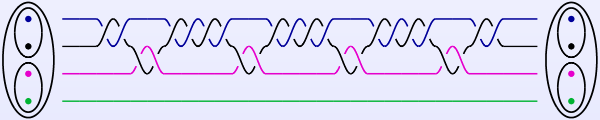
Contents

Non-Abelian Anyons

Nayak, Simon, Stern, Freedman, Das Sarma, *Rev. Mod. Phys* **80** (2008)

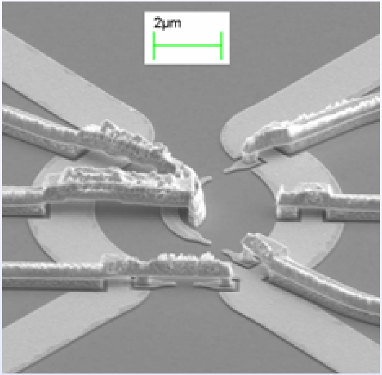
- **Non-Abelian anyons** are particles whose exchange is described by non-trivial unitary operators.
- They are the key for topological quantum computation: qubits can be stored in systems of non-abelian anyons and quantum gates can be obtained by their exchanges (braidings).



$$\sigma_3^{-2}\sigma_2^{2}\sigma_3^{-4}\sigma_2^{2}\sigma_3^{-4}\sigma_2^{2}\sigma_3^{-4}\sigma_2^{2}\sigma_3^{-2} \cong -iX \pm 0,0031$$

Non-Abelian Anyons

- It seems that certain fractional quantum Hall states have excitations which are non-Abelian anyons $(\nu = 5/2, 12/5)$.
- However, so far, there is no direct observation of their non-abelian nature.
- To control single excitation in a solid state FQH device seems to be a very difficult task.



Ultracold atoms to simulate FQHE

'And it grew wondrous cold'
The Rime of the Ancient Mariner

In order to simulate the quantum Hall physics with cold atoms the following elements are needed:

- A 2-dimensional confinement;
- Strong and possibly tunable interactions among atoms (eventually involving an inner degree of freedom);

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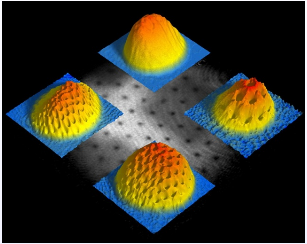
In order to simulate the quantum Hall physics with cold atoms the following elements are needed:

- A 2-dimensional confinement;
- Strong and possibly tunable interactions among atoms (eventually involving an inner degree of freedom);
- A fictious **magnetic field** for the atoms (which are neutral!).

Artificial magnetic fields for cold atoms:



With **rotating traps.**

See the review: N. Cooper, Adv. Phys. (2008)

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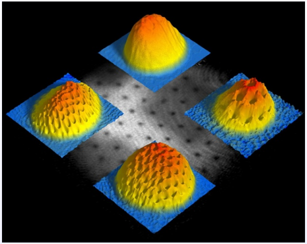

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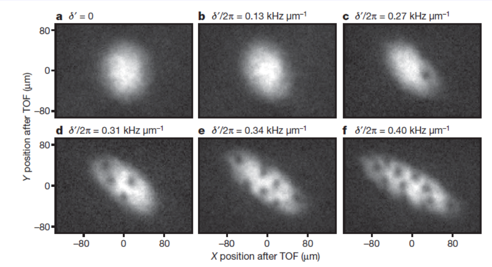


With spatially dependent optical couplings of **internal states** of the atoms (Berry phases).

See Y.-J. Lin et al., Nature (2009)

The duality between FQHE and rotating traps is based on the following Hamiltonian in the rotating frame:

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) - \Omega L_z + H_{int}$$

Rotating Cold Atomic Gases / Quantum Hall Systems

The duality between FQHE and rotating traps is based on the following Hamiltonian in the rotating frame:

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{1}{2} m \omega^2 r_i^2 \right) - \Omega L_z + H_{int}$$

Let us define a vector potential $A = (A_x, A_y)$:

