Partition Functions of Non-Abelian Quantum Hall States

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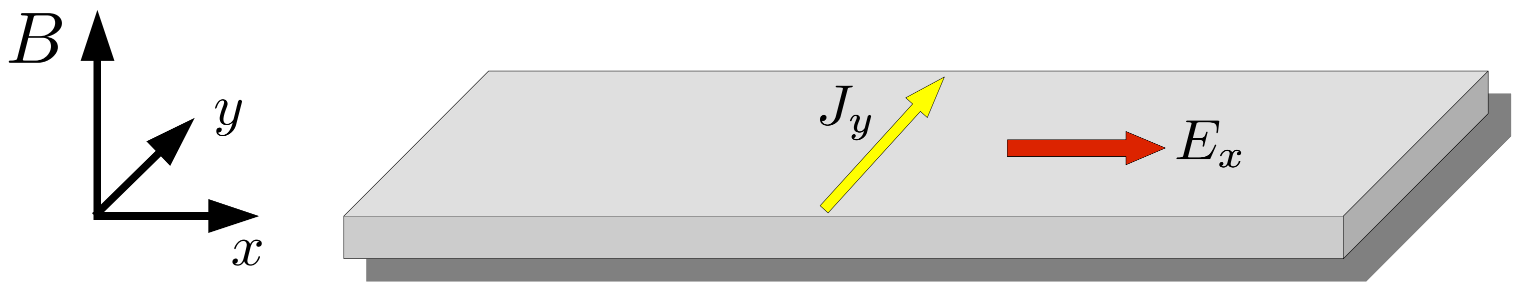
Outline

- CFT description: bulk & edge excitations
- Non-Abelian statistics
- Partition function & modular invariance
- Signatures of non-Abelian statistics:
  - thermopower
  - Coulomb blockade conductance peaks

Quantum Hall Effect

- 2 dim electron gas at low temperature T ~ 10 mK

  and high magnetic field B ~ 10 Tesla



- Conductance tensor $J_i = \sigma_{ij} E_j, \quad \sigma_{ij} = R_{ij}^{-1}, \qquad i,j = x,y$
- Plateaux: $\sigma_{xx} = 0, \; R_{xx} = 0$ no Ohmic conduction → gap

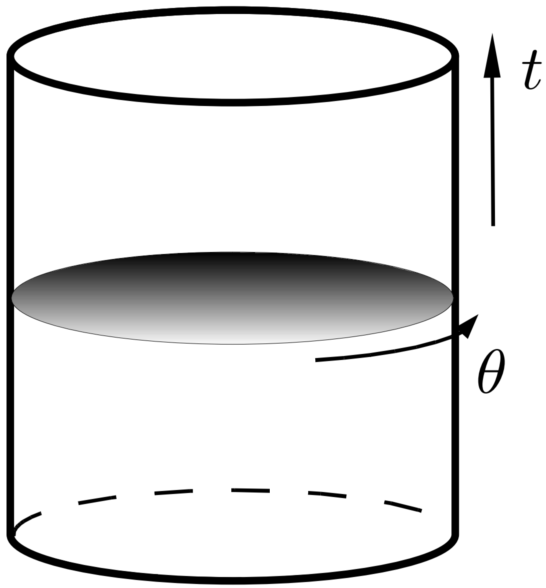
  $$\sigma_{xy} = R_{xy}^{-1} = \frac{e^2}{h}\nu, \quad \nu = 1(\pm 10^{-8}), 2, 3, \ldots \frac{1}{3}, \frac{2}{5}, \ldots, \frac{5}{2},$$

