presented at: INSTANS/NORDITA conference " Quantum Matter in Low Dimensions: Opportunities and Challenges"
Albanova University Center Stockholm, September 6-10, 2010.

Incompressibility, Quantum Geometry and Hall viscosity in the FQHE.

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- some new light on incompressibility in FQHE
- a "metric" that measures the "shape" of the "Mott-Hubbard-like" structures that underly FQHE incompressibility.
- "Hall viscosity" as fundamental FQHE property.

- see FDMH, arXiv: 0906:1854, and "in preparation"

previous history

- the dissipationless antisymmetric term in the viscosity tensor has been discussed in classical magnetic fluids, plasmas in magnetic fields (e.g. Lifschitz and Pitaevskii text "Physical Kinetics")
- Zograf, Avron and Seiler (1995) considered the "odd viscosity" in integer QHE
- Tokatly and Vignale (2008) called it "Lorentz shear modulus" but treated it incorrectly in FQHE.
- Read (2009) gives a corrected formulation for rotationally-symmetric FQHE states, calls it "Hall viscosity", emphasizes cft conformal block model wavefunctions

This work

- generalizes discussion to FQHE fluids without rotational invariance (e.g. with "tilted" magnetic fields)
- obtains a clear separation between integer QHE (cyclotron motion) and FQHE (guiding center fluid) parts
- finds surprising relations to other aspects of FQHE

FQHE states are **incompressible topologically-ordered** states of 2D electrons in a magnetic field

- Analogous to Mott-Hubbard physics on a lattice, an energy gap prevents adding more electrons to an "already occupied" region.

- This incompressibility is a consequence of SHORT-RANGE components of the Coulomb interaction, analogous to Hubbard "U".

- Can be viewed as a consequence of "non-commutative geometry" of electron "guiding centers" moving on the "quantum plane"

$$[R^x, R^y] = -i\ell_B^2$$

- Much work on FQHE emphasizes **topological properties:**
  - **edge states** (described by (1+1)-D conformal field theories, cft) and
  - vortex-like excitations with fractional and non-Abelian **braiding statistics**(described by (2+1)-D topological quantum field theories (TQFT))
- These aspects do not depend on any **metric** or short-distance lengthscale, and take the incompressibility **as given**.
- New results on "**Hall viscosity**" of FQHE states give new insight into lengthscales, shapes, and short-distance properties of incompressible states.

some questions

- many properties of FQHE depend on a fundamental quantum area, the area through which a quantum h/e of magnetic flux passes.
- elementary unit of FQHE state is a **droplet** of p particles in an area containing q flux quanta. What is the **shape** of this droplet (if rotational symmetry is present, it must be circular; if not, what determines it?)
- "shape" involves lengthscales, and hence a **metric** (note, topological properties and areas are metric-independent). What determines this metric?

area: $2\pi q \ell_B^2$

$2\pi q \ell_B^2$?

summary of new results

- The "Hall viscosity" is defined by a rank-2 symmetric tensor $\eta_H^{ab} = \eta_{H0}^{ab} + \bar{\eta}_H^{ab}$
- it is the sum of two distinct parts, one associated with **cyclotron motion** and the integer QHE, the other with **guiding centers** and the FQHE
- The guiding-center part is **odd under particle-hole transformations** of the Landau level
- It defines a metric associated with incompressibility: $\bar{\eta}_H^{ab} = \frac{1}{2}\bar{\eta}_H g^{ab}$
- the discontinuity of the Hall viscosity at QHE edges gives an intrinsic electric dipole per unit length on the boundary. $dp^a = e(\Delta\eta_H^{ab}\frac{\ell_B^2}{\hbar})\epsilon_{bc}dL^c$
- The magnitude of the guiding center part provides a lower bound to the $O(q^4)$ behavior of the "guiding-center structure factor", a fundamental property of incompressibility identified by Girvin, Macdonald and Platzman, 1986.
- The inequality is **satisfied as an equality** for cft-based model wavefunctions (Laughlin, Moore-Read, Read-Rezayi).
- It is related to a SO(2,1) Lie algebra of area-preserving deformations, can be numerically calculated by adiabatic variation of pbc's (torus) on finite-size systems.

$$\eta_H^{ab} = \frac{1}{A}\langle\Lambda^{ab}\rangle_0 \quad [\Lambda^{ab},\Lambda^{cd}] = \frac{1}{2}i\hbar\left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc} + \epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{ac}\right)$$

some applications

- The Pfaffian and anti-Pfaffian (next talk) are distinguished by opposite signs of their guiding-center Hall viscosity (they are related by particle-hole transformation)
- This leads to an intrinsic electric dipole moment on domain walls between these states. (excitations of the neutral CFT modes on these wall are fluctuations of this dipole moment around its ground-state value)
- Can be used to investigate the mysterious "non-unitary cft" models (Haldane-Rezayi, "Gaffnian," etc.) which appear to represent systems at transitions between FQHE states with same filling, different Hall viscosity.

Laughlin state for 1/m FQHE

holomorphic form

symmetric gauge

$$\Psi_L^{1/m}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \prod_{i<j} (z(\boldsymbol{r}_i) - z(\boldsymbol{r}_j))^m \prod_i e^{-z^*(\boldsymbol{r}_i) z(\boldsymbol{r}_i)/4\ell_B^2}$$

magnetic area
$2\pi\ell_B^2 = \frac{e}{\hbar}\boldsymbol{B}\cdot\hat{\boldsymbol{n}}$

$$z(\boldsymbol{r}) = x + iy$$

- Has rotational invariance under

$$z = (x + iy) \rightarrow e^{i\phi} z$$

- rotational invariance implies a "hidden" dependence on a "metric" derived from the cyclotron effective mass tensor of the Landau levels

Landau levels

$$H = \sum_i \frac{1}{2m} g^{ab} \pi_{ia} \pi_{ib}$$

"metric" (det$|g|=1$) derived from Galileian mass tensor

$$\boldsymbol{\pi}_i = \boldsymbol{p}_i - e\boldsymbol{A}(\boldsymbol{r}_i)$$

dynamical momentum

$$R_i^a = r_i^a - \epsilon^{ab} \hbar^{-1} \pi_{ib} \ell_B^2$$

guiding center

2D Levi-Civita antisymmetric symbol

$$[R_i^a, \pi_{ib}] = 0$$

Euclidean covariant notation

$$[\pi_{ia}, \pi_{ib}] = i \frac{\hbar^2}{\ell_B^2} \epsilon_{ab}$$

lower 2D indices $a = 1,2$

$$[R_i^a, R_i^b] = -i \ell_B^2 \epsilon^{ab}$$

upper 2D indices $a = 1,2$

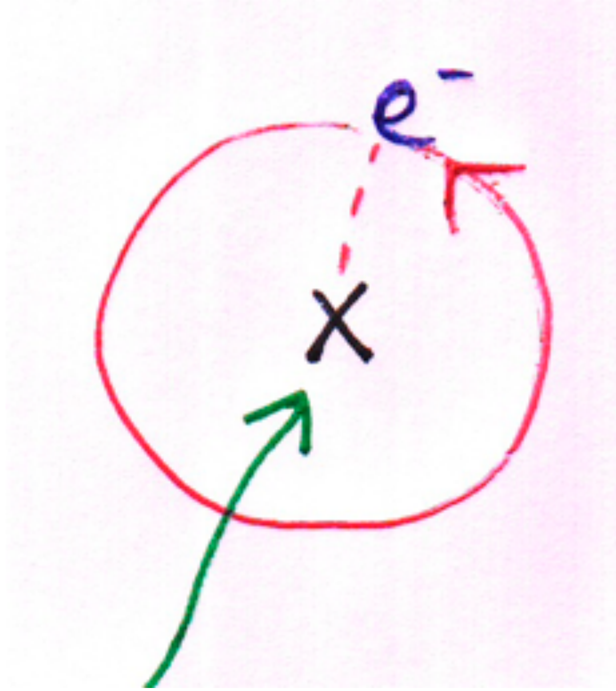
- generator of rotations is defined by metric:

