QCD thermodynamics at intermediate coupling

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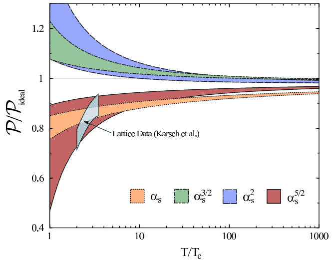
[1] In collaboration with Eric Braaten, Emmanuel Petitgirard, Lars Kyllingstad, Lars Leganger, Mike Strickland, and Nan Su. PRL **83**, 2139 (1999), PRD61 **014017** (2000), PRD **61**, 074016 (2000), PRD **63**, 105008 (2001), PRD **64**, 105012 (2001), PRD **66**, 085016 (2002), PRD **70**, 045001 (2004), PRD **71**, 025011 (2005), PRD **78**, 076008,(2008), JHEP **0908**, 066 (2009), PRD **80**, 085015 (2009).

Weak-coupling expansion



Perturbative free energy vs temperature for QCD with $N_F = 2$ and $N_c = 3$. Lattice results from Karsch et. al. 03.

- The weak-coupling expansion of the free energy of QCD has been calculated to order $\alpha_s^3 \log \alpha_s$ [a].
- Temperatures expected at RHIC energies are $T \sim 0.3$ GeV corresponds to $\alpha_s(2\pi T) \sim 1/3$ or $g_s \sim 2$.
- Successive terms contributing to $\mathcal{F}$ form a decreasing series only if $\alpha_s \simeq 1/20$ or $T \sim 10^5$ GeV.

[a] Arnold and Zhai, 94/95, Kastening and Zhai 95, Braaten and Nieto 96, Kajantie, Laine, Rummukainen, and Schröder 02.

Anharmonic Oscillator

- Consider an anharmonic oscillator with potential

$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{g}{4}x^4 \, .$$

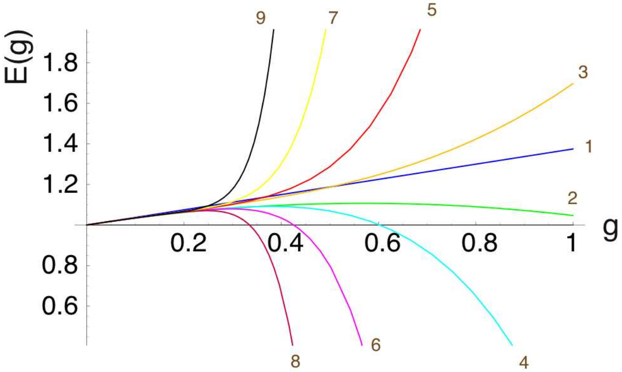
- The perturbative expansion of the ground-state energy $E(g)$ is known to all orders in g [2]

$$E(g) = \omega \sum_{n=0}^{\infty} c_n^{BW} \left(\frac{g}{4\omega^3}\right)^n \, , \quad c_n^{BW} = \left\{ \frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16} \right\} \, .$$

- Behavior for large n: $c_n^{BW} \sim 3^n(n-\frac{1}{2})!$
- Asymptotic series with zero radius of convergence.

[2] Bender and Wu, PR **184** 1231 (1969), PRD **7**, 1620 (1973).

Anharmonic Oscillator



Variational perturbation theory

- Split the harmonic term into two pieces and the treat the subtracted piece as an interaction[3]:

$$\omega^2 \to \Omega^2 + \left(\omega^2 - \Omega^2\right) \Longrightarrow E_N(g,r) = \Omega \sum_{n=0}^{N} c_n(r) \left(\frac{g}{4\Omega^3}\right)^n \,,$$