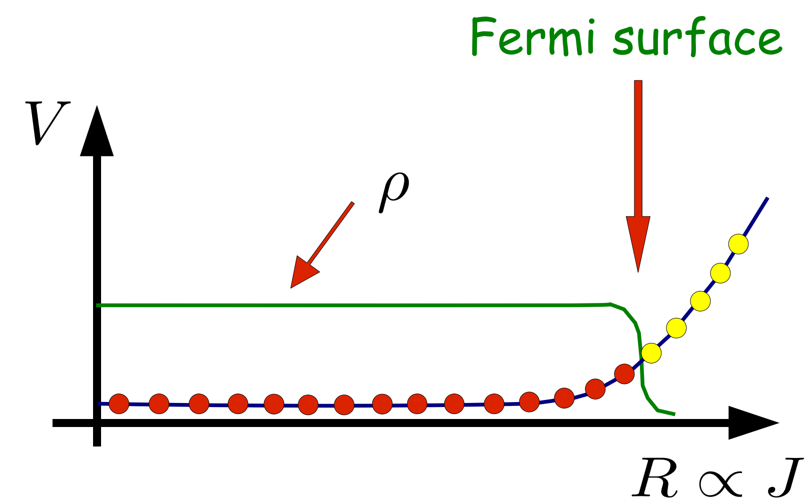
- High precision & universality
- Uniform density ground state: $\rho_o = \frac{eB}{hc}\nu$

Incompressible fluid

Conformal field theory of edge excitations

The edge of the droplet can fluctuate: edge waves are massless

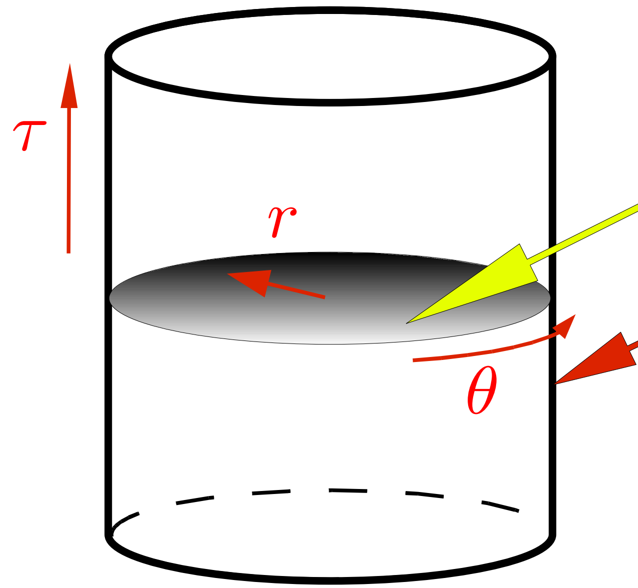




- edge ~ Fermi surface: linearize energy $\varepsilon(k) = \frac{v}{R}(k - k_F), \quad k = 0, 1, \ldots$
- relativistic field theory in 1+1 dimensions, chiral (X.G.Wen '89)

→ chiral compactified c=1 CFT (chiral Luttinger liquid)

CFT descriptions of QHE: bulk & edge



$z = re^{i\theta}$ plane (bulk excit)

$\zeta = e^{\tau} e^{i\theta}$ cylinder (edge excit)

$\langle\ \phi_1 \quad \phi_2\ \rangle_{CFT}$

$(z_1 - z_2)^{\alpha^2}$ anyon wavefunction

$(\zeta_1 - \zeta_2)^{\alpha^2}$ edge-excit. correlator

- same function by analytic continuation from the circle:
  - both equivalent to Chern-Simons theory in 2+1 dim (Witten '89, X.G.Wen '89)
- simplest theory for $\nu = 1/p$ is chiral Luttinger liquid (U(1) CFT):
  - wavefunctions: spectrum of anyons and braiding
  - edge correlators: physics of conduction experiments

Non-Abelian fractional statistics

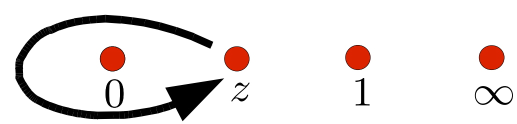
- $\nu = \frac{5}{2}$ described by Moore-Read "Pfaffian state" ~ Ising CFT x U(1)
- Ising fields: I identity, ψ Majorana = electron, σ spin = anyon
- fusion rules:
  - $\psi \cdot \psi = I$ 2 electrons fuse into a bosonic bound state
  - $\sigma \cdot \sigma = I + \psi$ 2 channels of fusion = 2 conformal blocks

$$\langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle = a_1 F_1(z) + a_2 F_2(z)$$ hypergeometric