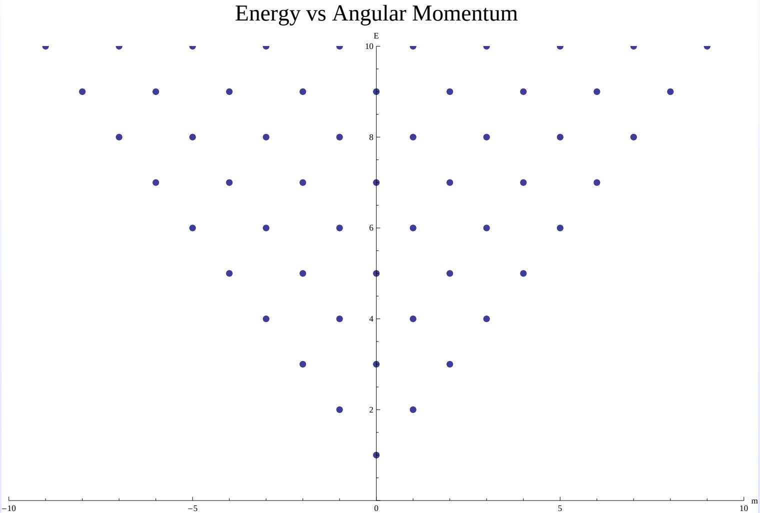
$$\vec{A} = m\omega \hat{z} \times \vec{r} \quad \Rightarrow \quad \vec{B}_{eff} = \vec{\nabla} \times \vec{A} = 2m\omega \hat{z}$$

We can rewrite H in the form:

$$H = \sum_{i=1}^{N} \frac{1}{2m} \left(\vec{p}_i - \vec{A} \right)^2 + (\omega - \Omega) L_z + H_{int}$$

which coincides with the Landau levels Hamiltonian in the limit of fast rotation $\Omega \rightarrow \omega$

From the 2D harmonic oscillator to the Landau levels:



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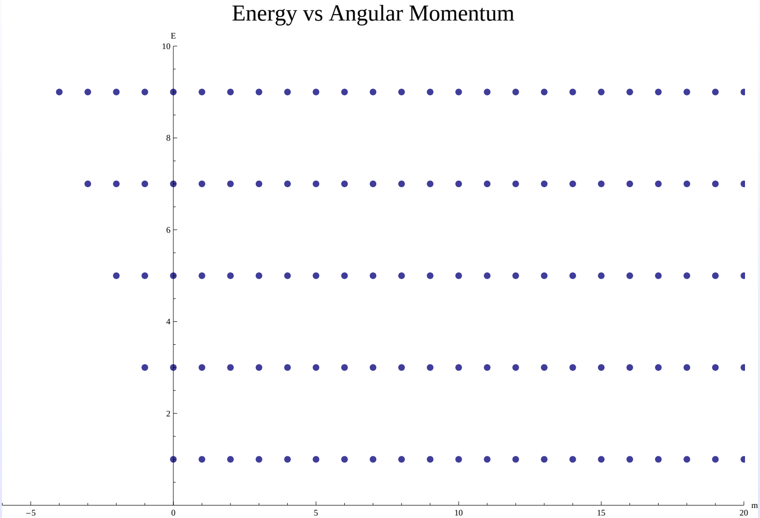

Lowest Landau Level

The angular momentum degeneracy implies that the lowest Landau level is spanned by the single-particle wavefunctions:

$$\psi_{0,m}(z) = z^m e^{-\frac{B}{4}|z|^2} \qquad \text{with} \quad z = x + iy$$

The many particle states are defined by:

$$\Psi\left(z_1, \ldots, z_N\right) = P\left(z_1, \ldots, z_N\right) e^{-\frac{B}{4}\sum_i |z_i|^2}$$

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Consider a **strong repulsive contact interaction**:

$$H = \sum_{i=1}^{N}\left(\vec{p}_i - \vec{A}\right)^2 + \left(\omega - \Omega\right) L_z + g \sum_{i<j} \delta\left(z_i - z_j\right)$$

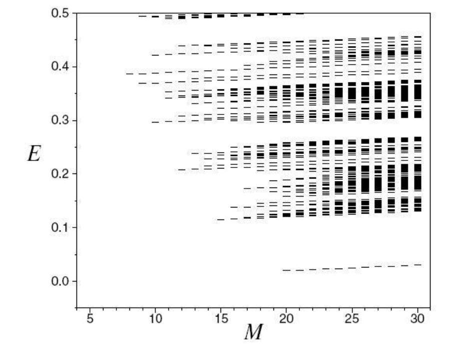
To have zero interaction energy one must introduce a Jastrow factor:

$$\Rightarrow P\left(z_1, \ldots, z_N\right) \propto \prod_{i<j}\left(z_i - z_j\right)$$

Laughlin States (Strong interaction regime):

$$\psi_L(z) = \prod_{i<j}^{N} (z_i - z_j)^m \prod_{k}^{N} e^{-\frac{|z_k|^2}{2}}$$

m **even: Bosonic gas**



m **odd: Fermionic gas**
($m = 1$: Slater determinant)

- The rotational term breaks the LLL degeneracy.
- The total angular momentum of ψ_{gs} is $M_0 = N(N-1)m/2$.
- Elementary excitations in the LLL must have $M > M_0$.