$$L^z(g) = \frac{\ell_B^2}{2\hbar} \sum_i g^{ab} \pi_{ia} \pi_{ib} + \frac{1}{2\hbar} \sum_i g_{ab} R_i^a R_i^b$$

$$g^{ac} g_{bc} = \delta_b^a$$

cyclotron motion around guiding center

$$\ell_B = \left|\frac{\hbar}{eB_n}\right|^{1/2}$$



coordinates have "upper" indices

momenta have "lower" indices

- guiding center R

antisymmetric symbol

$$r^a = R^a + (\ell_B^2/\hbar)\epsilon^{ab}\pi_b$$

electron coordinate

dynamical momentum (of cyclotron motion)

$$[R^a, \pi_b] = 0$$

factorizing a 2D metric

$$g_{ab} = \omega_a \omega_b^* + \omega_b \omega_a^*$$
$$i\epsilon_{ab} = \omega_a \omega_b^* - \omega_b \omega_a^*$$

Hermitian generalized eigenproblem

$$\omega_a = g_{ab}\omega^b = i\epsilon_{ab}\omega^b$$
$$g_{ab}\omega^{*a}\omega^b = 1$$
$$g_{ab}\omega^a\omega^b = 0$$

a complex normalized 2D (2 component) vector

- for example

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \boldsymbol{\omega}(g) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\boldsymbol{\omega} \cdot \boldsymbol{r} = z = x + iy$$

Another look at the Laughlin state:

$$a_i^\dagger(g) = \frac{\ell_B}{\hbar}\omega^a(g)\pi_{ia}$$

- Landau-level raising operator $[a_i(g), a_i(g)^\dagger] = 1$
- guiding center "z" operator

$$b_i^\dagger(g) = \frac{1}{\ell_B}\omega_a(g)R_i^a$$

$[b_i(g), b_i(g)^\dagger] = 1$

$$|\Psi_L^{1/m}(g')\rangle \propto \prod_{i<j}\left(b_i^\dagger(g') - b_j^\dagger(g')\right)^m |0\rangle$$

- Laughlin state is an eigenstate of

$$L^z(g') = \frac{\hbar}{2\ell_B^2}\sum_i g'_{ab}R_i^a R_i^b$$

$$L^z = \frac{\hbar}{2}NN_{\text{orb}}$$

$$N_{\text{orb}} = mN - (m-1)$$

The number of orbitals occupied by the N particles

The "shift"

metric defined by cyclotron effective mass

$$a_i(g)|0\rangle = 0$$
$$b_i(g')|0\rangle = 0$$

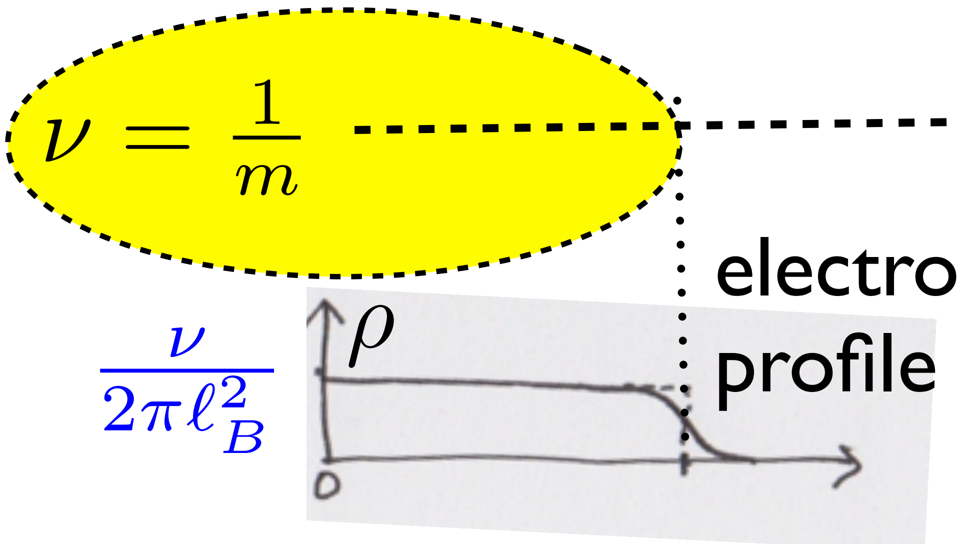
This "metric" g' is a freely-choosable variational parameter!

holomorphic form

$$\psi_L(\alpha) = \prod_i e^{-\frac{1}{2} z_i^* z_i / \ell_B^2} \prod_{i<j} \left((z_i - z_j) + \alpha \ell_B^2 \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \right)^m \prod_i e^{-\alpha^* z_i^2 / \ell_B^2}$$

$$|\alpha| < \frac{1}{2}$$

N-particle Laughlin droplet with “metric” g'



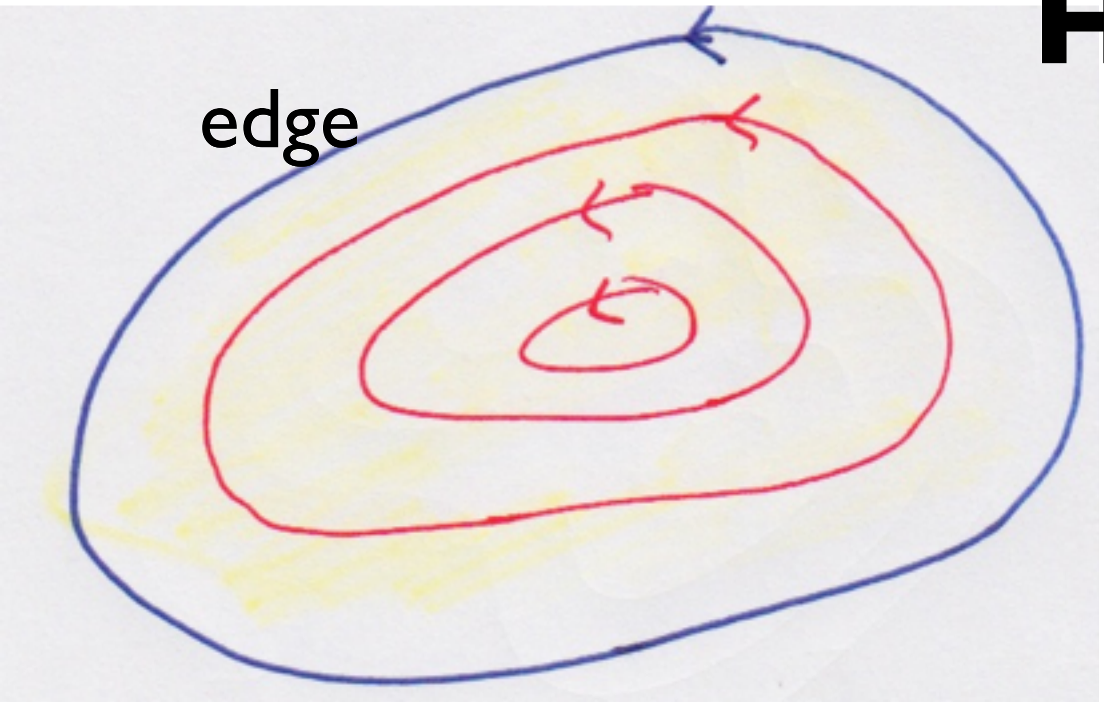
- elliptical shape, with “fuzzy edges” along the nominal boundary $\frac{1}{2}g'_{ab}r^a r^b = N_{\text{orb}}\ell^2$
- This is NOT just a change of edge shape; the “elementary droplets” of this FQHE fluid are also elliptical regions of m orbitals containing 1 particle each.