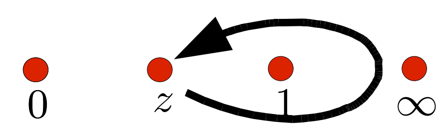
→ state of 4 anyons is two-fold degenerate (Moore, Read '91)

- statistics of anyons ~ analytic continuation → 2x2 matrix

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z e^{i2\pi}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z)$$



$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} ((z-1) e^{i2\pi}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z)$$



(all CFT redone for Q. Computation: M. Freedman, Kitaev, Nayak, Slingerland,...., 00'-10')

Topological quantum computation

- qubit = two-state system $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$
- QC: perform $U(2^n)$ unitary transformations in n qubit Hilbert space
- Proposal: (Kitaev; M. Freedman; Nayak; Simon; Das Sarma '06)

use non-Abelian anyons for qubits and operate by braiding

4-spin system $\alpha|F_1\rangle + \beta|F_2\rangle$ is 1 qubit (2n-spin has dim 2^{n-1})

- anyons topologically protected from decoherence (local perturbations)
- more stable but more difficult to create and manipulate

→ great opportunity

→ new experiments and model building

Models of non-Abelian statistics

- Study Rational CFTs with non-Abelian excitations:
  - best candidate: Pfaffian & its generalization, the Read-Rezayi states

$$\nu = 2 + \frac{k}{k+2}, \quad \begin{cases} k = 2, 3, \ldots \\ M = 1 \end{cases} \qquad U(1)_{k+2} \times \frac{SU(2)_k}{U(1)_{2k}}$$

  - alternatives: other (cosets of) non-Abelian affine groups $U(1) \times \frac{G}{H}$

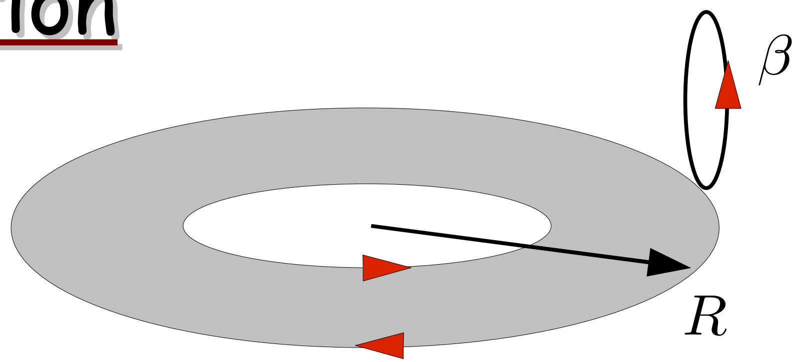
- Identify their N sectors of fractional charge and statistics
  - Abelian (electron) & non-Abelian (quasi-particles)

- Compute physical quantities that could be signatures of non-Abelian statistics:
  - Coulomb blockade conductance peaks
  - thermopower & entropy

use partition function

- quantity defining Rational CFT (Cardy '86; many people)
- complete inventory of states (bulk & edge)
- modular invariance as building principle:
  - S matrix and fusion rules
  - further modular conditions for charge spectrum
  - straightforward solution for any non-Abelian state $U(1) \times \frac{G}{H}$
  - useful to compute physical quantities
- Inputs:
  - non-Abelian RCFT (i.e. $\frac{G}{H}$)
  - Abelian field representing the electron "simple current"
- Output is unique

