Dynamics of 1D correlated systems with and without disorder

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Outline

1. What does entanglement reveal about quantum critical states?

Quantum critical states have infinitely more
entanglement than "typical" many-body states.
When is there universal behavior in entanglement?

(a) how does better understanding of entanglement entropy connect to numerical methods for correlated quantum ground states? "*finite-entanglement scaling*"

Frank Pollmann, S. Mukerjee, Ari Turner, JEM, PRL 2009; F. Pollmann and JEM, NJP 2010

(b) dynamical results on 1D correlated systems

F. Pollmann, S. Mukerjee, A. Green, JEM, PRE 2010

Jonas Kjäll, F. Pollmann, JEM, arXiv 2010

2. What can we say about dynamics in disordered correlated systems and "many-body localization at infinite temperature"?

Jens Bardarson, F. Pollmann, JEM, in preparation

Quantum entanglement

Sometimes a pure quantum state of a bipartite system *AB* is also a pure state of each subsystem separately:

Example: S_z=1 state of two s=1/2 spins, A and B

$$|\Psi_{AB}\rangle = |\uparrow_A\rangle \otimes |\uparrow_B\rangle$$

a "product" state

Sometimes a pure quantum state of a bipartite system *AB* is **not** a pure state of each subsystem separately:

Example: singlet state of two s=1/2 spins

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an "entangled" state

"Maximal knowledge of the whole does not imply maximal knowledge of the parts"

Entanglement entropy

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle - |\downarrow_A\rangle \otimes |\uparrow_B\rangle\right)$$

an “entangled” state

In an entangled state, the state of subsystem A or B is not a pure quantum state, but rather a **density matrix**

For the singlet

$$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_B$$

A classical uncertainty or **entropy** has been created by the operation of looking at only part of the system.

Entanglement entropy

Definition: the entanglement entropy of a pure state,
with respect to a given partition into A and B,
is the von Neumann entropy of the partial density matrices

$$\langle\phi_1|\rho_A|\phi_2\rangle = \sum_j (\langle\phi_1| \times \langle\psi_j|)|\psi\rangle\langle\psi|(|\phi_2\rangle \times |\psi_j\rangle)$$

$$S(\rho) = -\mathrm{Tr}\rho_A \log_2 \rho_A = -\mathrm{Tr}\rho_B \log_2 \rho_B$$

The singlet generates one bit of classical entropy when the two spins are separated

Note that the partial density matrix for subsystem A gives the results of *all* experiments limited to A

How much entanglement entropy occurs in ground states of local Hamiltonians?

To get some intuition for how entanglement behaves in statistical physics, consider "valence bond states" of $s=1/2$ systems:

Rule: every spin forms a singlet with some other spin

Short-ranged VBS
(= "dimer covering")

Long-ranged VBS

In these states, entanglement entropy S just counts singlets:
S = 1 bit for each singlet crossing the AB boundary. (But real states are usually a bit more complicated.)

How much entanglement entropy occurs in ground states of local Hamiltonians?

Consider partitions of a *d*-dimensional infinite system *AB* into a subregion *A* of linear size *L* and an infinite subregion *B*.

How should entanglement entropy scale with *L*?

If we can ignore entanglement between points farther apart than some length scale ξ, then entanglement entropy should be determined by a shell of thickness ~ ξ around the *AB* boundary:

$$S \sim L^{d-1}\xi \Rightarrow S \sim L^{d-1} \text{as } L \to \infty \text{ with system parameters fixed}$$

the "area law"

If there is no notion of locality, any site in *A* is as likely to be entangled with a site in *B* as with another site in *A*, and $S \sim L^d$

What the area law tells us

In one dimension, the area law has been established for gapped systems (Verstrate&Cirac; Hastings; for a review, see Eisert et al., arXiv 2008)

This area law also explains why gapped systems in 1D are well approximated by "matrix product states" (to be introduced in a moment) and can be simulated efficiently on a classical computer.

We can go beyond the area law

1. by looking at gapless states (subject of this talk)

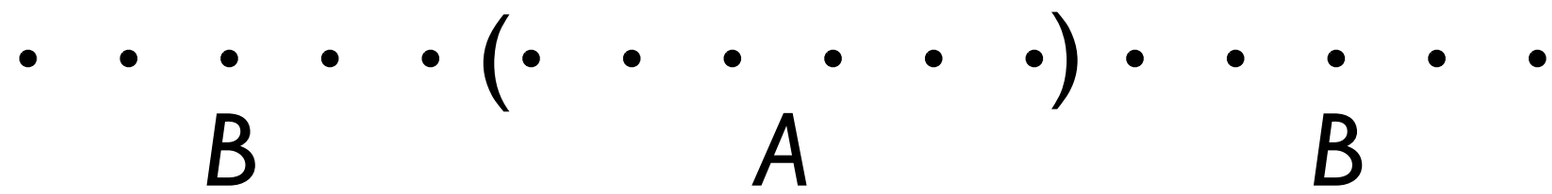
2. by looking at interesting subleading terms

In higher dimensions, there can be area laws even in gapless systems, with interesting subleading parts (Ryu-Takayanagi, Fradkin-JEM, Metlitski et al.)

How much entanglement entropy occurs in ground states of local Hamiltonians?

We start with "pure" (translation-invariant), local Hamiltonians in one dimension.

Consider a partition for which *A* is a contiguous set of *N* spins inside an infinite chain:

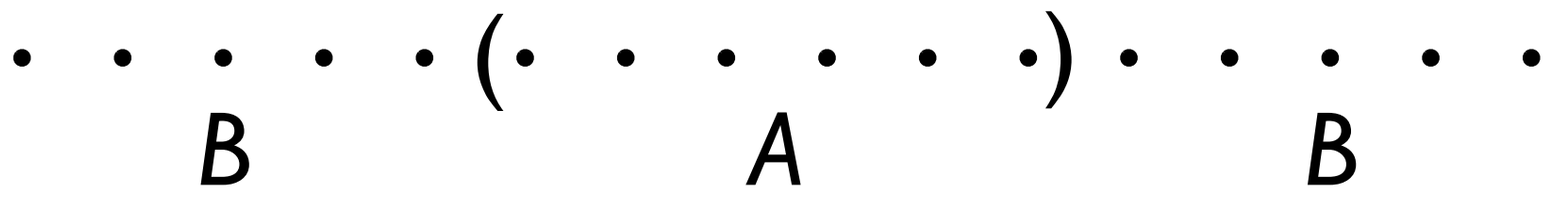


Away from critical points (i.e., when correlations are short-ranged), entanglement is localized in the vicinity of the boundary and the "area law" is satisfied:

$$\lim_{N\to\infty} S = C < \infty$$

But what about quantum critical states? Is there qualitatively more entanglement?

How much entanglement entropy occurs in critical states of local Hamiltonians?



Example of a quantum critical ground state: (c=1) Heisenberg AF

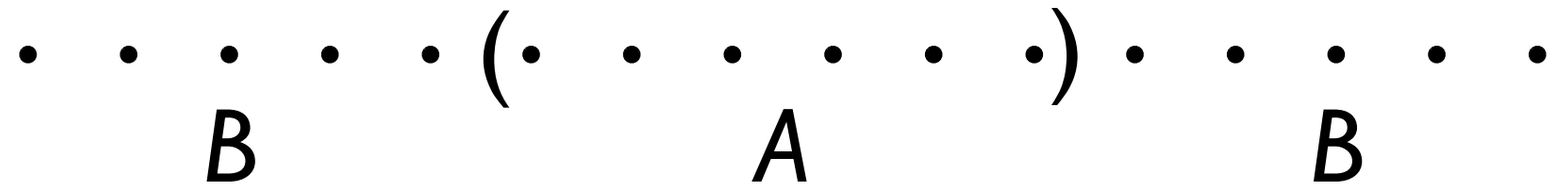
$$H = J \sum_i \mathbf{s}_i \cdot \mathbf{s}_j, \quad J > 0$$

At criticality, the entanglement of a connected subset of N spins, with the remaining spins, is (note: violates area law)

$$\lim_{N\to\infty} S \sim \frac{c}{3} \log N \to \infty$$

At clean and conformally invariant quantum critical points in d=1, there is logarithmically divergent entanglement with a coefficient related to the "central charge" of associated CFT. (Holzhey, Wilczek et al. 94, Vidal et al. 03, Calabrese and Cardy 04).

Uses of entanglement entropy in d=1



For the subset of 1D quantum critical points that are described by 2D conformal field theories:

The appearance of the central charge in the ground-state entanglement is consistent with its appearance in other quantities related to entropy, such as the free energy at finite temperature

$$f = \frac{F}{L} = f_0 - \frac{\pi}{6} c (kT)^2 \hbar v$$

The central charge is an important quantity, but only defined for a subset of quantum critical points.

(Entanglement entropy can be defined at *any* quantum critical point. Does it still show similar behavior, with a universal coefficient?)

What about “applications”?

We want to apply knowledge about entanglement to improve our understanding of old-fashioned condensed matter: correlations, phase diagrams, etc.

1. Quantum critical states have increased entanglement, sometimes with universal properties, both in 1D and higher dimensions.

2. It is believed that entanglement entropy underpins the best algorithms for correlated states and dynamics in 1D (and ground states in 2D) not amenable to quantum Monte Carlo.
Feiguin & White, Vidal, Verstraete & Cirac, Kollath, Schollwoeck, ...

Connection between (1) and (2): (1) Critical states of local Hamiltonians have “a moderate amount” of entanglement;
(2) efficient numerical methods should notice this property.

Studying quantum correlations with classical algorithms: applied entanglement entropy

Basic (hazy) concept: "Entanglement entropy determines how much classical information is required to describe a quantum state."

Example:
how many classical real numbers are required to describe a *product* (not entangled) state of N spins?

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Answer: ~ N (versus exponentially many for a general state)

How do we efficiently manipulate/represent moderately entangled states?

Applied entanglement entropy

The remarkable success of the density-matrix renormalization group algorithm in one dimension (White, 1992; Ostlund and Rommer, 1995) can be understood as follows:

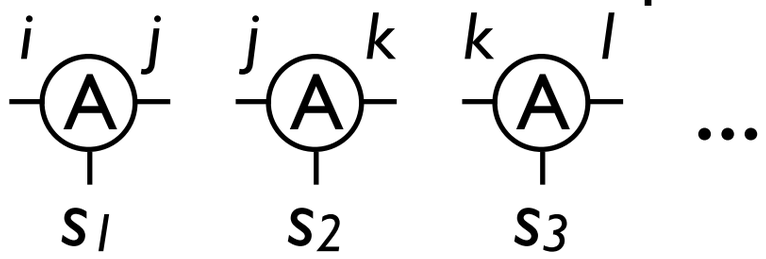
DMRG constructs "matrix product states" that retain local entanglement but throw away long-ranged entanglement.

Example states for four spins:

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Graphical tensor network representation:



Application: finite-entanglement scaling

All numerical methods have difficulty with quantum critical points. In DMRG-type approaches, this can be understood from the divergence of entanglement entropy at such points: the entanglement in a matrix product state is limited by dim *A*.