What fixes the shape of elementary droplet?

- If there is rotational symmetry around the normal to the 2D "Hall surface", the elementary droplet has the same (circular) shape as the cyclotron orbits
- If not (e.g., if the magnetic field is "tilted") it is a compromise between the shape of the cyclotron orbit (through a **form factor**) and the shape of equipotentials of the **Coulomb potential** of a point charge (determined by the dielectric tensor)

$$H^{\text{eff}} = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \tilde{V}(\boldsymbol{q}) f(\boldsymbol{q})^2 \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}$$

Hall viscosity



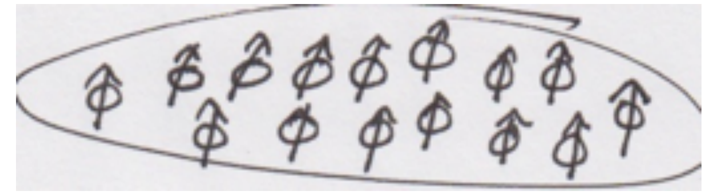
drift velocity flow lines along equipotentials

droplet of incompressible "Hall fluid"

- dissipationless flow means that in steady state, stress forces are normal to flow lines
- "viscosity" is linear response of stress tensor to non-uniform drift velocity field
- "**Hall viscosity**" is dissipationless part of viscosity tensor, just as "**Hall conductivity**" is dissipationless part of conductivity tensor

Isotropic fluid (special case, earlier work only considered this)

- Use Cartesian coordinates with indices i = 1,2, don't distinguish upper and lower indices, metric $g_{ij} = \delta_{ij}$.
- stress tensor $\sigma_{ij} = \sigma_{ji}$, symmetric
- viscosity $\sigma_{ij} = p\delta_{ij} + \eta_{ijkl}\nabla_k v_l + O(v^2)$



(e.g., fluid of spinning molecules has a "Hall viscosity" contribution to the stress tensor)

$$\eta_{ijkl} = \eta^L_{ijkl} + \eta^S_{ijkl} + \eta^H_{ijkl}$$

$$\eta^L_{ijkl} = \eta^L \delta_{ij}\delta_{kl}$$

longitudinal, vanishes if incompressible, dissipative

$$\eta^S_{ijkl} = \eta^S(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})$$

shear, dissipative

$$\eta^H_{ijkl} = \tfrac{1}{2}\eta^H(\delta_{ik}\epsilon_{jl} + \delta_{il}\epsilon_{jk} + \delta_{jk}\epsilon_{il} + \delta_{jl}\epsilon_{ik})$$

Hall, **non-dissipative odd under time-reversal**

$$\epsilon_{ij} \to -\epsilon_{ij}$$

$$\eta^H = \frac{1}{2}\bar{\ell}^z$$

← "internal" angular momentum of fluid per unit area (Read 2009)

Without isotropy.......

- stress tensor is not symmetric, has indices of opposite type: force across a boundary line element in a fluid is

$$dF_a = \sigma_a^b \epsilon_{bc} dL^c$$

- continuity relation for momentum transport in a translationally-invariant fluid:

$$\partial_t \pi_a(\boldsymbol{r}, t) + \nabla_b \sigma_a^b(\boldsymbol{r}, t) = 0$$

momentum density stress tensor (momentum current density)

$$\pi_a(\boldsymbol{r}) = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} \tilde{\pi}_a(\boldsymbol{q})$$

$$\tilde{\pi}_a(\boldsymbol{q}) = \sum_i e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i/2} p_i e^{i\boldsymbol{q}\cdot\boldsymbol{r}_i/2}$$

Fourier transform of momentum density

Long-wavelength expansion..

$$\tilde{\pi}_a(\boldsymbol{q}) = P_a + \tfrac{1}{2} i q_a \mathcal{D} - i\epsilon_{ab}\Lambda^{bc} q_c + O(q^2)$$

- generator of translations $P_a = \sum_i p_{ia}$
- generator of dilatations $\mathcal{D} = \frac{1}{2}\sum_i \{r_i^a, p_{ia}\}$
- SO(2,1) generators of area-preserving deformations $\Lambda^{ab} = \Lambda^{ba} = \left(\Lambda^{ab}\right)^\dagger = \frac{1}{2}\sum_i \left(\epsilon^{ac} r_i^b + \epsilon^{bc} r_i^a\right) p_{ic}$

$$[\mathcal{D}, \Lambda^{ab}] = 0$$

$$[\Lambda^{ab}, \Lambda^{cd}] = \tfrac{1}{2} i\hbar \left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc} + \epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{ac}\right)$$

- The quadratic Casimir is: $C_2 = -\det|\Lambda| \equiv -\frac{1}{2}\epsilon_{ac}\epsilon_{bd}\Lambda^{ab}\Lambda^{cd}$

$L^z(g) = g_{ab}\Lambda^{ab}$

generator of rotations, if isotropic

Can derive the general relation for 2D Hall viscosity;

- stress tensor in presence of fluid flow (covariant form):

$$\sigma^a_b = p\delta^a_b - \eta^{ac}_{bd}\nabla_c v^d + O(v^2)$$

- Dissipationless Hall viscosity term (A = total area covered by fluid):

$$(\eta_H)^{ab}_{cd} = -\frac{i}{A\hbar}\epsilon_{be}\epsilon_{df}\langle[\Lambda^{ae},\Lambda^{df}]\rangle_0$$ antisymmetric!

$$-\frac{i}{A\hbar}\langle[\Lambda^{ae},\Lambda^{df}]\rangle_0 = \frac{1}{2}\left(\epsilon^{ac}\eta^{bd}_H + \epsilon^{bc}\eta^{ad}_H + \epsilon^{ad}\eta^{bc}_H + \epsilon^{bd}\eta^{ac}_H\right)$$

- Defines a new symmetric rank-2 **Hall-viscosity tensor**:

$$\eta^{ab}_H = \frac{1}{A}\langle\Lambda^{ab}\rangle_0 \quad = \tfrac{1}{2}g^{ab}\langle\ell^z(g)\rangle_0$$

in the case of rotational invariance (c.f. Read 2009) application to FQHE

- The SO(2,1) deformation algebra splits into two independent pieces: cyclotron orbits and guiding centers:

$$\Lambda^{ab} = \frac{\ell_B^2}{2\hbar}\epsilon^{ac}\epsilon^{bd}\sum_i \tfrac{1}{2}\{\pi_{ic}\pi_{id}\} \quad + \quad \frac{\hbar}{2\ell_B^2}\sum_i \tfrac{1}{2}\{R_i^a, R_i^b\}$$

cyclotron motion ↓

Results of zograf et al for integer QHE case (cyclotron-motion form factors) (not so interesting)

guiding centers ↓

relates to incompressibility, new FQHE results!

summary of new results:

- The guiding-center Hall viscosity **defines** the natural metric g^{ab} associated with incompressibility through

$$\eta_H^{ab} \equiv \eta_H g^{ab}$$

- The Hall viscosity gives an **intrinsic electric dipole moment per unit length** on the (unreconstructed) boundary of a Hall fluid:

$$dp^a = \frac{e}{\hbar}\eta_H^{ab}\epsilon_{bc}dL^c \ell_B^2$$

- The guiding-center Hall viscosity provides a lower bound to a fundamental measure of incompressibility provided by the "guiding-center structure function", which is **satisfied as an equality** by ideal model FQHE states such as Laughlin, Moore-Read, Read-Rezayi...