Annulus partition function



$$i2\pi\ \tau = -\beta\frac{v}{R} + it, \quad \beta = \frac{1}{k_B T}$$

$$i2\pi\zeta = \beta(-V_o + i\mu)$$

$$Z_{\text{annulus}} = \sum_{\lambda=1}^{p} |\theta_\lambda(\tau,\zeta)|^2, \quad \theta_\lambda(\tau,\zeta) = \text{Tr}_{\mathcal{H}^{(\lambda)}}\left[e^{i2\pi\tau(L_0 - c/24) + i2\pi\zeta Q}\right]$$

modular invariance conditions

geometrical properties & physical interpretation (A. C., Zemba, '97)

$$T^2 : \ Z(\tau+2,\zeta) = Z(\tau,\zeta), \quad L_0 - \overline{L}_0 = \frac{n}{2}$$ half-integer spin excitations globally

$$S : \ Z\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = Z(\tau,\zeta),$$ completeness $$\theta_\lambda\left(\frac{-1}{\tau}\right) = \sum_{\lambda'} S_{\lambda\lambda'}\theta_{\lambda'}(\tau)$$ S matrix

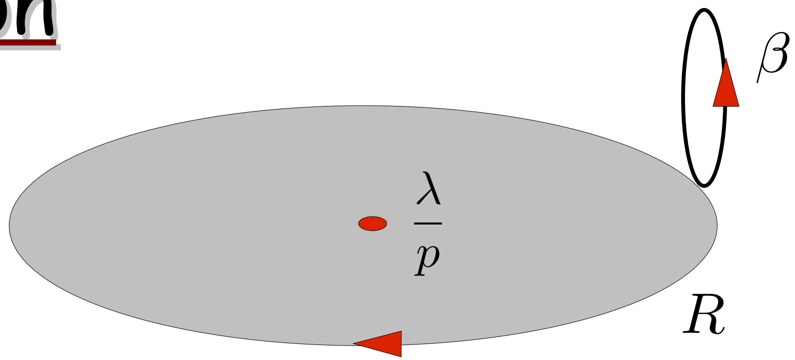
$$U : \quad Z(\tau,\zeta+1) = Z(\tau,\zeta), \quad Q - \overline{Q} = n$$ integer charge excitations globally

$$V : \quad Z(\tau,\zeta+\tau) = Z(\tau,\zeta), \quad \Delta Q = \nu$$ add one flux: spectral flow

$$\theta_\lambda(\zeta+\tau) \sim \theta_{\lambda+1}(\tau)$$

Disk partition function

Annulus -> Disk (w. bulk q-hole $\bar{Q} = \frac{\lambda}{p}$)



$$Z_{\text{annulus}} \;\rightarrow\; Z_{\text{disk},\;\lambda} = \theta_\lambda(\tau,\zeta)$$

$$\theta_\lambda(\tau,\zeta) = K_\lambda(\tau,\zeta;p) = \frac{1}{\eta}\sum_n e^{i2\pi\left[\tau\frac{(np+\lambda)^2}{2p}+\zeta\frac{np+\lambda}{p}\right]}, \quad \nu = \frac{1}{p}, \quad c = 1$$

- $U:\; Q - \overline{Q} = n$ sectors with charge $Q = \frac{\lambda}{p} + n$

  basic quasiparticle + n electrons

- T^2 : electrons have half-integer dimension (=J),

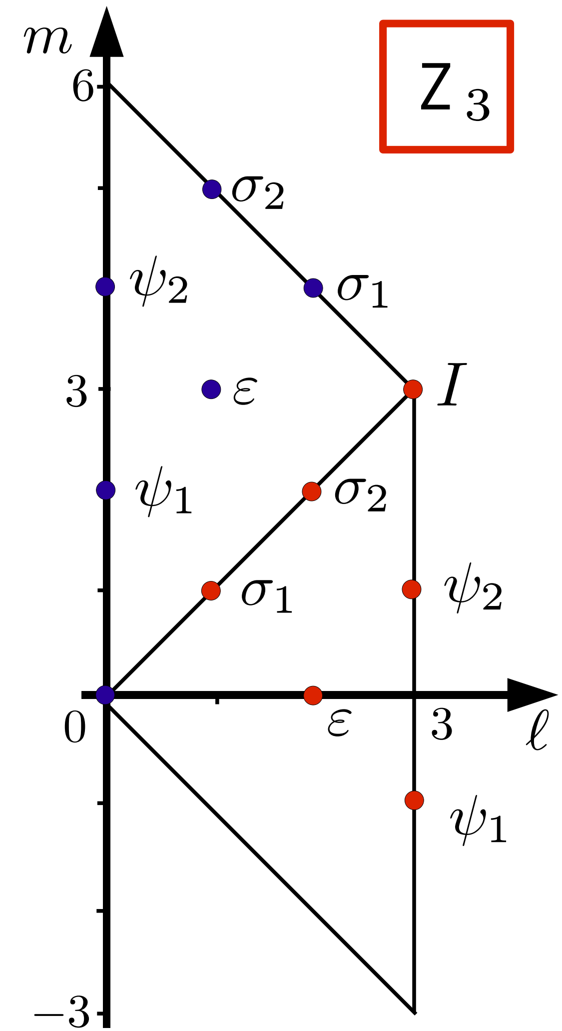
  and integer relative statistics with all excitations