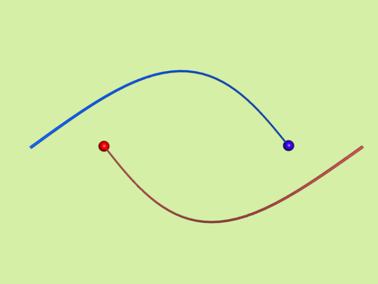
Figure: $N = 5$ bosons, $\Omega = 0,999\,\omega$

Excitations of the Laughlin state

With a repulsive localized potential (off–resonant laser) we can obtain excitations:

$$\psi_L \longrightarrow \psi_{\zeta_0,\zeta_1,\ldots,\zeta_M} = \prod_i (z_i - \zeta_0) \prod_i (z_i - \zeta_1) \ldots \prod_i (z_i - \zeta_M)\, \psi_L$$

- These quasi-holes can be moved one around the other.
- Once two excitations are counterclockwise exchanged the wavefunction acquires a phase π/m.
- They are **abelian anyons**.



- Laughlin ground-states have abelian excitations: the Berry phase is a *number*.
- How can we make it a *matrix*?

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- First naive guess: we add a degree of freedom, i.e. we consider a *two-component fermionic or bosonic gas*.

The Quest for Non-Abelian Anyons with cold atoms

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The question we want to address can be rephrased as:

*'Is it possible to obtain **non-abelian excitations** using a **non-abelian gauge potential**?'*

- Laughlin ground-states have abelian excitations: the Berry phase is a *number*.
- How can we make it a *matrix*?
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The question we want to address can be rephrased as:

'Is it possible to obtain ***non-abelian excitations*** *using a* ***non-abelian gauge potential****?'*

The general answer is *'No'*. But...

Non-Abelian Gauge Potentials

- What is a non-abelian gauge potential?

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Non-Abelian Gauge Potentials

- What is a non-abelian gauge potential?
- How can we simulate it with cold atoms?
- What are the features of the corresponding system?

Classical Non-Abelian Gauge Potentials

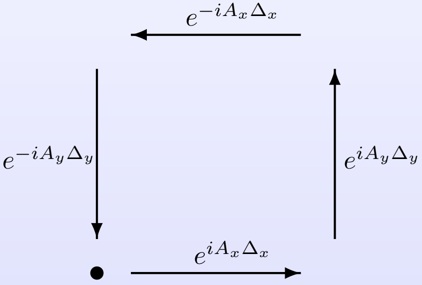
We consider a two-component gas ($|\uparrow\rangle$ and $|\downarrow\rangle$).
The single-particle Hamiltonian reads:

$$H = (p_x + A_x)^2 + (p_y + A_y)^2$$

where A_x and A_y are 2×2 matrices and:

$$[A_x, A_y] \neq 0 \qquad \text{Example :} \quad A_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Moving an atom around a closed path a non-trivial unitary operator is applied:



Wilson loop:

$$W = \text{Tr}\left(e^{iA_x\Delta_x} e^{iA_y\Delta_y} e^{-iA_x\Delta_x} e^{-iA_y\Delta_y}\right)$$

In order to be truly non-abelian:

$$|W| < 2$$

Non-Abelian Gauge Potentials

Dalibard et al., arXiv:1008.5378

How can non-abelian vector potentials be obtained?

- 4-level tripod atomic systems (Laser induced vector potential)
- Cavity QED (Larson and Levin)
- Optical lattices (Zoller, Lewenstein,...)
- Bose-Einstein condensates with laser induced potentials (Larson and Sjoqvist)

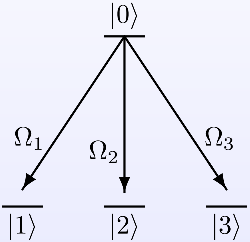
The essential characteristic of these systems is the presence of different internal atomic levels (usually hyperfine states) coupled by external potentials.