VOLUME 54, NUMBER 6 PHYSICAL REVIEW LETTERS 11 FEBRUARY 1985

Collective-Excitation Gap in the Fractional Quantum Hall Effect

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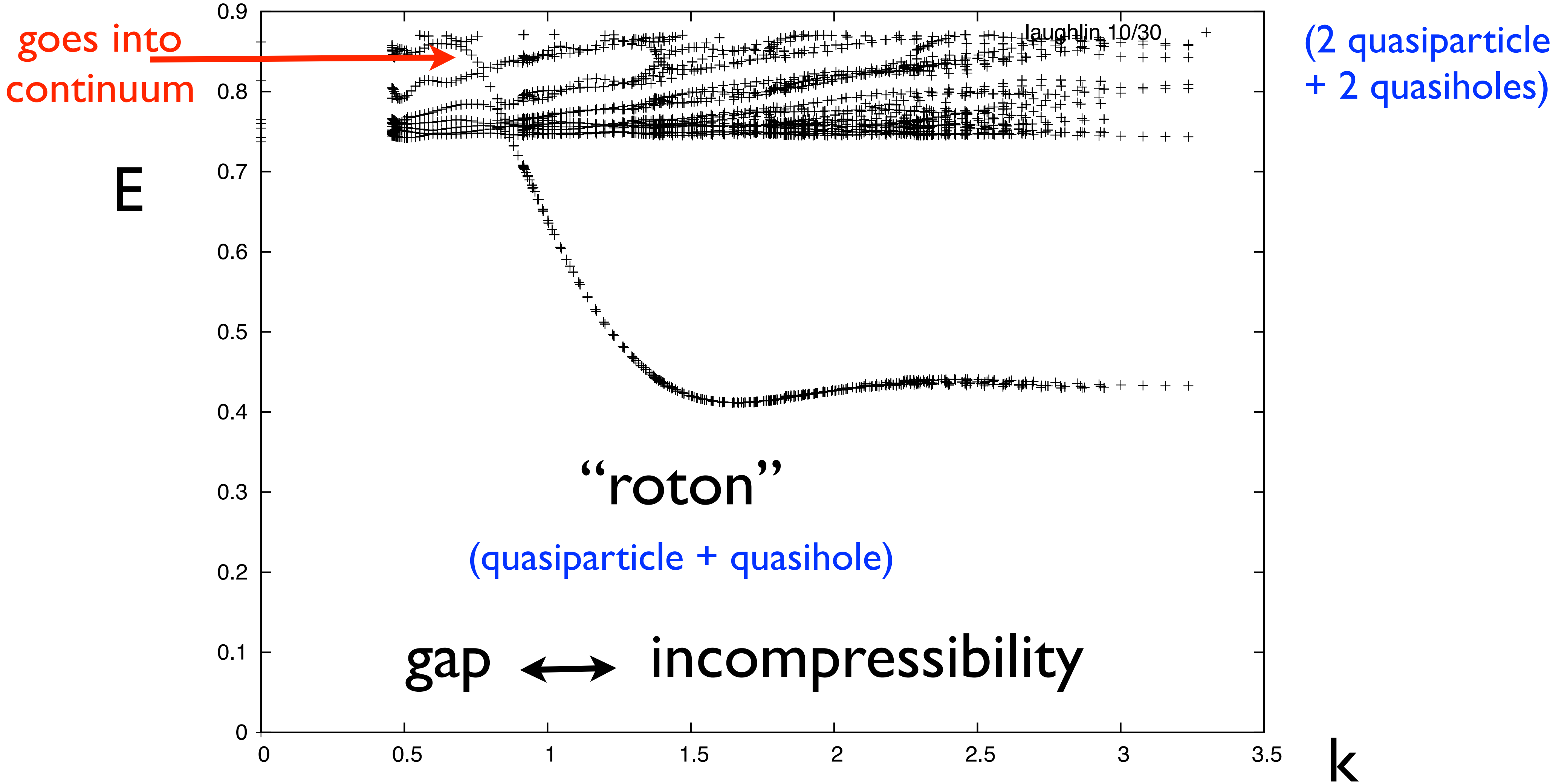
We present a theory of the collective excitation spectrum in the fractional quantum Hall-effect regimes, in analogy with Feynman's theory for helium. The spectrum is in excellent quantitative agreement with the numerical results of Haldane. *Within this approximation* we prove that a finite gap is generic to any liquid state in the extreme quantum limit and that in this single-mode *approximation* gapless excitations can arise only as Goldstone modes for ground states with broken translation symmetry.

- crucial point: for an incompressible FQHE state, the "guiding center structure factor" s(k) vanishes as k^4 as $k \to 0$



Collective mode with short-range V_1 pseudopotential, 1/3 filling (Laughlin state is exact ground state in that case)

(behavior of mode at small k disagrees with recent prediction by Vignale and Toklaty based on a conjectured property involving "Hall viscosity" that appears to be false)

$$[R^a, R^b] = i\epsilon^{ab}\ell^2$$

guiding-center algebra

$$\rho(\boldsymbol{q}) = \sum_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$

Fourier components of guiding-center density

filling factor

$$\langle\rho(\boldsymbol{q})\rangle = 2\pi\nu\delta^2(\boldsymbol{q}\ell)$$

$$\langle\rho(\boldsymbol{q})\rho(\boldsymbol{q}')\rangle - \langle\rho(\boldsymbol{q})\rangle\langle\rho(\boldsymbol{q}')\rangle = 2\pi s(\boldsymbol{q})\delta^2(\boldsymbol{q}\ell)$$

$$s(\boldsymbol{q}) = \frac{1}{N_{\text{orb}}}\sum_{ij}\langle e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\rangle - \langle e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}\rangle\langle e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\rangle$$