- # sectors $p = dim\,(S_{\lambda\lambda'})$ = Wen's topological order

→ we recover phenomenological conditions on the spectrum

Pfaffian & Read-Rezayi states

$$\nu = 2 + \frac{k}{k+2}, \quad \begin{cases} k = 2, 3, \ldots \\ M = 1 \end{cases} \qquad U(1)_{k+2} \times \frac{SU(2)_k}{U(1)_{2k}}$$



- Z_k parafermion sectors (ℓ, m) and characters

  $\chi^\ell_m, \quad \ell = 0, 1, \ldots, k, \qquad m \bmod 2k$

- electron is Abelian $\quad \Psi_e = e^{i\alpha\varphi}\psi_1, \quad (\ell, m) = (0, 2)$
- $Q = \frac{q}{p}$ + electron: $\quad (q, m, \ell) \rightarrow (q+p, m+2, \ell)$

  $p = 2 + k$; parity rule $\quad q = m \bmod k$

$$\theta^\ell_a = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \; \chi^\ell_{a+2\beta}$$

- sectors labeled by (a, ℓ)

  $a = 0, \ldots, k+1, \quad \ell = 0, \ldots, k, \quad a = \ell \bmod 2$

- # sectors = topological order $\quad (k+2) \times \frac{k(k+1)}{2} \times \frac{1}{k} = \frac{(k+2)(k+1)}{2}$

$$Z^{RR}_{annulus} = \sum_{\ell=0}^{k} \sum_{a=0}^{\hat{p}-1} \left| \, \theta^\ell_a(\tau,\zeta) \, \right|^2 , \qquad Z^{RR}_{disk} = \theta^\ell_a = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \; \chi^\ell_{a+2\beta}$$

Ex: Pfaffian (k=2)

ground state + electrons

$$Z^{\text{Pfaffian}}_{\text{annulus}} = |K_0 I + K_4 \psi|^2 \; + |K_0 \psi + K_4 I|^2 \; + |(K_1 + K_{-3})\,\sigma|^2$$
$$+ |K_2 I + K_{-2}\psi|^2 \; + |K_2 \psi + K_{-2} I|^2 \; + |(K_3 + K_{-1})\,\sigma|^2$$

non-Abelian quasiparticle

- K_λ charge parts $Q = \frac{\lambda}{4} + 2n$
- $I, \; \psi, \; \sigma$ Ising parts (Majorana fermion)
  - 6 sectors
  - also $Q = 0, \pm\frac{1}{2}$ Abelian excitations

(Milavanovich, Read '96; AC, Zemba '97)