- New coefficients can be obtained from the old ones

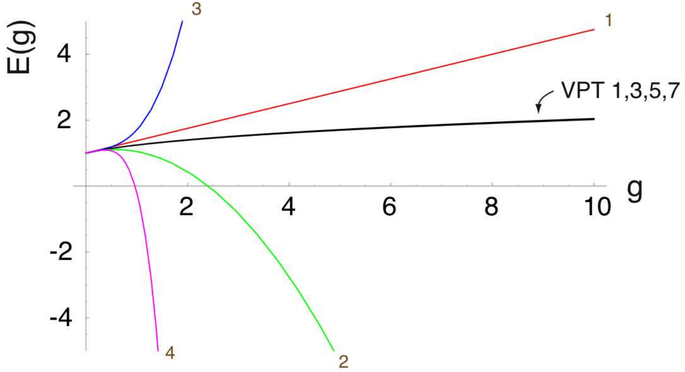
$$c_n(r) = \sum_{j=0}^{n} c_j^{\rm BW} \begin{pmatrix} (1-3j)/2 \\ n-j\,. \end{pmatrix} (2r\Omega)^{n-j} \,, r = \frac{2}{g}\left(\omega^2 - \Omega^2\right)$$

- Fix Ω_N by a variational principle

$$\left.\frac{\partial E_N}{\partial \Omega}\right|_{\Omega=\Omega_N} = 0\,.$$

Variational perturbation theory

- W



Screened perturbation theory

- Consider massless ϕ^4-theory [4] :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{g^2}{24}\phi^4$$

- Add and subtract a (thermal) mass terms

$$\mathcal{L} \;=\; \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{2}m_1^2\phi^2 - \frac{g^2}{24}\phi^4$$

- $m_1 = m$ and we recover the original theory.
- Treat $\frac{1}{2}m_1^2\phi^2$ and $\frac{g^2}{24}\phi^4$ as interactions on equal footing.

[4] F. Karsch, A. Patkos, and P. Petreczky, PLB401, 69(1997).

Screened perturbation theory

$$\mathcal{L} = \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int},$$

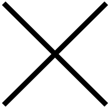
where

$$\begin{aligned} \mathcal{L}_{\rm free} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{\rm int} &= \frac{1}{2}m_1^2\phi^2 - \frac{g^2}{24}\phi^4 \end{aligned}$$

We are expanding around an ideal gas of massive particles.

Feynman rules

Quartic interaction:



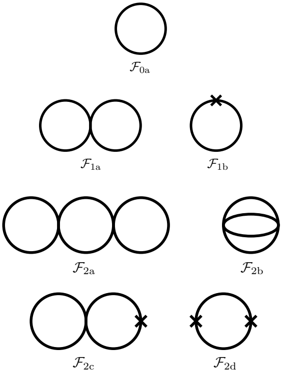
$$= -\frac{g^2}{24}$$

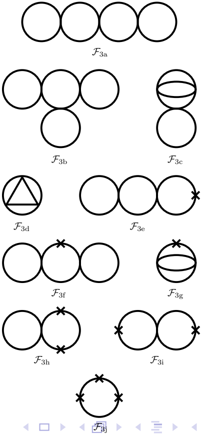
Mass insertion:

$$= \frac{1}{2}m_1^2$$

The diagrams

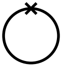
The free energy $\mathcal{F}$ is the sum of these diagrams:





Calculating $\mathcal{F}$

- Expand in powers of g^2 (loop expansion).
- Expand each diagram in powers of $m/T \sim g$:



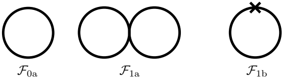
$$= -\frac{1}{2}m_1^2 T \sum_{p_0} \int_p \frac{1}{P^2+m^2}$$

$$= -\frac{1}{2}m_1^2 T \left[\int_p \frac{1}{p^2+m^2} + \sum_{p_0 \neq 0} \int_p \frac{1}{P^2+m^2} \right]$$

$$= -\frac{1}{2}m_1^2 T \left[\int_p \frac{1}{p^2+m^2} + \sum_{p_0 \neq 0} \int_p \frac{1}{P^2} \left(\frac{m^2}{P^2} + \frac{m^4}{P^4} + \cdots \right) \right]$$

- Truncate expansion at g^7.