- guiding-center structure factor

Note:
smaller O(k4) term
means
more incompressible

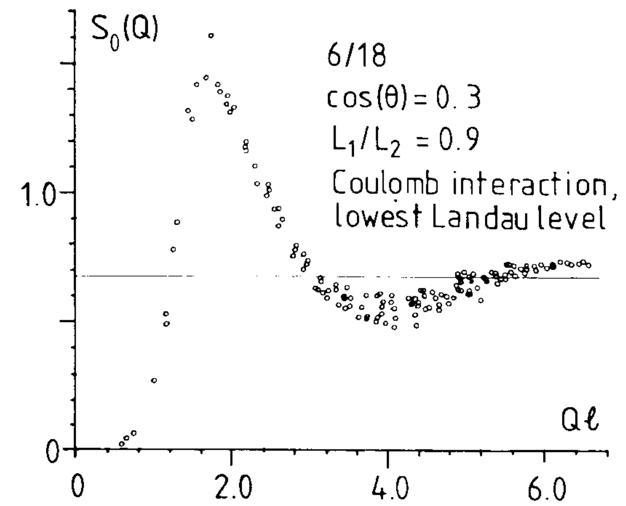


Figure 8.8 Same quantities as Fig. 8.7, but calculated for the true ground state of the lowest-Landau-level Coulomb interaction rather than the Laughlin state.

$$S_0(\boldsymbol{q}) = \frac{1}{N}\sum_{ij}\langle e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i} e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\rangle - \langle e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}\rangle\langle e^{-i\boldsymbol{q}\cdot\boldsymbol{R}_j}\rangle$$

$$\nu = \frac{N}{N_{\mathrm{orb}}}$$

number of orbitals = number of flux quanta through 2D surface

$$s(\boldsymbol{q}) = \nu S_0(\boldsymbol{q})$$

(per flux , instead of per particle)

variational upper bound to collective excitation energy

$$0 < \Delta E(k) < \frac{O(k^4)}{s(k)}$$

must be $O(k^4)$ if gapped

Girvin et al. 1985:

In order to evaluate Eq. (5) using (10) and (11) we need a specific model for the ground state. We have chosen to use the Laughlin ground state (LGS) for $\nu = \frac{1}{3}, \frac{1}{5}$.[2] For the LGS $\bar{s}(k)$ does vanish as $|k|^4$ with a coefficient which may be calculated exactly, i.e.,[8,9]

$$\bar{s}(k) = |k|^4(1-\nu)/8\nu . \qquad (16)$$

what determines this coefficient?

from a classical plasma sum rule!

- Laughlin state:

$$\nu = \frac{1}{m} \qquad s_0' = \frac{(m-1)}{2}$$

$$s(k) = |s_0'|\nu \left(\frac{k^2\ell^2}{2}\right)^2$$

This will be identified as the "guiding center spin" of the "elementary droplet" of Laughlin state.

- example 1/3 Laughlin state:

$$s'_0 = 1$$

m	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$					
Laughlin state	1	0	0	1	0	0	1	0	0
uniform reference state	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Laughlin state: $L^z = \frac{3}{2} - s'_0$

uniform reference state: $L^z = \frac{3}{2}$

elementary droplet

- example:2/3 "anti-Laughlin" state:
  01101101101 1

$$s'_0 = -1$$

Odd under particle-hole transformation!

PHYSICAL REVIEW B VOLUME 33, NUMBER 4 15 FEBRUARY 1986

Magneto-roton theory of collective excitations in the fractional quantum Hall effect

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We present a theory of the collective excitation spectrum in the fractional quantum Hall effect which is closely analogous to Feynman's theory of superfluid helium. The predicted spectrum has a large gap at $k=0$ and a deep magneto-roton minimum at finite wave vector, in excellent quantitative agreement with recent numerical calculations. We demonstrate that the magneto-roton minimum is a precursor to the gap collapse associated with the Wigner crystal instability occurring near $\nu=\frac{1}{7}$. In addition to providing a simple physical picture of the collective excitation modes, this theory allows one to compute rather easily and accurately experimentally relevant quantities such as the susceptibility and the ac conductivity.

In order to evaluate (4.12) it is convenient to note that the projected density operators obey a closed Lie algebra defined by

$$[\bar{\rho}_{\mathbf{k}},\bar{\rho}_{\mathbf{q}}]=(e^{k^*q/2}-e^{kq^*/2})\bar{\rho}_{\mathbf{k}+\mathbf{q}} \; . \qquad (4.13)$$

The fundamental Lie algebra:

$$[\rho(\boldsymbol{q}), \rho(\boldsymbol{q}')] = 2i \sin(\tfrac{1}{2}\boldsymbol{q} \times \boldsymbol{q}'\ell^2)\rho(\boldsymbol{q} + \boldsymbol{q}')$$

- The regularized form (in the thermodynamic limit) is the **fluctuation** relative to uniform background density:

$$\delta\rho(\boldsymbol{q}) = \rho(\boldsymbol{q}) - \langle\rho(\boldsymbol{q})\rangle \qquad \lim_{\lambda\to 0} \delta\rho(\lambda\boldsymbol{q}) = 0$$

$$[\delta\rho(\boldsymbol{q}), \delta\rho(\boldsymbol{q}')] = 2i \sin(\tfrac{1}{2}\boldsymbol{q} \times \boldsymbol{q}'\ell^2)\delta\rho(\boldsymbol{q} + \boldsymbol{q}')$$

obeys same Lie algebra as unregularized form!

- The momentum (generator of translations):

$$P_a = \lim_{\lambda\to 0} \lambda^{-1} \frac{\hbar}{\ell^2} \epsilon_{ab} \nabla_q^b \delta\rho(\lambda\boldsymbol{q})$$

$$[P_a, P_b] = 0 \qquad [P_a, \delta\rho(\boldsymbol{q})] = \quad \hbar q_a \delta\rho(\boldsymbol{q})$$

components now commute!

$$\rho(0) = N$$

- without regularization, we just get

$$P_a = \frac{\hbar}{\ell^2}\epsilon_{ab}\sum_i R_i^b$$

$$[P_a, P_b] = \frac{\hbar^2}{\ell^2}\epsilon_{ab}\rho(0) \neq 0$$

(components of unregularized momentum do <u>not</u> commute)

unregularized momentum	=	regularized momentum	+	center-of-mass momentum

SO(2,1) deformation algebra:

$$\Lambda^{ab} = \Lambda^{ba}$$

$$[\Lambda^{ab}, \Lambda^{cd}] = \frac{1}{2}\left(\Lambda^{ac}\epsilon^{bd} + \Lambda^{ad}\epsilon^{ac} + \Lambda^{bd}\epsilon^{ac} + \Lambda^{bc}\epsilon^{ac}\right)$$

Casimir: $$C_2 = \det|\mathbf{\Lambda}| = \frac{1}{2}\epsilon_{ac}\epsilon_{bd}\Lambda^{ab}\Lambda^{cd}$$

$$\{\Lambda^{11}, \Lambda^{12} = \Lambda^{21}, \Lambda^{22}\}$$

Three generators

- unregularized form:

$$\Lambda^{ab} = \frac{1}{4\ell^2}\sum_i \{R_i^a, R_i^b\}$$

quadratic, symmetric

- regularized form (center-of-mass part removed):

$$\Lambda^{ab} = \lim_{\lambda\to 0}\frac{1}{2\lambda^2}\nabla_q^a\nabla_q^b\delta\rho(\lambda\boldsymbol{q})$$

- The SO(2,1) “deformation algebra” is the Lie subalgebra of generators of linear area-preserving deformations of the guiding centers.
- The full Girvin-MacDonald-Platzman Lie algebra is the full algebra of arbitrary area-preserving deformations of the guiding centers.

Rotational symmetry

- requires a metric
- angular momentum is sum of guiding center and dynamical momentum part:

$$L^z = g_{ab}\Lambda^{ab} - \bar{L}^z \qquad \bar{L}^z = \frac{\ell^2}{2\hbar} g^{ab}\pi_a\pi_b$$

metric

cyclotron motion kinetic energy = $\omega_c \bar{L}^z$

(separately conserved in high-field limit)

Not present when field is tilted!