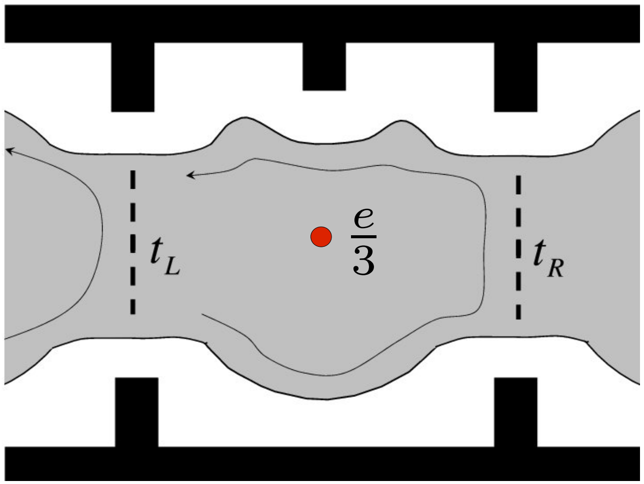
$$Z^{RR}_{annulus} = \sum_{\ell=0}^{k} \sum_{a=0}^{\hat{p}-1} \left| \theta^{\ell}_{a}(\tau,\zeta) \right|^2 , \qquad Z^{RR}_{disk} = \theta^{\ell}_{a} = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \, \chi^{\ell}_{a+2\beta}$$

- charge and neutral q. #'s are coupled by "parity rule"
- but S-matrix for θ^{ℓ}_{a} is factorized: $S_{a\ell,a'\ell'} \sim e^{i2\pi aa' N/M} \; s_{\ell\ell'}$

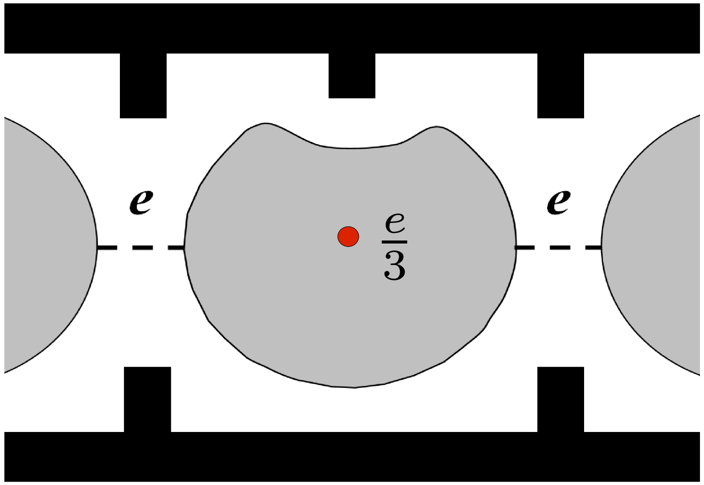
- generalization to other N-A models: (A.C, G. Viola, '10)
  - Wen's non-Abelian Fluids $U(1) \times SU(2)_k$
  - Anti-Read-Rezayi $U(1) \times \overline{SU(2)_k}$
  - Bonderson-Slingerland $U(1) \times \mathrm{Ising} \times SU(n)_1$
  - N-A Spin Singlet state $U(1)_q \times U(1)_s \times \dfrac{SU(3)_k}{U(1)^2}$

→ unique result once N-A CFT and electron field have been chosen

Experiments on non-Abelian statistics



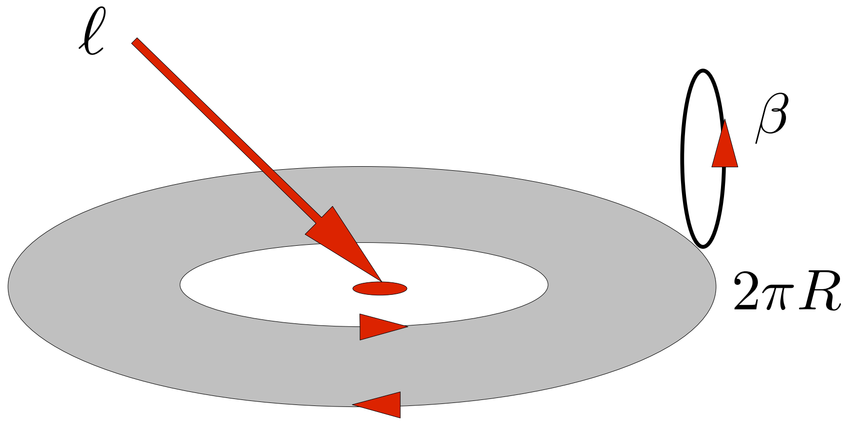
(a)