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Quantitatively, it is found that dim *A* plays a role similar to imposing a finite system size:

(Tagliacozzo et al., PRB 2008). $$L_{\text{eff}} \propto \chi^{\kappa}, \quad \chi = \dim A$$

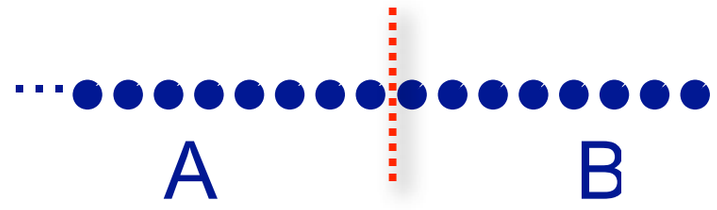
Finite matrix dimension effectively moves the system away from the critical point, just like a relevant field in the Hamiltonian.

What determines this "finite-entanglement scaling"?
Is it like "finite-size scaling" of CFT's (cf. Blöte, Cardy, & Nightingale)

Another way to picture the entanglement of a state

- **Schmidt decomposition** of the state (**SVD**):
-

$$
\begin{aligned}
|\psi\rangle &= \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B \\
&= \sum_{\alpha=1}^{\min(N_A,N_B)} \lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B
\end{aligned}
$$



with $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$

- a natural measure of the entanglement is the **entropy:**

$$S_A = S_B = S = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

- equivalent to definition in terms of the reduced density matrices $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ (von Neumann):

$$S_A = -\mathrm{Tr}\rho_A \log(\rho_A)$$

- entropy is **maximal** if all λ_α are equal
- entropy is **minimal** if $\lambda_1 = 1$ (with $\lambda_\alpha = 0,\ \alpha > 1$)

- **entanglement entropy** is interesting because...
  - ➡measure of the amount of classical information needed to represent the state (at least in **DMRG**, **TEBD**, ...)
  - ➡basis-independent characterization of phase transitions
  - ➡important more generally for quantum information theory

Efficient representation of quantum states?

- Hilbert-space dimension of many-body problems increases **exponentially** with number of sites
  example: spin 1/2 system on “classical” computers
  (store **one state** in double precision)

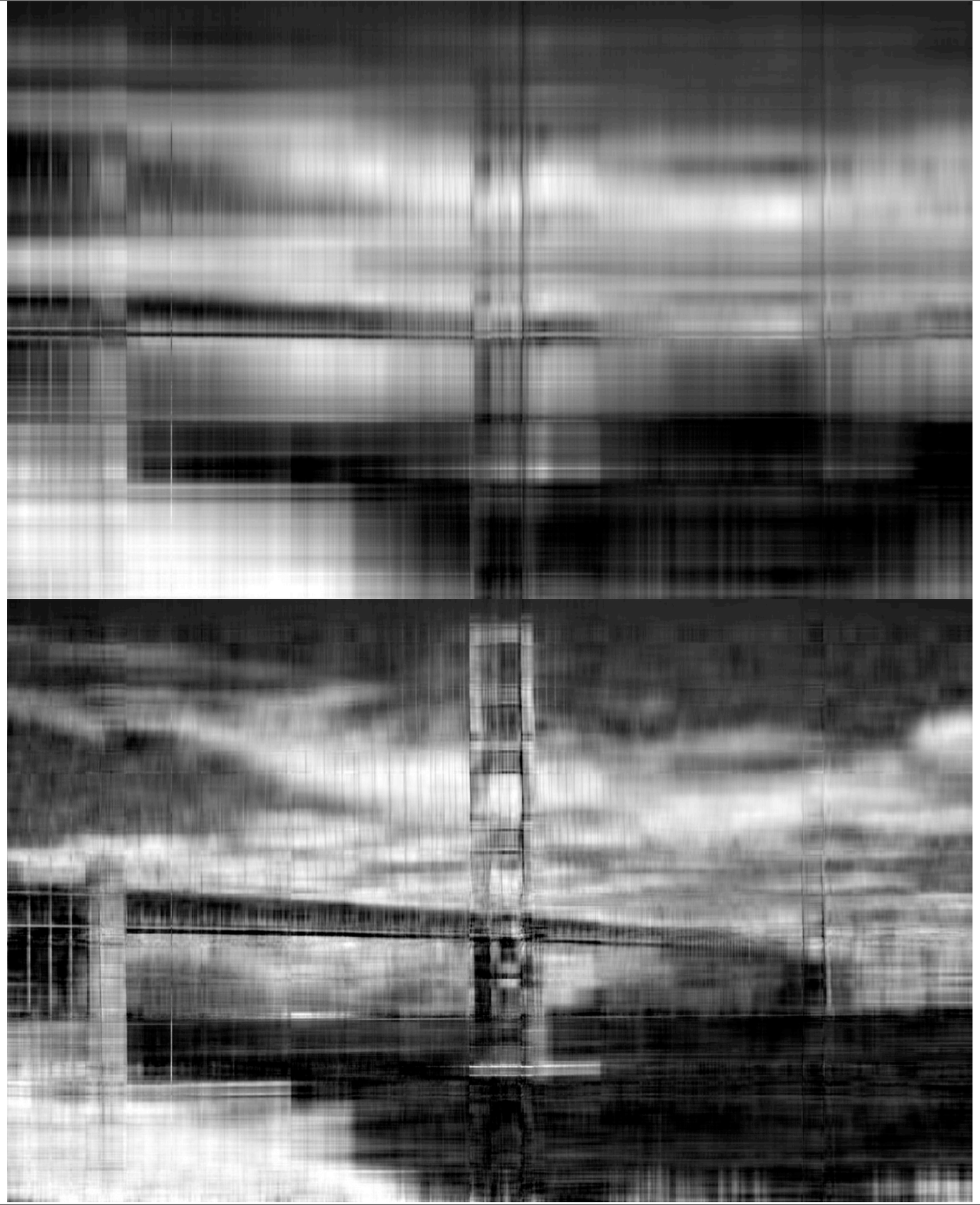
- need an efficient way to “**compress**” quantum states so that the matrices studied remain finite-dimensional

  ➡slightly entangled 1D systems: **Matrix Product States**

  ➡DMRG, TEBD, ...

  We will assume here that the optimal MPS description does not depend strongly on which algorithm is used.

$\chi = 4$

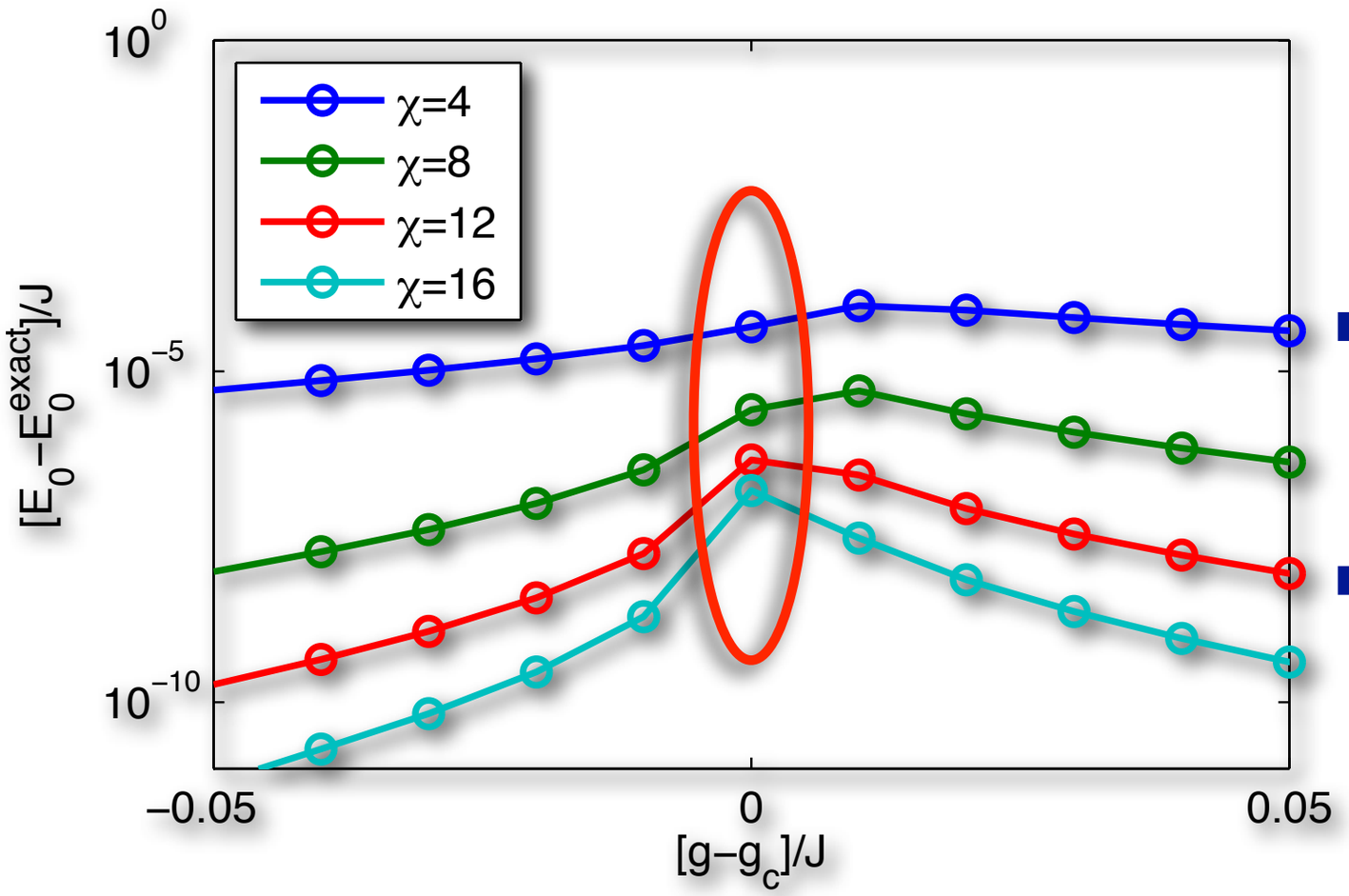


$\chi = 16$

$\chi = 64$

$\chi = 256$

- find the ground state of a system by using **imaginary time evolution** (almost unitary for small time steps)
- parallel updates for **infinite/translational invariant** systems: **iTEBD** [Vidal '07]
- **example**, transverse Ising model: $H = \sum_i \left(J\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x \right)$