- In the absence of rotational symmetry, (e.g. with a tilted field):

$$\lim_{\lambda \to 0} s(\lambda \boldsymbol{q}) \to \frac{1}{4} \lambda^4 \Gamma_S^{abcd} q_a q_b q_c q_d \ell^4$$

- new result, derived using translational invariance without assuming rotational invariance:

$$\Gamma_S^{abcd} = \frac{1}{2N_{\mathrm{orb}}} \langle \{ \Lambda^{ab}, \Lambda^{cd} \} \rangle - \langle \Lambda^{ab} \rangle \langle \Lambda^{cd} \rangle$$

symmetric in $ab \leftrightarrow cd$

4th-rank tensor

Fluid dynamics of the incompressible state

guiding-center density

$$H = \int d^2\boldsymbol{r}\ (h_0(\boldsymbol{r}) + V(\boldsymbol{r})\bar{\rho}(\boldsymbol{r}))$$

(quasi)-local translationally-invariant Hamiltonian that gives rise to incompressibility

Slowly-varying "external" potential (including Hartree potential from long-range part of Coulomb force).

$\partial_t \rho + \nabla_a J^a = 0$ continuity equation for particle density

$J^a = \rho v^a$ definition of flow velocity field

Drift velocity field

$$v^a(\boldsymbol{r}) = \frac{\ell_B^2}{\hbar}\epsilon^{ab}\nabla_b V(\boldsymbol{r})$$

$$\frac{\partial}{\partial t}\pi_a + \nabla_b \sigma_a^b + \rho \nabla_a V = 0$$

V(r) violates momentum conservation

stress tensor

$$\pi_a = \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} \langle \tilde{\pi}_a(\boldsymbol{q})\rangle$$ momentum density

$$\tilde{\pi}_a(\boldsymbol{q}) = \frac{\hbar}{\ell_B^2}\epsilon_{ab}\sum_{i=1}^{N}(e^{i\frac{1}{2}\boldsymbol{q}\cdot\boldsymbol{R}_i})\boldsymbol{R}_i^b(e^{i\frac{1}{2}\boldsymbol{q}\cdot\boldsymbol{R}_i})$$

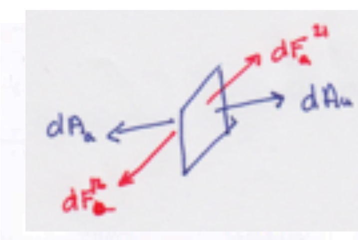
Viscosity (linear response)

$$\sigma_b^a = -p\delta_b^a + \eta_{bd}^{ac}\nabla_c v^d + O(v^2)$$

- stress tensor → σ_b^a
- hydrostatic pressure → p
- viscosity tensor → η_{bd}^{ac}
- gradient of flow velocity field → $\nabla_c v^d$

Stress tensor

force across boundary:



$$dF_a = \sigma_a^b \, dA_b$$

↑ area element

There is no "symmetry" of the Stress tensor because it has one (position-type) upper index and one (momentum-type) lower index

- special feature of an incompressible Hall fluid:

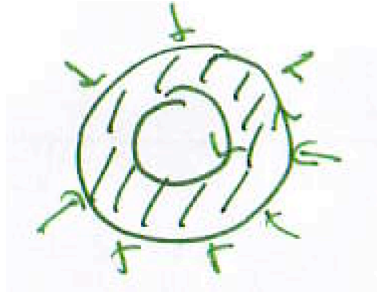
$$p = 0$$

vanishing hydrostatic pressure!

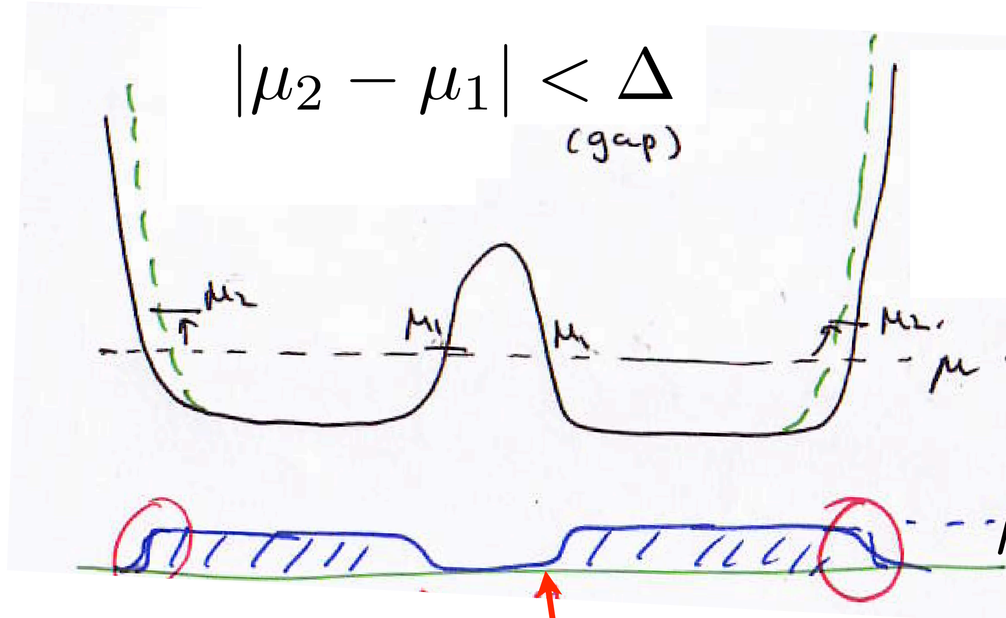
IF rotational invariance with metric g_{ab} is present

$$\sigma^{ab} \equiv g^{ac}\sigma_c^b = \sigma^{ba} \text{ (symmetric)}$$

(no such relation without rotational invariance)



edge-currents screen interior from forces applied at the edge



supports different chemical potentials at inner and outer edges

$$\rho = \frac{\nu}{2\pi\ell_B^2}$$

no change

- apply pressure to outer edge

  - the edge current increases, generating a balancing force that compensates the applied "pressure"
  - no transmission of force to interior (unlike a classical incompressible fluid which is gapless and transmits pressure)

- rewrite viscosity tensor as a dimensionless rank-4 tensor with 4 upper indices:

$$\eta^{ac}_{bd} = \frac{\hbar}{\ell_B^2}\epsilon_{be}\epsilon_{df}\Gamma^{acef}$$

$$v^d_{\text{drift}} = \frac{\ell_B^2}{\hbar}\epsilon^{de}\nabla_e V$$

dimensions of viscosity

dimensionless

sym sym

Γ is symmetric in the first pair and in the second pair of indices

so $\eta^{ac}_{ad} = 0 \; (p = 0)$

no dissipation: $\sigma^a_b \nabla_a J^b = 0$

$$\eta^{ac}_{bd} = -\eta^{ca}_{db}$$

"Hall viscosity"

$$\Gamma^{abcd} = -\Gamma^{cdab}$$

"dimensionless Hall viscosity"

Odd!