Their effective Hamiltonian can be described by a non-abelian gauge potential.

Tripod scheme

Ruseckas et al., PRL **95**, 010404 (2005)
Jacob et al., NJPh **10**, 045022 (2008)

Optical couplings among 3 quasi-degenerate ground states and one excited state:



Rabi frequencies:

$$\begin{aligned}\Omega_1 &= \Omega \sin\theta \cos\phi \, e^{iS_1} \\ \Omega_2 &= \Omega \sin\theta \sin\phi \, e^{iS_2} \\ \Omega_3 &= \Omega \cos\theta \sin\phi \, e^{iS_3}\end{aligned}$$

There are two dark states whose effective Hamiltonian is described by a 2×2 fictious non-abelian potential A:

$$H = (p_x + A_x)^2 + (p_y + A_y)^2$$

The system we want to analyze is characterized by the following potential:

$$A_x = q\sigma_x - \frac{B}{2} y\mathbb{I}\,, \qquad A_y = q\sigma_y + \frac{B}{2} x\mathbb{I}\,;$$

The corresponding Hamiltonian can be written as:

$$H = \left(p_x + A_x\right)^2 + \left(p_y + A_y\right)^2 = H_a\mathbb{I} + H_{na}$$

Introducing the ladder operators $d^\dagger = B\bar{z} - 4\partial_z$ and $d = Bz + 4\partial_{\bar{z}}$ we can rewrite:

$$H_a = 2q^2 + B + \frac{1}{4} d^\dagger d\,, \qquad H_{na} = q\begin{pmatrix} 0 & -id \\ id^\dagger & 0 \end{pmatrix}$$

which corresponds to the Jaynes-Cummings model.

- The non-Abelian term splits the Landau levels in two sublevels and couples subsequent Landau levels.
- The eigenstates of the Hamiltonian can be written in the form:

$$\chi_n^{\pm} = \left(B + 2q\sqrt{2Bn} \mp \sqrt{B^2 + 8q^2Bn}\right)\psi_{n-1}\left|\uparrow\right\rangle + \\ + \left(B - 2q\sqrt{2Bn} \pm \sqrt{B^2 + 8q^2Bn}\right)\psi_{n}\left|\downarrow\right\rangle$$

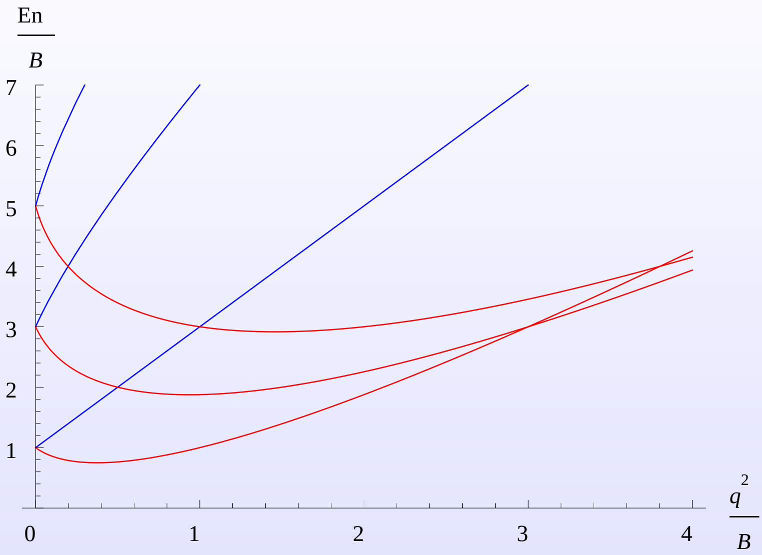
- The energy eigenvalues are:

$$\varepsilon_n^{\pm} = 2Bn + 2q^2 \pm \sqrt{B^2 + 8q^2Bn}$$

  and in the limit of small q we get the splitting $\sim E_{n-1} \pm 4q^2 n$
- There is only one family of uncoupled states: $\psi_0 \left|\downarrow\right\rangle$
- The ground state for $q^2 < 3$ is χ_1^-.