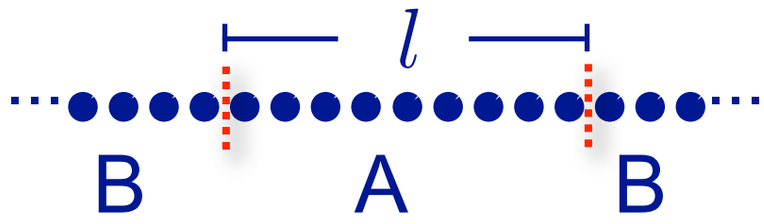


➡convergence of wave function is worst at the **critical point**

➡**conformal invariance**

- scaling relation for the **block entropy** in critical systems with conformal invariance [Holzhey et al. ‘94]

$$S = \frac{c}{3} \log l$$

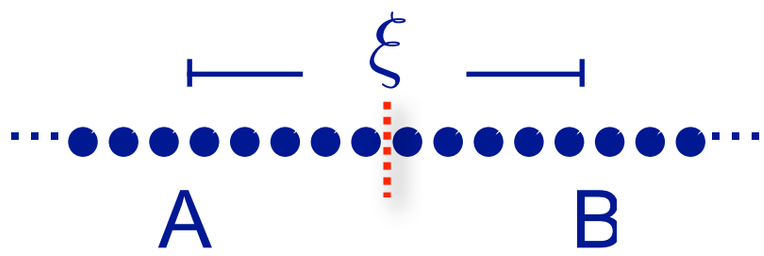


➡**entanglement entropy diverges logarithmically** as $l \to \infty$

➡ **half chain entropy** is $S = c/6 \log l$

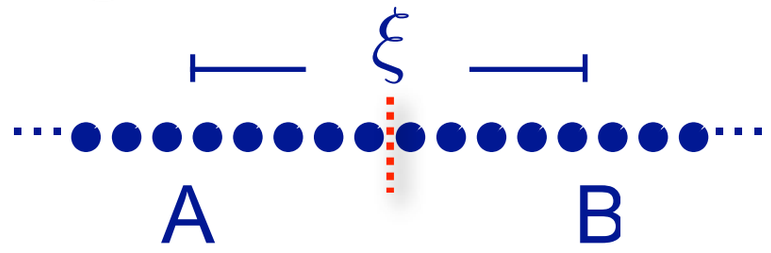
- entanglement entropy is finite away from criticality [Calabrese et al. ‘04]

$$S = \frac{c}{6} \log \xi$$



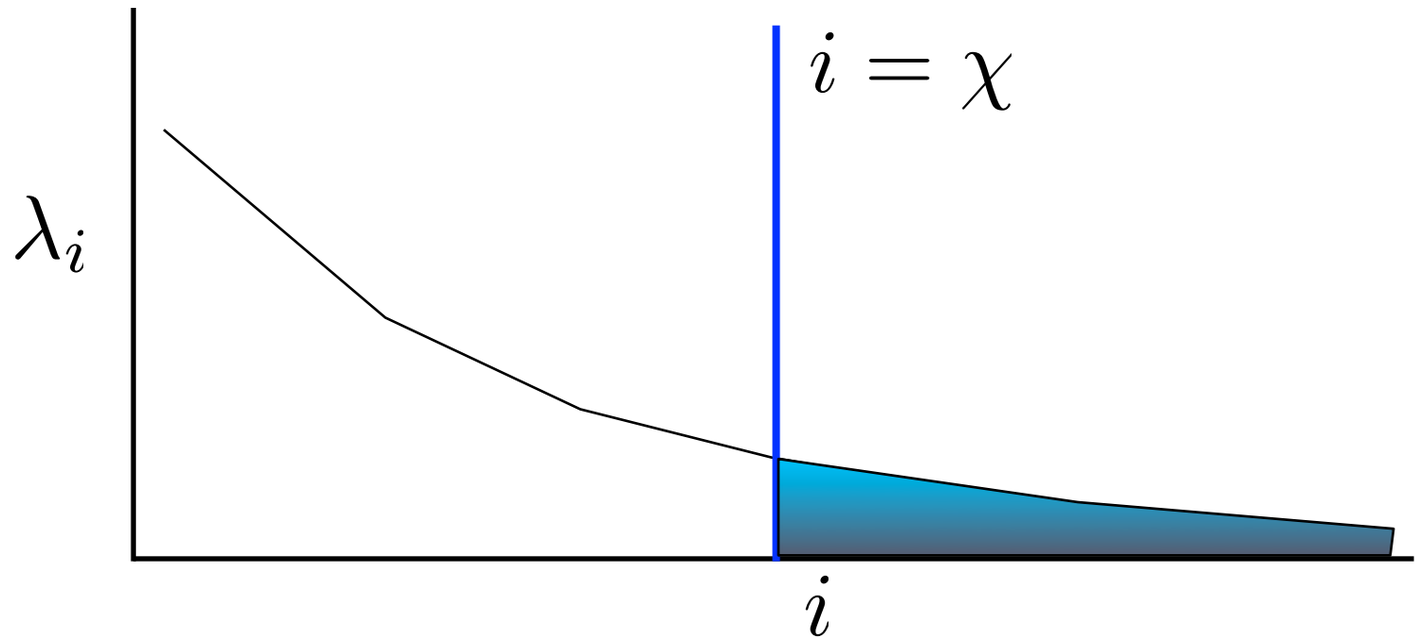
Physical picture for how "finite entanglement" determines a correlation length:

$$S = \frac{c}{6} \log \xi$$



1. At the critical point, ordinary energetics (χ = ∞) favors maximizing ξ.

2. But there is a long tail of Schmidt eigenvalues for large ξ.
So the energy increase from truncating the SVD to keep only the lowest χ eigenvalues is largest for large ξ.



Balancing these two energies leads to a prediction for ξ.

- Take “entanglement spectrum” [Calabrese, Lefebvre ‘08]

  (see Pollmann, JEM, NJP ’10 for numerical tests; generally a good approximation)

$$n(\lambda) = I_0\left(2\sqrt{-b^2 - 2b\log\lambda}\right)$$

of $\hat{\lambda}$ ‘s greater than λ

with $b = \dfrac{S}{2} = \dfrac{c}{12}\log\xi = -2\log\lambda_{\max}$

continuum of Schmidt values $|\psi\rangle = \sum_{\alpha=1}^{\infty}\lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B$

- **at a critical point**: all values are equal ($\lambda_\alpha \to 0$)

Finite χ approximations at critical points will give **universal scaling forms**, e.g. $S = f(c)\log(\chi)$

Derive a universal scaling form for the entropy

- ground state of critical system at finite χ has **finite correlation length** ξ
- energy density $E(T) = F - T\left(\frac{\partial F}{\partial T}\right)_L = E_0 + \frac{\pi c T^2}{6u}$ at finite T

$$E(\xi) = E_0 + \frac{A}{\xi^2},\ \xi = \frac{u\nu}{T\pi}$$

- $|\psi^\xi\rangle$ still needs an **infinite** χ to be represented exactly in terms of MPS: **entanglement spectrum**
  $[n(\lambda) = I_0(2\sqrt{-b^2 - 2b\log\lambda}),\ b = c/12\log\xi\ ]$

  ➡**further corrections to the energy density**

- let's look at the **effect of the truncation** at **one** bond

$$|\psi^{\xi}\rangle = \sum_{\alpha=1}^{\infty} \lambda_\alpha |\phi_\alpha\rangle_A |\phi_\alpha\rangle_B \Rightarrow |\psi^{\xi}_{\chi,1}\rangle = \frac{\sum_{\alpha=1}^{\chi} \lambda_\alpha |\phi_\alpha\rangle_A |\phi_\alpha\rangle_B}{\sum_{\alpha=1}^{\chi} \lambda_\alpha^2}$$

- **energy difference** to the non-truncated wavefunction

$$\delta E_{\chi,1} = \left(E_\xi^{\mathrm{ex}} - E_\xi\right)\left(1 - |\langle\psi^{\xi}|\psi^{\xi}_{\chi,1}\rangle|^2\right)$$

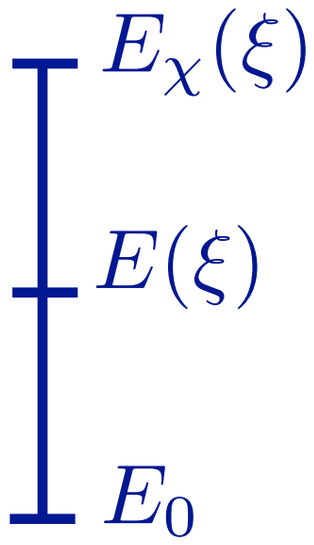
and $\left(E_\xi^{\mathrm{ex}} - E_\xi\right) = \Delta \propto 1/\xi$

E_ξ^{ex} = measure of energy of the excited states

$$\Rightarrow \quad \delta E_\chi = \frac{B}{\xi} P_r(\xi,\chi), \;\; P_r(\xi,\chi) = \sum_{n=\chi+1}^{\infty} \lambda_n^2$$

- energy density of a truncated state

$$E_\chi(\xi) = E_0 + \frac{A}{\xi^2} + \frac{B}{\xi} P_r(\xi, \chi)$$



- $E_\chi(\xi)$ is a non-monotonic function

➡minimize the energy and find the **optimal correlation length for a fixed matrix dimension**

➡scaling relation $S = (c/6) \log \xi$ yields the entropy, etc.

- **we can find the best approximation of the critical state for a given number of states we keep**

- **analytical solution** for the asymptotic case
  (using a continuum of Schmidt values and $\chi \to \infty$)

➡**universal finite-entanglement scaling relations**

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}+1}\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}+1}} \log \chi$$

We believe this finite-entanglement scaling will result in any approach with finite matrix dimension, in the same way that finite-size scaling is "universal".

Now we try to check this nonlinear *c* dependence:

(some more checks are in

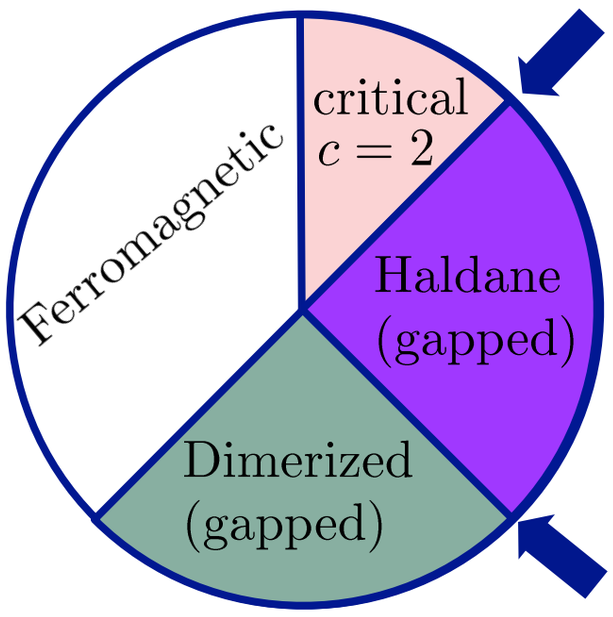
[F. Pollmann, S. Mukerjee, A. Turner, and J.E. Moore, PRL 2009]

and we agree with Tagliacozzo et al. for the cases studied there)

- test the scaling relation on various critical points using the **iTEBD** method
- transverse Ising model: $H = \sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x\right)$

  ➡critical at $\mathrm{g}{=}1$ with $c = 1/2$
- XXZ model: $H = \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta\sigma_i^z \sigma_{i+1}^z\right)$