- after a little work, I obtain

$$\Gamma_A^{abcd} = \frac{1}{N_{\rm orb}} \langle \Psi_0 | \tfrac{1}{2i} [\Lambda^{ab}, \Lambda^{cd}] | \Psi_0 \rangle$$

unperturbed (uniform) ground state

$$\Lambda^{ab} = \frac{1}{4\ell_B^2} \sum_{i=1}^{N} \{R_i^a, R_i^b\} = \Lambda^{ba}$$

generators of linear deformations of the guiding centers

$$[\Lambda^{ab}, \Lambda^{cd}] = \tfrac{i}{2} \left(\epsilon^{ac} \Lambda^{bd} + \epsilon^{ad} \Lambda^{bc} + \epsilon^{bc} \Lambda^{ad} + \epsilon^{bd} \Lambda^{ac} \right)$$

- This is the SO(2,1) Lie algebra (like the Lorentz group in 2+1 dimensions)

- It has three generators Λ^{11}, Λ^{21} and Λ^{22}. Casimir is

$$C_2 = -\det|\Lambda| = (\Lambda^{12})^2 + \left(\frac{\Lambda^{11} - \Lambda^{22}}{2} \right)^2 - \left(\frac{\Lambda^{11} + \Lambda^{22}}{2} \right)^2$$

number of electron orbitals $N_{\rm orb} = \dfrac{A}{2\pi \ell^2}$

Structure factor:
new result (without invoking rotational invariance):

$$\lim_{\lambda \to 0} S_0(\lambda \boldsymbol{q}) = \frac{\lambda^4}{4} \Gamma_S^{abcd} q_a q_b q_c q_d + O(\lambda^6)$$

$$\Gamma_S^{abcd} = \frac{1}{N_{\text{orb}}} \left(\langle \Psi_0 | \tfrac{1}{2} \{ \Lambda^{ab}, \Lambda^{cd} \} | \Psi_0 \rangle - \langle \Psi_0 | \Lambda^{ab} | \Psi_0 \rangle \langle \Psi_0 | \Lambda^{cd} | \Psi_0 \rangle \right)$$

Combines naturally with Hall viscosity $i\Gamma_A^{abcd}$

$$\Gamma_S^{abcd} + i\Gamma_A^{abcd} = \frac{1}{N_{\text{orb}}} \left(\langle \Lambda^{ab} \Lambda^{cd} \rangle - \langle \Lambda^{ab} \rangle \langle \Lambda^{cd} \rangle \right)$$

can be calculated from adiabatic variation of periodic boundary condition geometry!

can write as a positive 3x3 Hermitian matrix

$$M_{(ab),(cd)} = \left(M_{(cd),(ab)} \right)^*$$

$(ab) = (11), (12), (22)$

•

$$\Gamma^{abcd} = \Gamma_A^{abcd} = 2\pi \tfrac{1}{2}\left(\epsilon^{ac}Q^{bd} + \epsilon^{ad}Q^{bc} + \epsilon^{bc}Q^{ad} + \epsilon^{bd}Q^{ac}\right)$$

antisymmetric
$\Gamma_A^{abcd} = -\Gamma_A^{cdab}$

A symmetric rank-2 tensor

- Read (2009) defines a scalar Hall viscosity of rotationally-invariant Hall fluids as

$$\eta^{(A)} = \tfrac{1}{2}\rho\bar{\ell}^z$$

fluid density

intrinsic L^z angular momentum per particle

$$\eta_H^{ab} = \frac{\hbar}{\ell_B^2} Q^{ab}$$

- then

$$Q^{ab} = \eta^{(A)}\frac{\ell^2}{\hbar} g^{ab}$$

constant in an incompressible region with translational invariance

metric defined by rotational invariance

Galilean metric (from mass tensor)

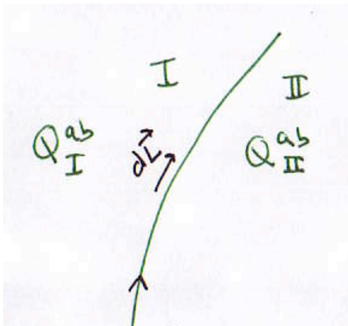
Avron et al result:

$$Q^{ab} = \left(\tfrac{1}{4\pi}\sum_n \nu_n s_n\right) g^{ab}$$

filling of n^{th} Landau level

$n + \frac{1}{2}$

significance of Q^{ab}:



static boundary (must be an equipotential = a flow line)

$$d\boldsymbol{L} \cdot \boldsymbol{\nabla} V = 0$$

$$dp^a = e\Delta Q^{ab}\epsilon_{bc}dL^c$$

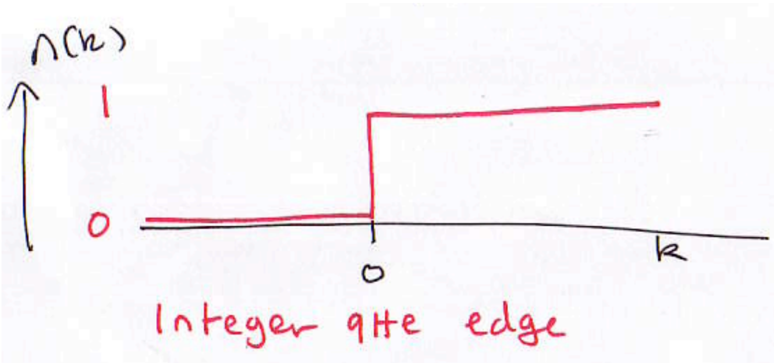
- electric dipole
- electron charge
- dimensionless tensor $\Delta Q^{ab} = Q^{ab}_I - Q^{ab}_{II}$
- length of boundary element

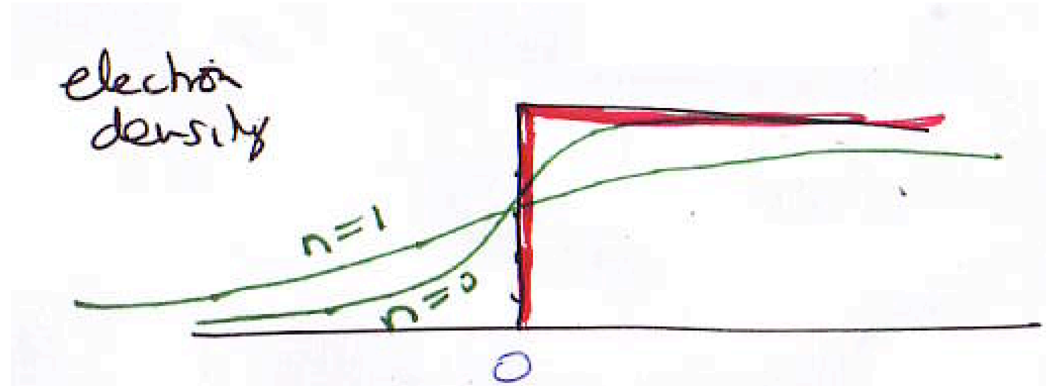
- Discontinuity of Q^{ab} across boundary means stress force from I to II does not balance that from II to I ! ($\nabla_a \nabla_b V$ is continuous)

- get an intrinsic dipole moment at the boundary so the stress anomaly is balanced by the force

$$dF_a = dP^b \nabla_b E_a$$
$$(eE_a = -\nabla_a V)$$

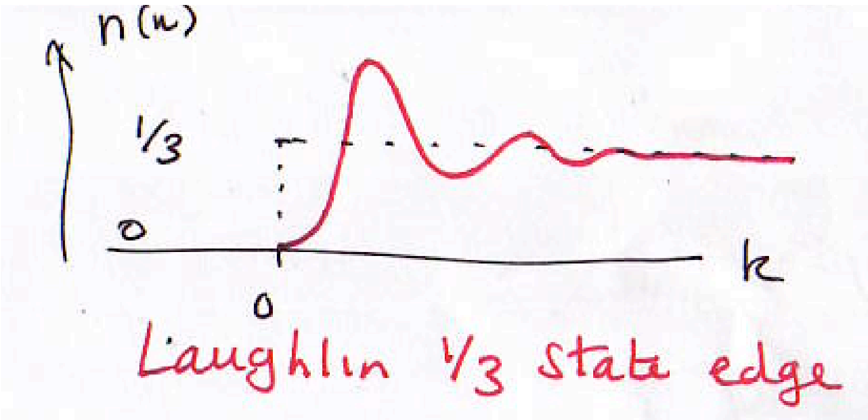
two separate contributions to edge dipole...