Non Abelian Symmetric Gauge

Single particle spectrum



- We can map the LLL states in QHE in the ground states χ_1^-:

$$z^m e^{-\frac{B}{4}|z|^2} |\downarrow\rangle \in \mathrm{LLL} \;\rightarrow\; \mathcal{G} z^m e^{-\frac{B}{4}|z|^2} |\downarrow\rangle \in \chi_1^- \,; \quad \mathcal{G} \equiv c_\uparrow \sigma_x + c_\downarrow d^\dagger$$

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- We introduce **intra-species repulsive delta interaction** ($\left|\uparrow\uparrow\right\rangle$ and $\left|\downarrow\downarrow\right\rangle$).

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Deformed Laughlin states (abelian!) are the new ground states:

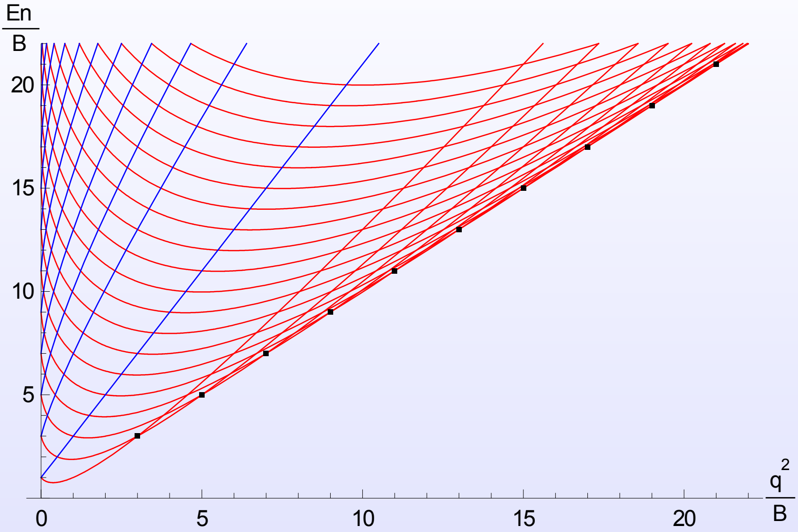
$$\Psi^m = \prod_j^N \mathcal{G}_j \prod_{i<j}^N (z_i - z_j)^m \, e^{-\frac{B}{4}\sum_i^N |z_i|^2} \left|\downarrow\downarrow \ldots \downarrow\right\rangle$$

Fermions: $m = 1, 3, 5, \ldots$
(Antisymmetric)

Bosons: $m = 4, 6, 8, \ldots$
(Intraspecies repulsion)

Degeneracy points

$q^2/B = 3, 5, 7, \ldots$



- In this point a single particle is in a superposition of states in χ_1^- and χ_2^-.
- We can define:

$$\underbrace{\mathcal{G}_1 \equiv c_{\uparrow,1}\sigma_x + c_{\downarrow,1}d^\dagger}_{\mathcal{G}_1 z^m e^{-\frac{B}{4}|z|^2}|\downarrow\rangle \in \chi_1^-} \quad \text{and} \quad \underbrace{\mathcal{G}_2 \equiv c_{\uparrow,2}\sigma_x d^\dagger + c_{\downarrow,2}d^{\dagger\,2}}_{\mathcal{G}_2 z^m e^{-\frac{B}{4}|z|^2}|\downarrow\rangle \in \chi_2^-}$$

Degeneracy point: $q^2/B = 3$

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- **Clustering**: to minimize the energy associated to an angular momentum term, $N/2$ atoms must be in χ_1^- and $N/2$ in χ_2^-.
- For fermions the highest density ground state is given by:

$$\Omega_c = \mathcal{A}\left[\prod_{k\in A}^{N/2}\mathcal{G}_{1;k}\prod_{i<j\in A}(z_i - z_j)\prod_{l\in B}^{N/2}\mathcal{G}_{2;l}\prod_{i<j\in B}(z_i - z_j)\right] e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}|\downarrow\downarrow\ldots\downarrow\rangle$$

Examples of ground states at the degeneracy points

Highest density state $\nu = 2$:

$$\Omega_c = \mathcal{A}\left[\prod_{k\in A}^{N/2}\mathcal{G}_{1;k}\prod_{i<j\in A}(z_i - z_j)\prod_{l\in B}^{N/2}\mathcal{G}_{2;l}\prod_{i<j\in B}(z_i - z_j)\right]e^{-\frac{B}{4}\sum_i^{2N}|z_i|^2}\left|\downarrow\downarrow\ldots\downarrow\right\rangle$$

Its elementary excitations are fermions but their superposition shows a non-Abelian behaviour.