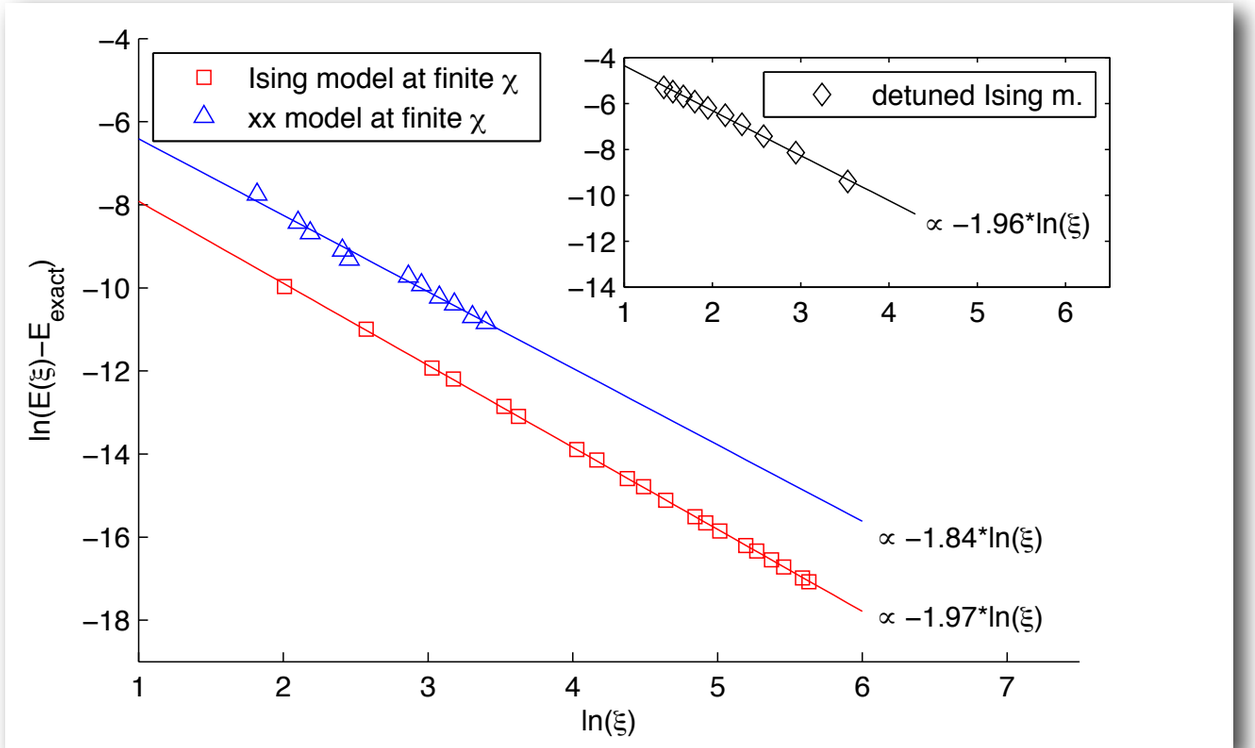
  ➡critical at $\Delta = 1$ with $c = 1$
- spin-1 model: $H = \sum_i \left(\cos\theta(\vec{\tau}_i.\tau_{i+1}^{\rightarrow}) + \sin\theta(\vec{\tau}_i.\tau_{i+1}^{\rightarrow})^2\right)$



$\theta = \pi/4, SU(3)_1$ with $\mathrm{c}{=}2$

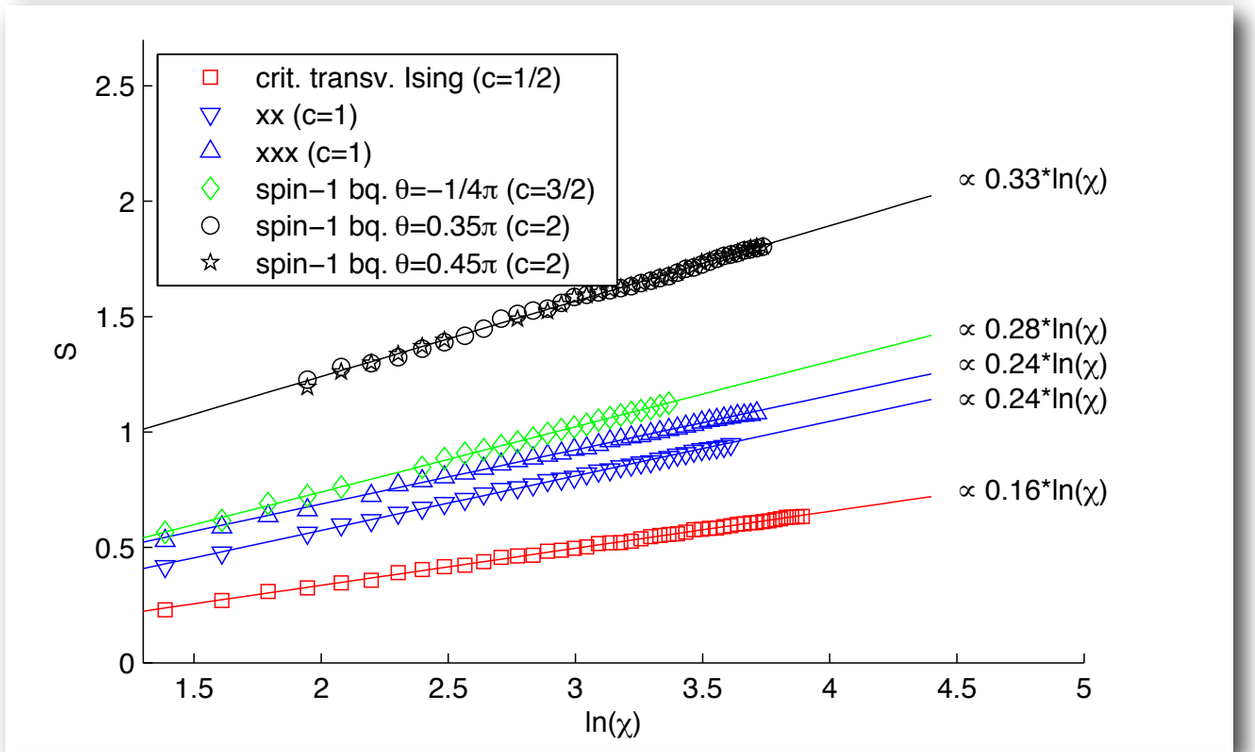
$\theta = -\pi/4, SU(2)_2$ with $\mathrm{c}{=}3/2$

- scaling of the **energy** and **entanglement entropy:**



$$E_\chi(\xi) = E_0 + \frac{A}{\xi^2} + \frac{B}{\xi} P_r(\xi, \chi)$$

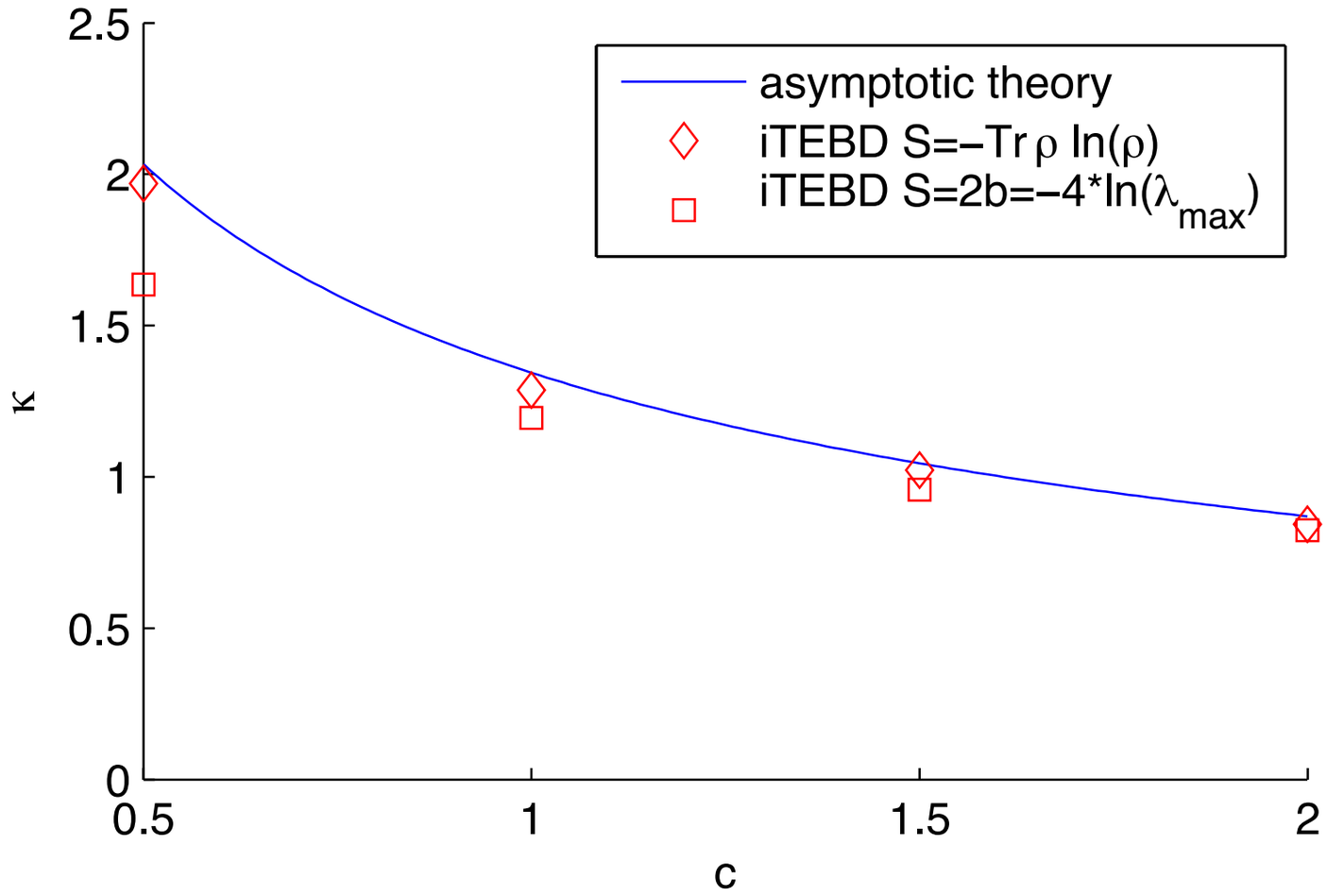
➡scales as $\propto 1/\xi^2$ for $\xi = \xi_{\text{opt}}$



$$S_\chi = \frac{1}{\sqrt{\frac{12}{c} + 1}} \log \chi$$

➡$S_\chi \propto \log \chi$ and depends only on c

- reasonable agreement of the **asymptotic theory** and **numerical results**



- Errors are no larger than differences between different definitions of entropy
- Another check: combine non-interacting copies; still get nonlinear dependence on total *c*

- new asymptotic scaling law for the finite-entanglement scaling of 1D quantum-critical systems

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}+1}\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}+1}} \log \chi$$

➡reasonable agreement with numerical results for accessible

- future directions:
  - ➡critical points with disorder (entanglement spectrum next)
  - ➡higher dimensions (?)
  - ➡can we improve on Calabrese-Lefebvre spectrum?

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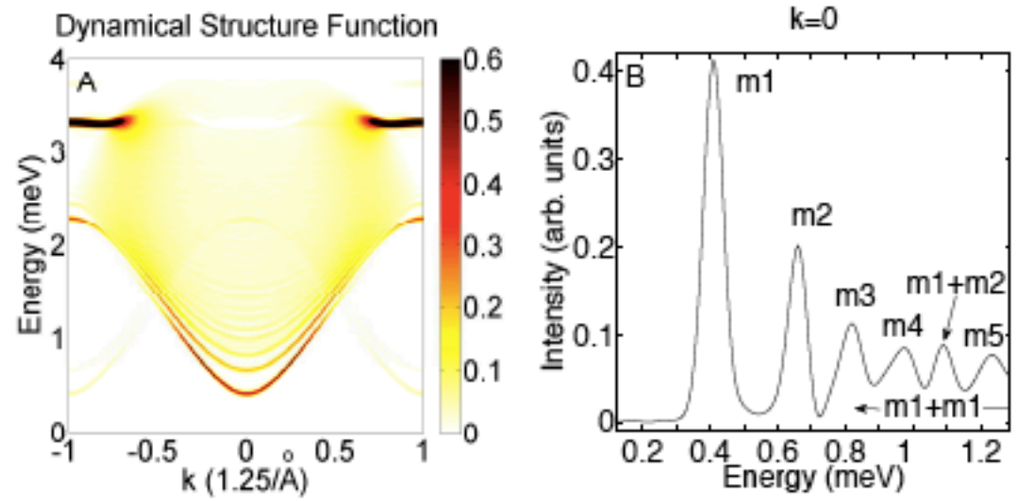
  - ➡can we improve on Calabrese-Lefebvre spectrum?