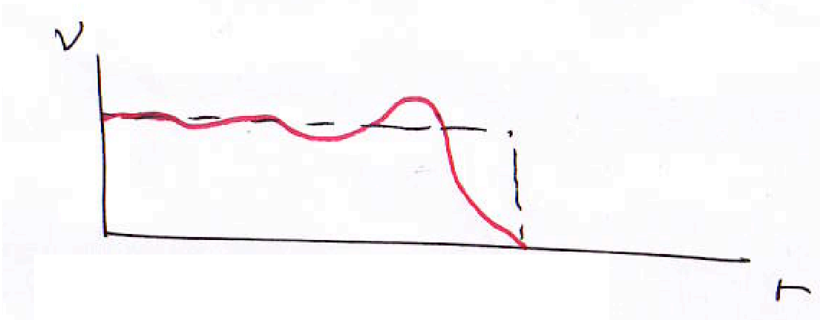
- smearing of electron density relative to guiding-center density by cyclotron-motion gaussian form-factor: (= the Avron et al. Hall viscosity term)

- non-trivial structure of guiding-center occupations near FQHE edges, required by conformal field theory..

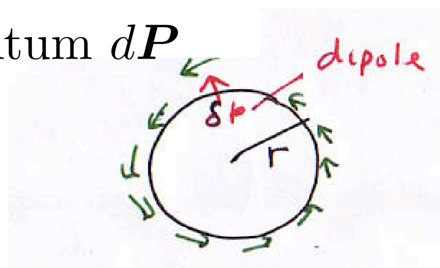


The two effects just add.

relation of edge dipole to "shift"



momentum $d\boldsymbol{P}$



circular droplet

The dipole at a segment of the edge has a momentum

$$dP_a = \frac{\hbar}{e\ell_B^2}\epsilon_{ab}dp^b$$

momentum dipole

doesn't contribute to total momentum:

$$\oint dP_a = 0$$

it does contribute an extra term to total angular momentum:

$$\Delta L^z(\boldsymbol{g}) = \hbar \oint \epsilon^{ab} g_{bc} r^c dP_a \neq 0$$

specialize to rotationally invariant case

- elementary droplet of Hall fluid has p particles in q orbitals and angular momentum $L^z = \frac{1}{2}pq - (s + s')$
- In the high-field limit, -s is the total Landau-orbit angular momentum of the droplet, and -s′ is the intrinsic guiding center angular momentum (a modified "shift"), (both are quantized)
- s′ is odd under particle-hole conjugation of a Landau level, and vanishes when all Landau levels are filled or empty (Integer QHE case).

For the guiding-center structure factor

$$S_0(q) \to \frac{1}{2\pi\ell^2}\gamma\left(\frac{q^2\ell^2}{2}\right)^2 \qquad \gamma \geq \frac{|s'|}{q}$$

absolute value of
Hall viscosity term,
which can have either sign

- for the known results for the Laughlin states, the bound is an equality.
- Numerical results for the Moore-Read state (adiabatic variation of periodic bc) shows that the Hermitian matrix M_{ab,cd} has a single eigenvalue, even for sizes too small for convergence to the quantized value given by the (modified) shift in rotationally-invariant geometries, showing that the bound is satisfied as a equality for model wavefunctions derived from conformal field theory.
- This is not true when corrections due to e.g. Coulomb interactions are included.

Two possibilities for the generic rotationally-invariant models

- the bound is an inequality, because the RPA-like dressing of the ground state by zero-point fluctuations of the collective mode make the system more compressible than the ideal model cft reference wavefunction.
- or, perhaps, the bound becomes an equality again in the thermodynamic limit (unlikey, but not ruled out yet).

numerical exact finite-size calculation of the fourth-rank tensors

1/3, Coulomb

	S(q)	Hall visc.
1 1 1 1	0.0938448048	0.0000000000
1 1 1 2	-0.0000000134	0.0875134467
1 1 2 1	-0.0000000134	0.0875134467
1 1 2 2	-0.0938448412	0.0000000002
1 2 1 1	-0.0000000134	-0.0875134467
1 2 1 2	0.0938448139	0.0000000000
1 2 2 1	0.0938448139	0.0000000000
1 2 2 2	-0.0000000146	0.0875134466
2 1 1 1	-0.0000000134	-0.0875134467
2 1 1 2	0.0938448139	0.0000000000
2 1 2 1	0.0938448139	0.0000000000
2 1 2 2	-0.0000000146	0.0875134466
2 2 1 1	-0.0938448412	-0.0000000002
2 2 1 2	-0.0000000146	-0.0875134466
2 2 2 1	-0.0000000146	-0.0875134466
2 2 2 2	0.0938448098	0.0000000000

0.093844 > 0.087513

N=11 particles in 33 orbitals, on torus
with 6-fold discrete rotational symmetry

1/3, Laughlin

1 1 1 1	0.0834031589	0.0000000000
1 1 1 2	0.0000000043	0.0834031566
1 1 2 1	0.0000000043	0.0834031566
1 1 2 2	-0.0834031496	0.0000000002
1 2 1 1	0.0000000043	-0.0834031566
1 2 1 2	0.0834031570	0.0000000000
1 2 2 1	0.0834031570	0.0000000000
1 2 2 2	0.0000000013	0.0834031565
2 1 1 1	0.0000000043	-0.0834031566
2 1 1 2	0.0834031570	0.0000000000
2 1 2 1	0.0834031570	0.0000000000
2 1 2 2	0.0000000013	0.0834031565
2 2 1 1	-0.0834031496	-0.0000000002
2 2 1 2	0.0000000013	-0.0834031565
2 2 2 1	0.0000000013	-0.0834031565
2 2 2 2	0.0834031526	0.0000000000

0.083403 = 0.083403

0.0833333.... in thermodynamic limit

*there is a simple analytic proof that Laughlin etc.
satisfy bound as an equality even in finite periodic systems

summary (Hall viscosity)

- add momentum continuity equation to supplement charge continuity of incompressible Hall fluids. Rotational invariance not assumed.
- linear response of stress tensor to non-uniform drift velocity
- unexpected relation to FQHE structure factor, gives lower bound that is an equality for cft model wavefunctions.
- significance of SO(2,1) deformation algebra.
- provides a natural "incompressibility metric", distinct from metrics derived from long coulomb interaction or cyclotron effective mass, gives length scale at which incompressibility is established in different directions.
- more in arXiv: 0906:1854