(b)

- (a) interference of edge waves (Chamon et al. '97; Kitaev et al 06)
  - ➡ Aharonov-Bohm phase, checks fractional statistics
  - – experiment is hard (Goldman et al. '05; Willett et al '09)
- (b) electron tunneling into the droplet (Stern, Halperin '06)
  - ➡ Coulomb blockade conductance peaks (Ilan, Grosfeld, Schoutens, Stern '08)
  - – check quasi-particle sectors (Stern et al.; A.C. et al. '09 - '10)
- ➡ Thermopower (Cooper, Stern; Yang, Halperin '09; Chickering et al. '10)

Thermopower



- fusion of n ℓ-type quasiparticles:
  - multiplicity $\sim (d_\ell)^n, \quad n \to \infty$
- Entropy

$$S(T=0) \sim n \log(d_\ell), \qquad d_\ell = \frac{s_{\ell 0}}{s_{00}} > 1$$ (quantum dimension)

- put temperature ΔT and potential ΔV_o gradients between two edges
- at equilibrium: $d\Omega = -SdT - QdV_o = 0$
- thermopower

$$\mathcal{Q} = -\frac{\Delta V_o}{\Delta T} = \frac{S}{Q}$$

(Cooper,Stern;Yang, Halperin '09)

- entropy from Z:

$$S = \left(1 - \tau \frac{d}{d\tau}\right) \log \frac{\theta_a^\ell(\tau + \Delta\tau, \zeta + \Delta\zeta)}{\theta_0^0(\tau, \zeta)} \sim \log \frac{s_{\ell 0}}{s_{00}}, \quad \tau \sim \frac{\beta}{R} \to 0$$

- it could be observable by varying B off the plateau center

$$\mathcal{Q} = \left| \frac{B - B_o}{e^* B_0} \right| \log(d_1)$$

(Chickering et al '10)