- Applications: sweep across a quantum critical point (either integrable or non-integrable)

F. Pollmann, S. Mukerjee, A. Green, JEM, PRE 2010

- Numerical spectral functions for E8 bound states in Ising-like chains: motivated by Coldea et al., Science 2010

J. Kjäll, F. Pollmann, JEM, arXiv 2010; Nordita talk last month



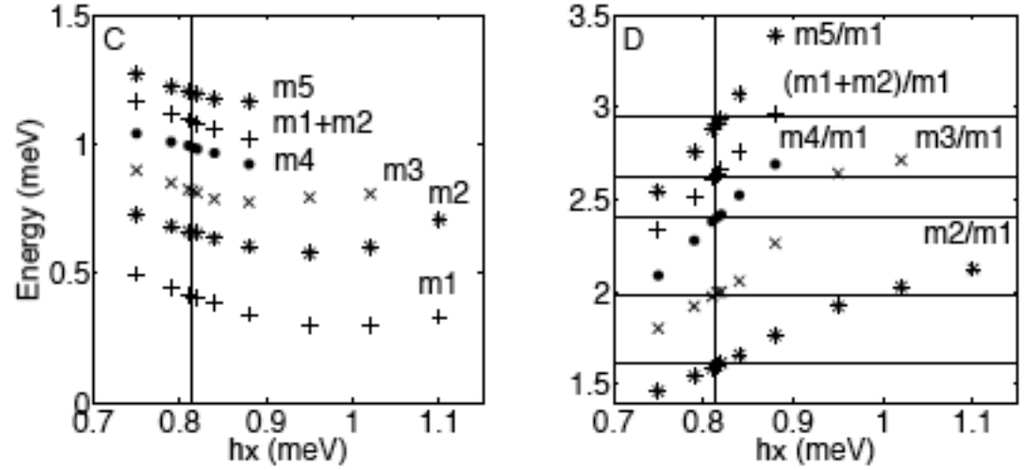


FIG. 3: (Color online) The full Hamiltonian describing $CoNb_2O_6$. (A) The dynamical structure function at h_c^x. (B) A cross section of (A) at zero momentum showing the masses of the first five bound states and two bound state pairs. (C) The bound state masses as a function of h^x. The minimum gap is above h_c^x and the bound state mass minimum decreases with increasing mass. (D) The ratio of bound state masses varies linearly around h_c^x and goes through the analytically calculated values at h_c^x.

Universality beyond conformal invariance: Entanglement at random quantum critical points

It turns out that even at random quantum critical points, universal entanglement exists and defines a critical entropy.

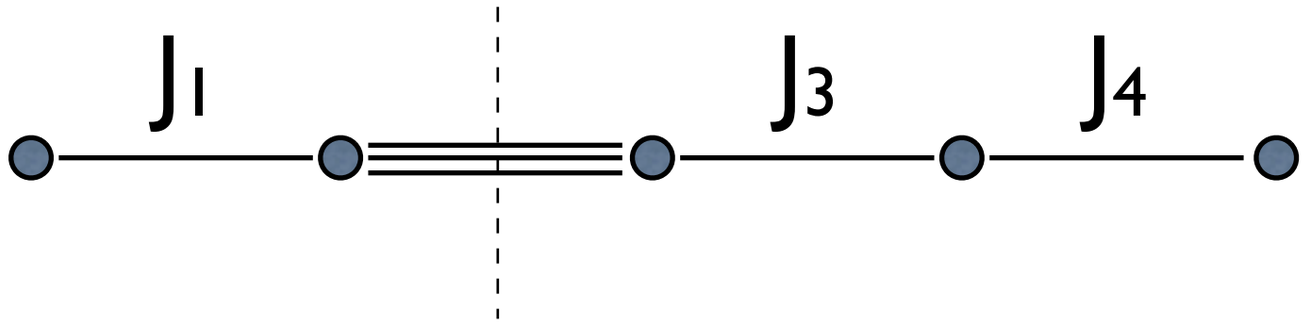
Example: **random** Heisenberg antiferromagnet (same as before, but now J on each bond is drawn from a random distribution)

P(J) = random distribution over J > 0 (antiferromagnetic couplings)

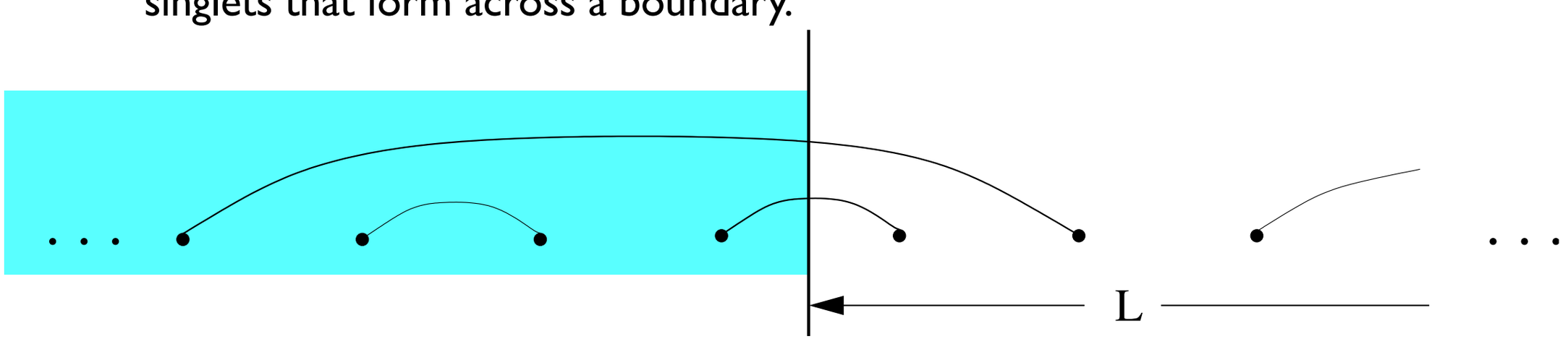
Almost all initial distributions flow to the same strongly random distribution of couplings, the random singlet phase.

These strongly random distributions have critical disorder-averaged correlations, but the system is not conformal in d+1-dim. spacetime.

Entanglement entropy and random criticality



For entanglement entropy, we need to obtain the mean number of singlets that form across a boundary.



Answer: the coefficient of the log divergence is reduced by ln 2 relative to the clean Heisenberg case (Refael & JEM, 2004; Laflorencie, 2005: numerical check on XX case).

But this similarity to the clean case is a little misleading: the full entanglement spectrum is rather different (Fagotti, Calabrese, JEM).

Many-body localization at infinite temperature

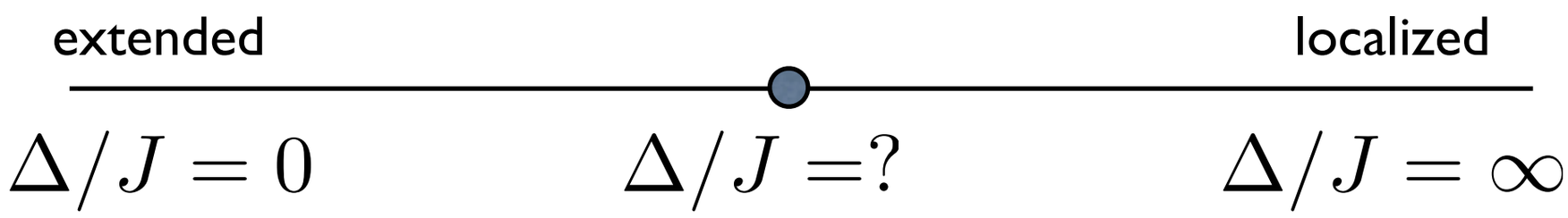
$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Clean XXZ chain + random z-directed Zeeman field

$$h_i \in (-\Delta, \Delta)$$

Claim: look at "infinite-temperature" dynamics but with no dephasing; evolve an arbitrary initial state by the Schrödinger equation

Heisenberg phase diagram:



Many-body localization at infinite temperature

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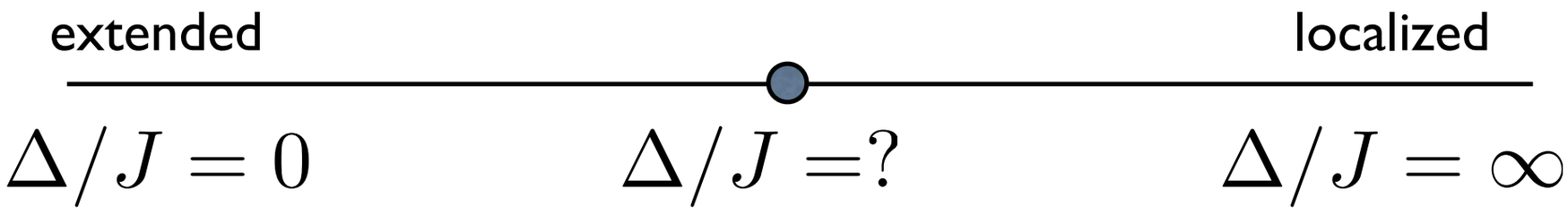
Heisenberg phase diagram:



Transition should be detectable in:

level statistics: (Wigner-Dyson vs. Poisson) Oganesyan & Huse, 2008
dynamical correlation functions
correlation distributions Pal & Huse, 2010
entanglement growth

This spin chain problem is a numerically easier reformulation of many-body localization in Fermi systems at nonzero T (Basko, Aleiner, Altshuler 2007)

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

"Extended phase": expect S grows linearly with t (Calabrese and Cardy)

"One-particle localized phase": (Jz = 0) eigenstates are Slater determinants of localized one-particle states; S saturates to a finite value.

What happens if we add interactions to the localized phase?