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Its elementary excitations are fermions but their superposition shows a non-Abelian behaviour.

Deformed Haffnian state $\nu = 1$:

$$\Omega_{\mathrm{Hf}} = \mathrm{Hf}\left((\mathcal{G}_{1;i}\mathcal{G}_{2;j} - \mathcal{G}_{2;i}\mathcal{G}_{1;j}) \frac{1}{z_i - z_j} \right) \prod_{i<j}^{N} (z_i - z_j)\, e^{-\frac{B}{4}\sum_i^{N}|z_i|^2} |\downarrow\downarrow \dots \downarrow\rangle$$

Deformed Moore and Read state $\nu = 1/2$:

$$\Omega_{\mathrm{MR}} = \mathcal{S}\left(\prod_{i\in A} \mathcal{G}_{1;i} \prod_{l\in B} \mathcal{G}_{2;l} \right) \mathrm{Pf}\left(\frac{1}{z_i - z_j} \right) \prod_{i<j}^{N} (z_i - z_j)^2\, e^{-\frac{B}{4}\sum_i^{N}|z_i|^2} |\downarrow\downarrow \dots \downarrow\rangle$$

Its excitations are non-abelian Ising anyons.

Conclusions and Perspectives

- Non-abelian gauge potentials do not generally give non-abelian excitations: rather deformed Laughlin states are found;
- These potentials split the degeneracy of the Landau levels, are experimentally realizable, and, in the symmetric gauge case, can be exactly solved;
- There are degeneracy points where non-abelian ground states can be recovered;
- Future work:
  - We can introduce more components ($SU(3)$ potentials);
  - We can analyze systems where lines of degeneracy are present;
  - We can study conductivity properties of the states in the degeneracy points;
  - We can investigate the case of attractive inter-species interactions;
  - The non-abelian states at the degeneracy points deserve further investigations.

References

- N.R. Cooper, Adv. Phys. **57**, 539 (2008)
- S. Viefers, *J. Phys.: Condens. Matter* **20**, (2008)
- J. Ruseckas et al., PRL **95**, 010404 (2005)
- N. Goldman et al. PRL **103**, 035301 (2009)
- N. Goldman et al. PRA **79**, 023624 (2009)
- M. Burrello and A. Trombettoni, to appear in PRL

From the tripod atoms to the symmetric gauge

A gas of tripod atoms in rotation is described by the following Hamiltonian:

$$H_{\mathrm{Rot}} = \left(p + \tilde{A}\right)^2 + \frac{1}{4}\omega^2 r^2 + \Omega \vec{r} \times \left(\vec{p} + \tilde{A}\right) + V_{\mathrm{rot}}$$

The effective potential for the two dark states is given, as a function of the Rabi frequencies, by:

$$\begin{aligned}
\mathcal{A}_{11} &= \cos^2\phi \nabla S_{23} + \sin^2\phi \nabla S_{13} \\
\mathcal{A}_{22} &= \cos^2\theta \left(\cos^2\phi \nabla S_{13} + \sin^2\phi \nabla S_{23}\right) \\
\mathcal{A}_{12} &= \cos\theta \left(\frac{1}{2}\sin 2\phi \nabla S_{12} - i\nabla\phi\right)
\end{aligned}$$

With a suitable choice of the parameters and a gauge transformation it is possible to rewrite the Hamiltonian as:

$$H = (p + A)^2 + (\Omega - \omega)\, \vec{r} \times \vec{p}$$

which is the single-particle Hamiltonian we analyzed.

About the excitations of Ω_c

Ω_c has the following excitations:

- σ_1 is a fermionic (skyrmion-like) quasi-hole in the χ_1^- sector.
- σ_2 is a fermionic (skyrmion-like) quasi-hole in the χ_2^- sector.
- $\psi = \sigma_1 \times \sigma_2$ is a true bosonic quasi-hole.

The correspondig fusion rules are the same of the toric code.
However, in the contest of cold atoms, one can consider the following superposition (excited atom):

$$\sigma \equiv \frac{\sigma_1 + \sigma_2}{\sqrt{2}}$$

The fusion rules for σ result (Ising-like but with different dimensions):

$$\sigma \times \sigma = \mathbb{I} + \psi \,, \qquad \sigma \times \psi = \sigma$$

Braiding rules:

$$\rho_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \rho_2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}; \qquad F = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$