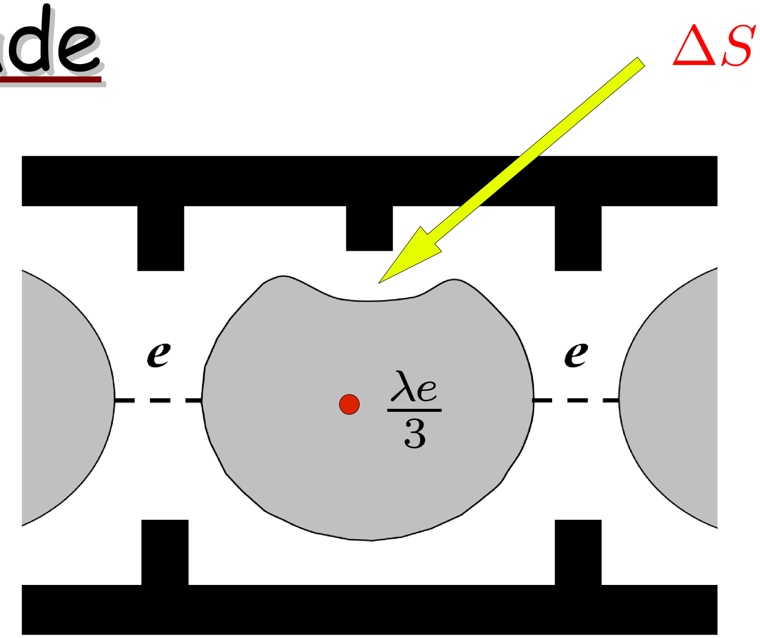
Coulomb blockade

- Droplet capacity stops the electron
- Bias & T ~ 0: needs energy matching

$$E(n+1, S) = E(n, S)$$

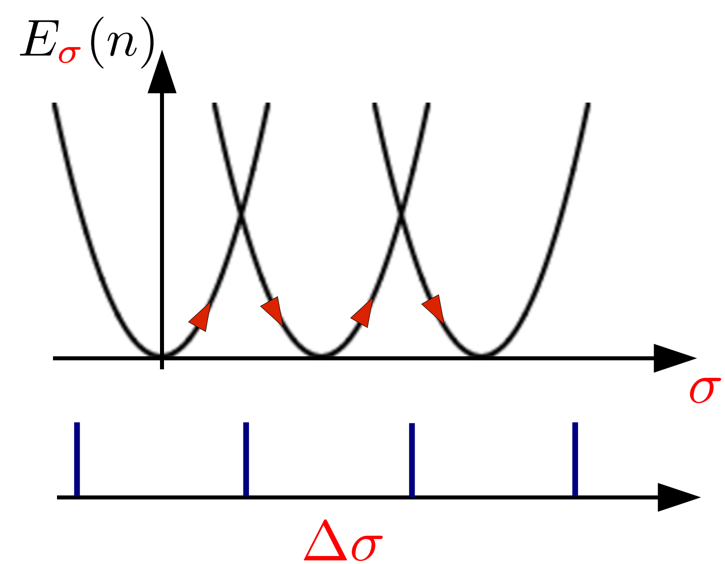
➡ current peak

- energy deformation by $\Delta S \sim \Delta Q_{\rm bkg}$



$$E(n, S) = \frac{v}{R}\frac{(\lambda + pn - \sigma)^2}{2p} \propto (Q - Q_{\rm bkg})^2$$

$$\Delta\sigma = \frac{B\Delta S}{\Phi_o} = \frac{1}{\nu}\ , \qquad \Delta S = \frac{e}{n_o}$$



$U(1)$ ➡ equidistant peaks

$U(1) \times \frac{G}{H}$ ➡ modulated pattern

$$\Delta\sigma^\ell_m = \frac{1}{\nu} + \frac{v_n}{v}\left(h^\ell_{m+2} - 2h^\ell_m + h^\ell_{m-2}\right)$$

$$\frac{v_n}{v} \sim \frac{1}{10}$$

- compares states in the same sector

$$\theta_a^\ell = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)}\ \chi_{a+2\beta}^\ell$$

- spectroscopy of lowest CFT states
- $T=0$: cannot distinguish NA state from "parent" Abelian state

(Bonderson et al. '10)

$T>0$ corrections

$$\langle Q\rangle_T \sim \frac{\partial}{\partial V_o}\log\theta_a^\ell$$

- two scales: $0 < T_n < T_{ch}, \qquad T_n = \frac{v_n}{R}\ , \quad T_{ch} = \frac{v}{R} \sim 10\ T_n$

$$T < T_n : \quad \Delta\sigma_m^\ell = \cdots + \frac{T}{T_{ch}} \log\left(\frac{(d_m^\ell)^2}{d_{m+2}^\ell\ d_{m-2}^\ell}\right),$$

d_m^ℓ multiplicity of neutral states

in (331) & Anti-Pfaff, not in Pfaff

$$T_n < T < T_{ch} : \qquad \cdots + \propto \frac{T}{T_{ch}}\ e^{-h_1^1 T/T_n}\ \frac{s_{\ell 1}}{s_{\ell 0}},$$

S matrix of non-Abelian part

→ test non-Abelian part of disk partition function

(Stern et al., Georgiev, AC et al. '09, '10)

$T > 0$ off-equilibrium

- energy offset $\Delta E_\sigma > 0$ and bias $\Delta V_o > 0$

  relevant regime: $T < \Delta V_o < \Delta E_\sigma$

→ thermal-activated Coulomb-Blockade conduction

$$\Gamma \sim d_a^\ell \left(\Delta E_\sigma^2 - \Delta V_o^2\right) e^{-\beta(\Delta E_\sigma - \Delta V_o)}$$

- real-time experiment of peak counting
- sensible to level multiplicity d_a^ℓ (but qualitative)

Conclusions

- non-Abelian anyons could be seen
- partition function:
  - it is simple enough
  - it defines the CFT, its sectors, fusion rules etc.
  - it is useful to compute observables
  - it can be the basis for further model building