Note: this is efficiently simulable because for early times the system has small entanglement (Prelovsek et al., 2007)

Many-body localization at infinite temperature

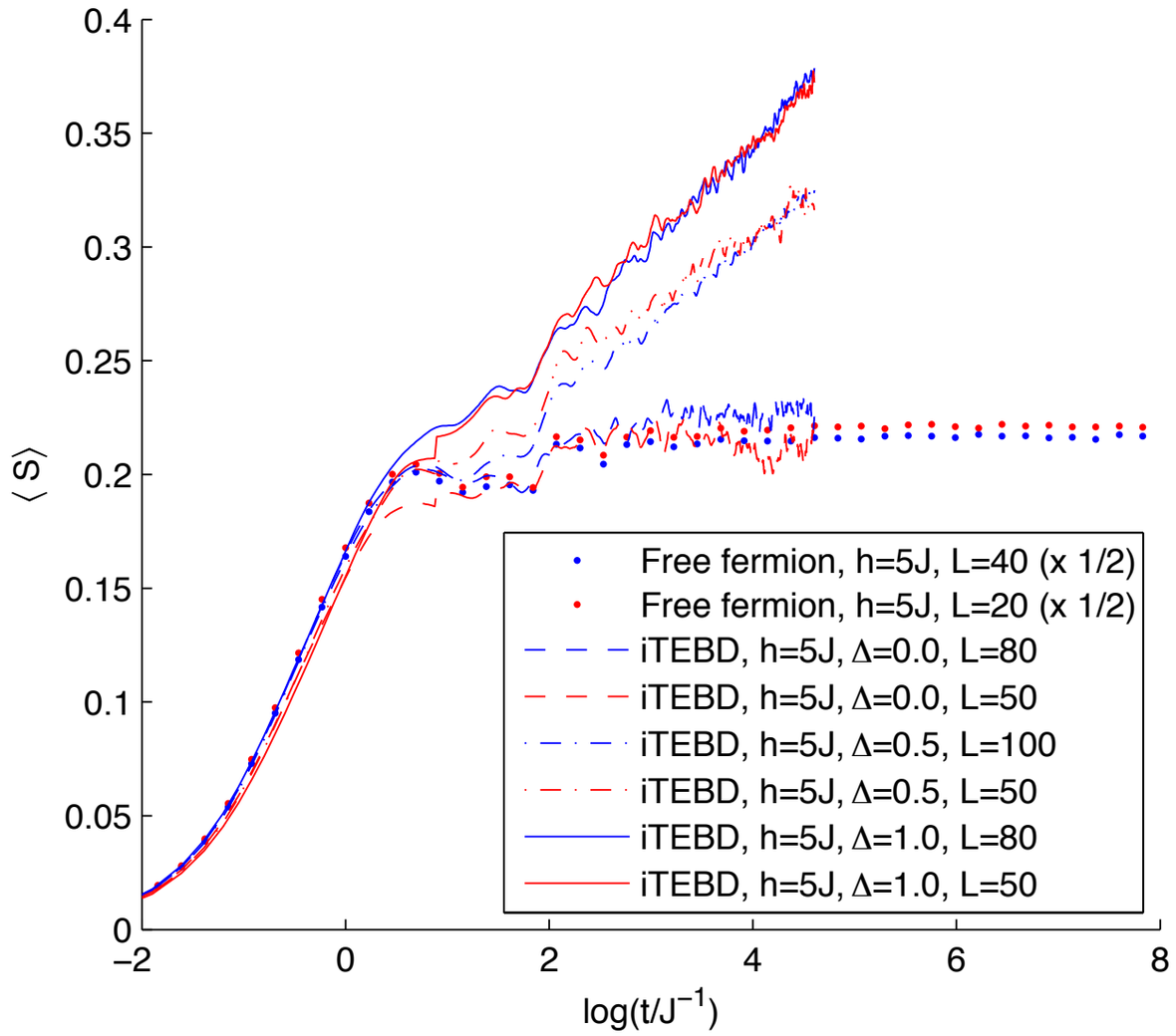
$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

$$S(t) \approx f(J_z) \log t$$

In an "excitation" picture, this looks like sub-diffusive but delocalized transport.

This occurs naturally in rate-equation models (e.g., Amir, Oreg, Imry (2010)).



Many-body localization at infinite temperature

$$S(t) \approx f(J_z) \log t$$

In an "excitation" picture,
this looks like sub-diffusive
but delocalized transport.

Conclusion: the entanglement growth, which is one measure of how fast the system returns to equilibrium, has a phase that is between strong localization and delocalization.

Speculation: If we assume S results from excitation motion, go to frequency *f* and use fluctuation-dissipation, this becomes 1/*f* noise, as empirically observed in many quantum and classical systems.

Entanglement entropy and understanding criticality

Numerical confirmation of modified coefficient for one case (XX chain->free fermions) by N. Laflorencie (PRB 2005)

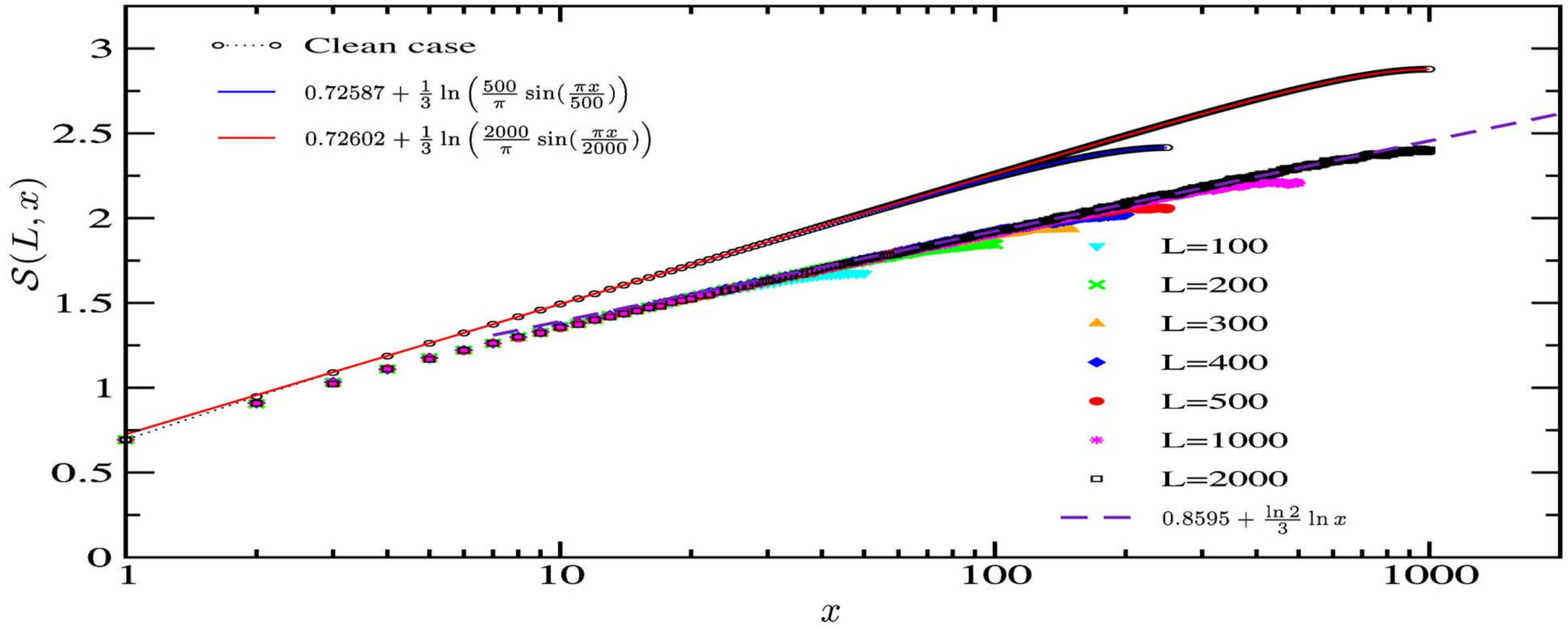
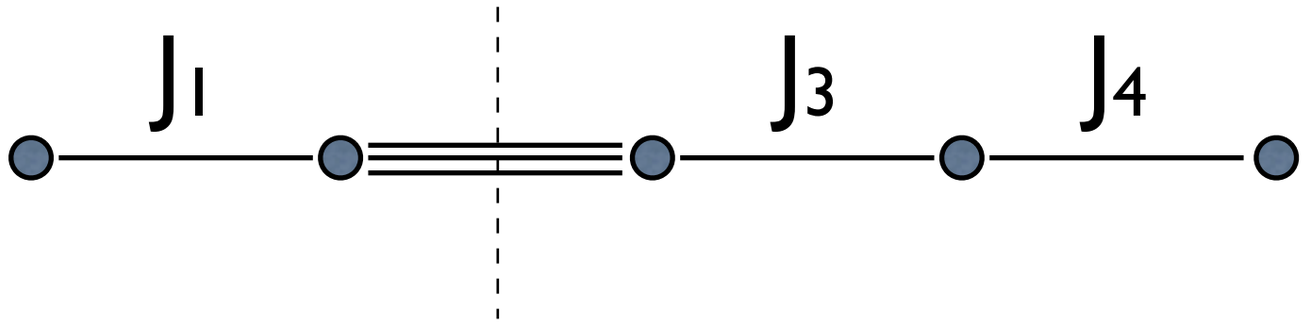


FIG. 2. (Color online) Entanglement entropy of a subsystem of size x embedded in a closed ring of size L, shown vs x in a log-linear plot. Numerical results obtained by exact diagonalizations performed at the XX point. For clean nonrandom systems with $L=500$ and $L=2000$ (open circles), $\mathcal{S}(x)$ is perfectly described by Eq. (3) (red and blue curves). The data for random systems have been averaged over 10^4 samples for $L=500$, 1000, 2000, and 2×10^4 samples for $100\leqslant L\leqslant 400$. The expression $0.8595+(\ln 2/3)\ln x$ (dashed line) fits the data in the regime where finite size effects are absent.

But strictly that only checks one number...



The key step in the RSRG is a memory (non-Poissonian) effect in RG time: the new bond created after a decimation is on average weaker than a typical bond.

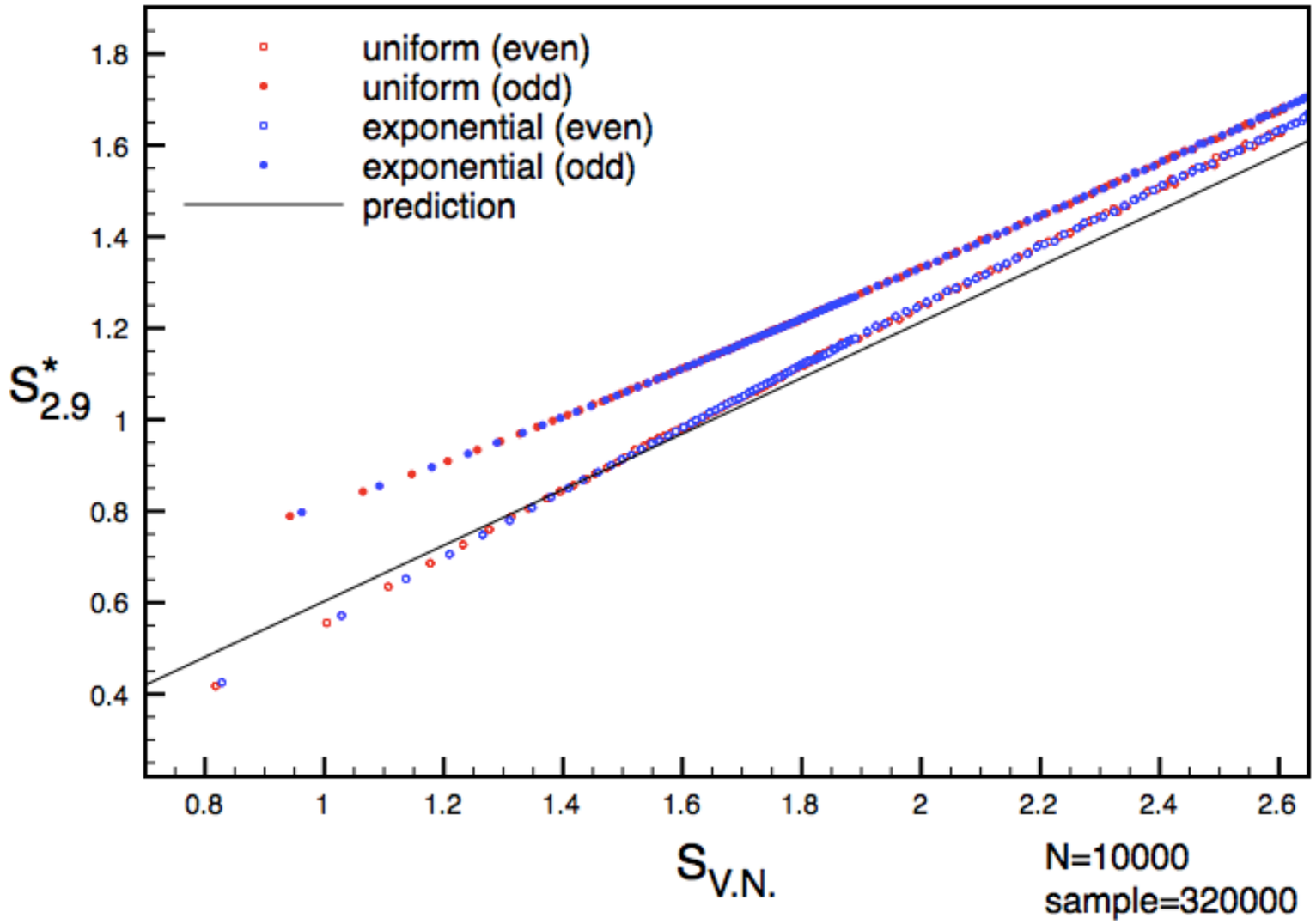
This non-Poissonian picture can be checked by computing the full entanglement spectrum, or analytically the average x^N where N is the number of singlets and x is a parameter, for all x.

$$g(t) = \frac{1}{\sqrt{5}}\left(e^{-(3-\sqrt{5})t/2} - e^{-(3+\sqrt{5})t/2}\right).$$

$$\langle x^N \rangle_t \equiv h(t) = \int_t^\infty g(t')\,dt' + x\int_0^t h(t')g(t-t')\,dt'.$$

Numerics on spectrum in XX chain:

(Fagotti, Calabrese, JEM, unpublished)



Details: what is shown is the "modified Renyi entropy"

$$S^*_\alpha = \frac{1}{1-\alpha} \log \overline{\mathrm{Tr}[\rho^\alpha]}$$

The RG calculation predicts that all these moments are simply related via the renewal equation.

Conclusions:

1. Finite-entanglement scaling: estimating the correlation length that emerges from finite matrix size in MPS leads to the unusual expression

$$\kappa = \frac{6}{\sqrt{12c} + c}$$

Does this actually help make better extrapolations to infinite size and infinite entanglement?

2. We can compute the entanglement spectrum for random-singlet critical points and check it numerically in the XX model.

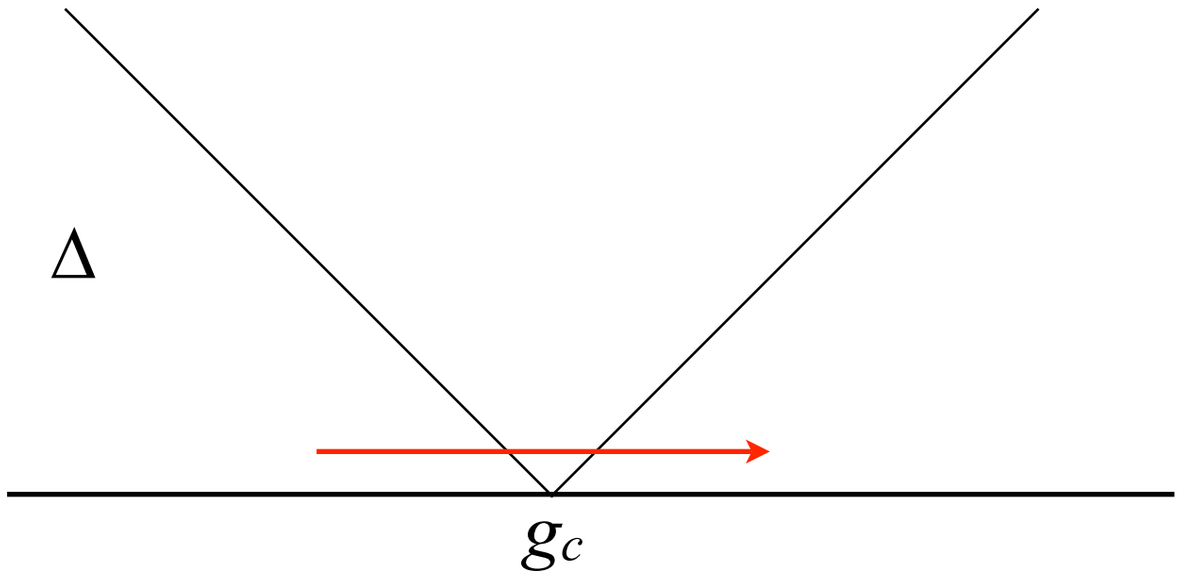
Can we use this to understand dynamics or other topical questions about random 1D models?

Application II: Dynamics near quantum critical points

Our motivation:

- We want to control how far a system is excited out of the instantaneous ground state of *H(t)*.

We sweep the Hamiltonian slowly through a 1D quantum critical point separating two gapped phases.



$$\begin{aligned} g(t) &= g_i + \Gamma t \\ g(0) &= g_i < g_c \\ g(t_f) &= g_i + \Gamma t_f > g_c \end{aligned}$$

The closing of the gap Δ means that deviations from the ground state are power-law in the sweep rate: e.g., $E_F - E_0 \sim \Gamma^\alpha$

rather than being exponentially small if $\Delta > 0$ everywhere.

Application 2: Dynamics near quantum critical points

- Sometimes we want to study quantities that are well-defined in the infinite system, and independent of a particular observable.

  We compare states using the spatial decay rate of the overlap (N = # of sites)

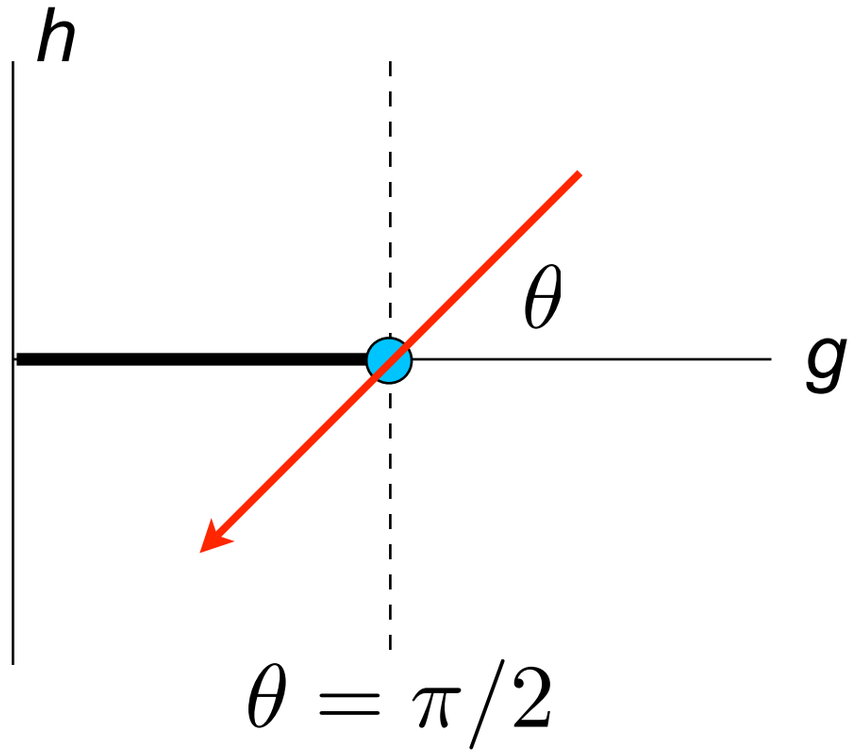
$$|\langle\psi_0|\psi_1\rangle|^2 \sim \exp(-\alpha N).$$

  which is easily computed from the matrix product state representation, and can be found exactly for the quantum Ising case.

Application 2: Dynamics near quantum critical points

- We would like to distinguish integrable and chaotic quantum dynamics, and spontaneous symmetry breaking from explicit symmetry breaking.

Our starting point: cross through the well-studied quantum Ising critical point at various angles in the phase diagram



(E8, Zamolodchikov)

$$\mathcal{H} = \sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\sigma_{ix} + h\sigma_i^z \right)$$

$$\sin\theta = \frac{h}{g - g_c}, \quad g_c = 1$$

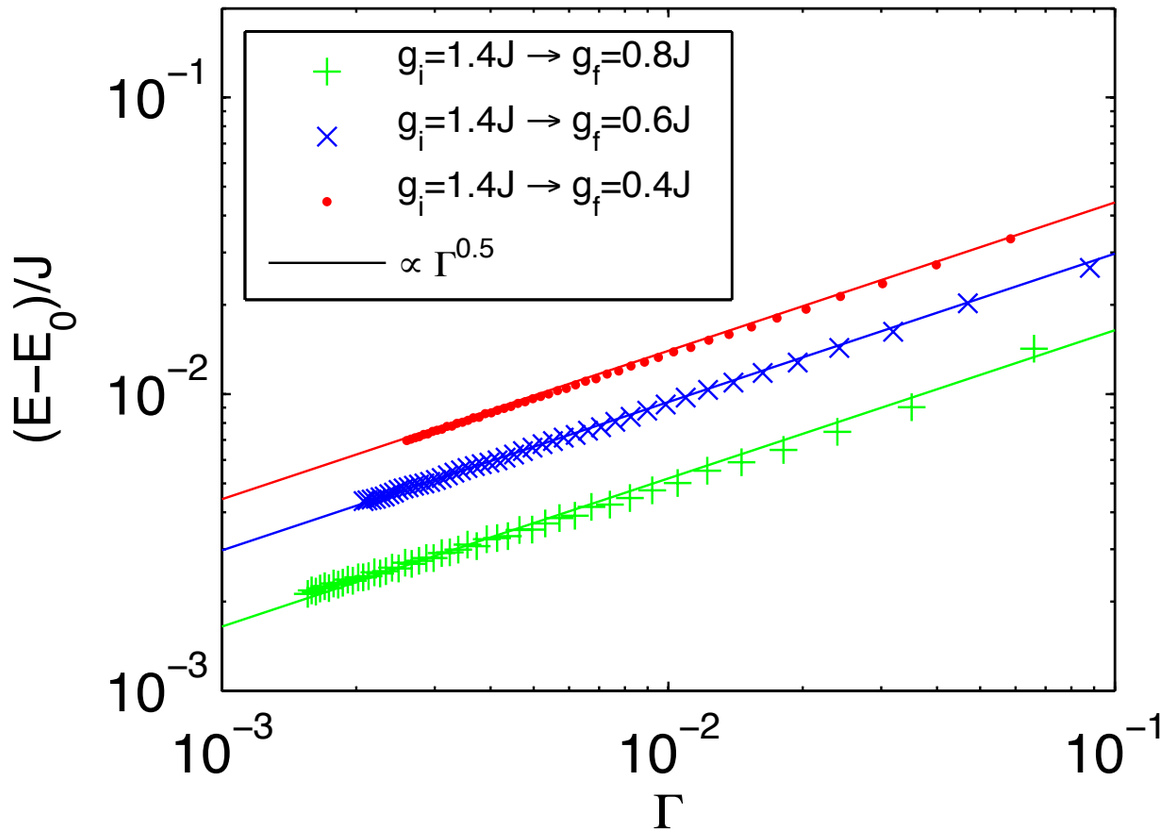
Case 1: quantum Ising sweep

We sweep *g* through the critical point at a constant rate.

How different is the resulting state from the ground state?

The energy difference and “number of excitations” are predicted to be related to the sweep rate by a simple scaling law

(Dziarmaga, Polkovnikov, ...)



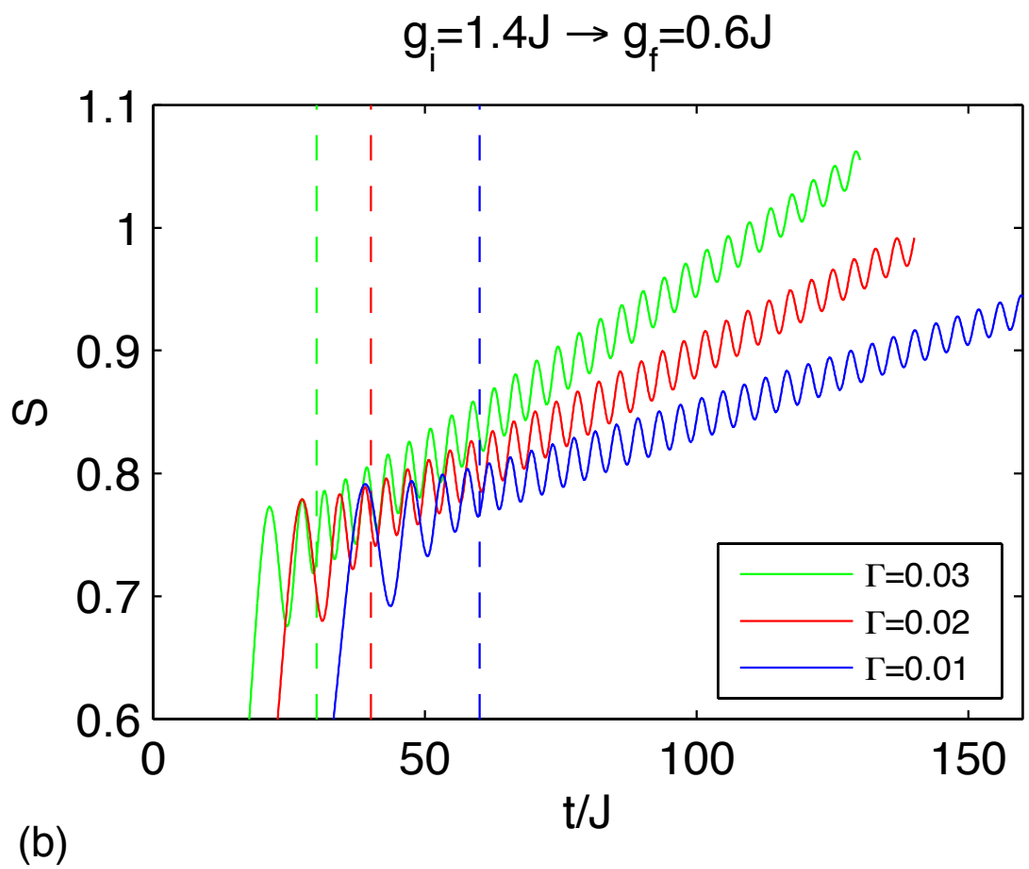
$$E_0'[g(t)] - E_0[g(t)] \sim n_{\rm ex}\Delta[g(t)] \sim \Gamma^{d\nu/(z\nu+1)}\Delta[g(t)],$$

Case 1: quantum Ising sweep

We sweep *g* through the critical point at a constant rate, then pause at a fixed final value g_f to observe evolution.

The quantum Ising model has well-defined linearly propagating excitations ("domain walls"). The propagation of these excitations leads to linearly increasing entanglement, even after the sweep has stopped.
(cf. Calabrese and Cardy)

This "light-cone" effect depends on the number of excitations created, and hence on the sweep rate.



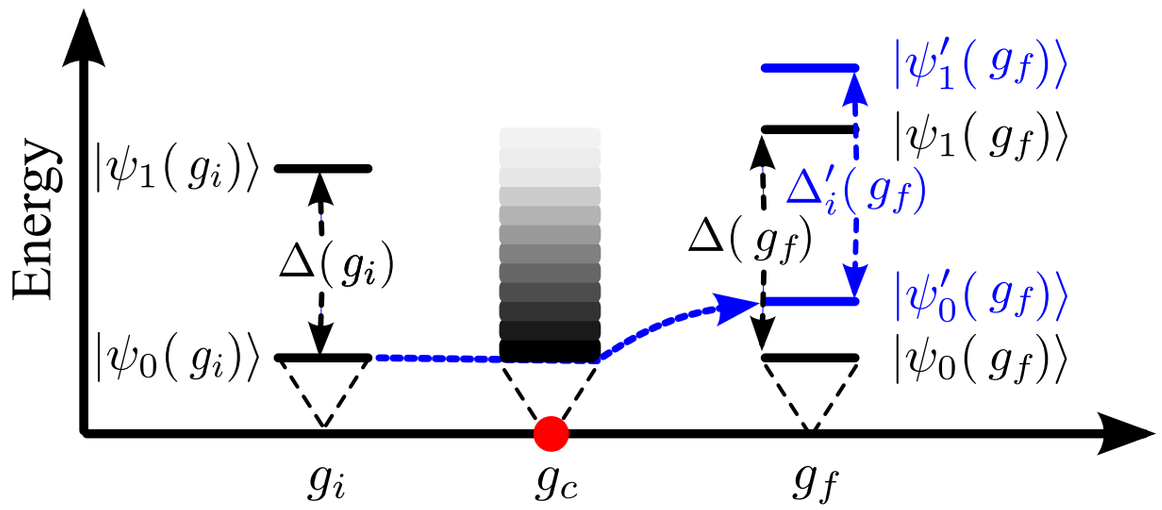
What causes these oscillations?

Case 1: quantum Ising sweep

We sweep g through the critical point at a constant rate, then pause at a fixed final value g_f to observe evolution.

The oscillations result because, after a slow sweep, the final state consists of a ground state plus excitations at multiples of the final gap.

The small dispersion in final energy leads to a slow decay of the oscillations.



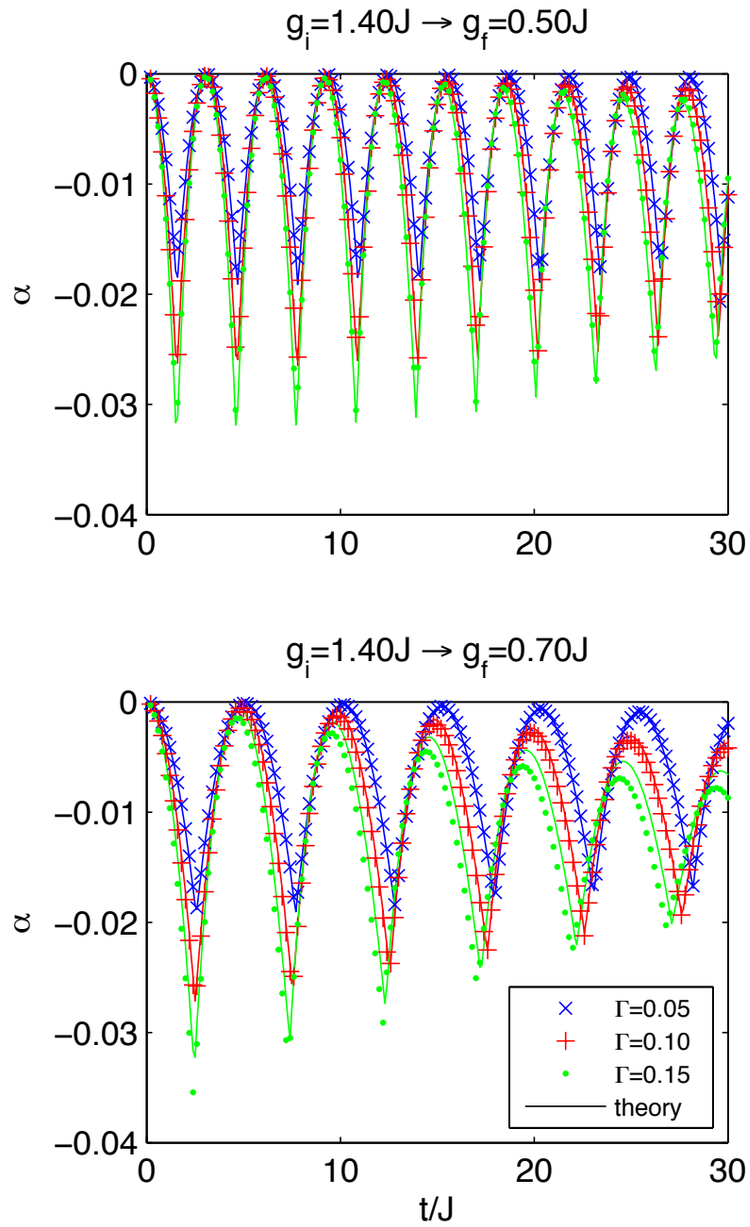
Case 1: quantum Ising sweep

We sweep *g* through the critical point at a constant rate, then pause at a fixed final value g_f to observe evolution.

We can use the overlap integral to focus on the oscillations and check the TEBD method.

Puzzle: why the nonanalytic dips at certain points in time?

For the quantum Ising model, can compute these exactly using a picture of Landau-Zener tunneling at each k independently (theory curves shown)...



Integrable versus non-integrable models

The nonanalytic dips result from a special *k* value where the tunneling probability is exactly 1/2. Since this model is integrable, the excitations at this *k* have sharp energy.

$$|\psi(0)\rangle = \prod_k (u_k|0_k\rangle + v_k|1_k\rangle)$$

$$|v_k|^2 = P_k = 1 - |u_k|^2 = \exp\left(-\frac{2\pi J^2 k^2}{\Gamma}\right)$$

$$|\psi(t)\rangle = \prod_k (u_k|0_k\rangle + e^{-i\Delta_f(k)t/\hbar} v_k|1_k\rangle)$$

Leads to universal 1/t "equilibration" (power-law rather than exponential), *in an integrable system,* resulting just from the continuum of excitation frequencies.

$$\alpha(t) = \frac{1}{2\pi}\int_0^\infty dk\, \log\left[(1-P_k)^2 + {P_k}^2 + P_k(1-P_k)\cos(\Delta_f(k)t)\right].$$

(Final state has a diagonal density matrix but is not thermal.)

Integrable versus non-integrable models

We see similar behavior with different exponents along the other integrable line (2D Ising model in a field).

Along other directions, the model is expected to be non-integrable. For a slow sweep, we see:

at short times the system looks like the integrable case;

beyond some time determined by theta and the sweep rate, the "excitations" begin to interact strongly and the cusps are washed out.

This leads to an "entanglement catastrophe" that makes the model difficult to study with our method.

Our current priority: understand what is universal in this process (and whether there can be violations of the energy scaling law given earlier).

Conclusions

We studied how entanglement and other quantities behave near quantum critical points in one dimension.

1. The finite-entanglement scaling at a conformally invariant critical point in 1D has a universal nonlinear dependence on the central charge.

2. Dynamics near a quantum critical point can show a weak type of "equilibration" (damped oscillations) in an infinite system even for an integrable system.

3. Entanglement growth, numerical accuracy, and physical properties such as oscillations all seem sensitive to non-integrability.

Some results (e.g., the energy scaling formula for sweeping through a quantum critical point; cf. Polkovnikov) are believed to be general to any dimension. Others will require some nontrivial development to reach d>1.