Calculating $\mathcal{F}$



$$
\begin{aligned}
\mathcal{F}_0 &= -\frac{\pi^2 T^4}{90}\left[1 - 15\hat{m}^2 + 60\hat{m}^3 + ...\right] \\
\mathcal{F}_1 &= -\frac{\pi^2 T^4}{90}\alpha\left[\frac{5}{4} - 15\hat{m} + ...\right] - \frac{\pi^2 T^4}{90}15\hat{m}_1^2\left[1 - 6\hat{m} + ...\right] , \\
\alpha &= \frac{g^2}{4\pi} \\
\hat{m} &= \frac{m}{2\pi T} .
\end{aligned}
$$

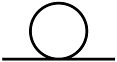
Calculating $\mathcal{F}$

- Ultraviolet divergences removed by renormalization of the vacuum, m^2 and g^2.
- UV-divergences and counterterms are temperature dependent(!).
- Temperature-dependent divergences are systematically substracted out.
- Final result obtained by setting $m_1 = m$.

Pressure

$$\mathcal{P} = -\mathcal{F}\big|_{m_1^2 = m^2}$$

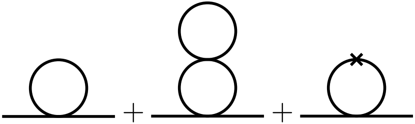
- m is an arbitrary parameter.
- If SPT calculations were carried out to all loop orders, the results would be independent of m.
- Need a prescription for m as a function of g and T.
- To obtain weak-coupling limit:

$m^2 =$  $= \dfrac{g^2 T^2}{24}$

Tadpole mass

We choose m to be the tadpole mass,

$$m^2 =$$

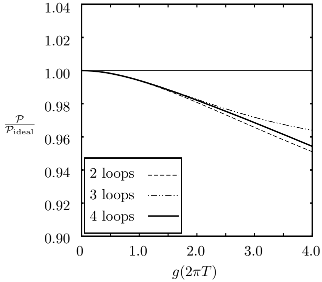


$$+ \cdots = g^2 \left. \frac{d\mathcal{F}}{dm^2} \right|_{m_1^2 = m^2}$$

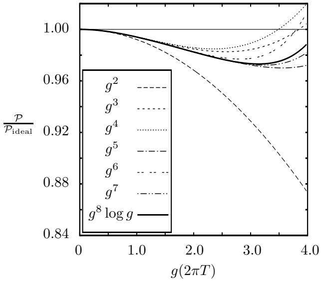
- Self-consistent gap equation for m.
- m is well-defined to all loop orders.
- Selective resummation of diagrams from all loop orders in the original (massless) theory.

Comparison

Screened perturbation theory:



Weak-coupling expansion:



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[5] JOA and L. Kyllingstad, PRD **78**, 076008 (2008), R.R. Parwani and H. Singh, PRD **511**, 4518(1995), E. Braaten and A. Nieto, PRD **51**, 6990 (1995), Gynther et al JHEP **04** 094 (2007), JOA, L. Kyllingstad, and L. Leganger JHEP **08**, 066 (2009).

Convergence properties

- Two-loop result:
  - Indistinguishable from exact numerical result[6].
  - Very fast convergence (g^4).
- Three-loop result:
  -
  - Fairly fast convergence (g^6).
- Four-loop result:
  - Exact numerical results not available.
  - Reasonable approximation?

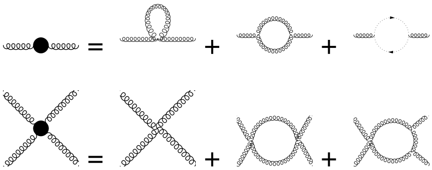
[6] JOA and M. Strickland PRD **63**, 105008 (2001)

Summary SPT

- Have calculated the pressure of a massless ϕ^4 theory to four-loop order in screened perturbation theory.
- Used double power expansion in g^2 and m/T, truncating series at g^7.
- Resummation of selected diagrams from all loop orders in the massless theory gives better convergence properties.
- Agreement with earlier results, both weak-coupling and exact numerical results.

Hard-thermal-loop perturbation theory

- Extension of SPT to gauge theories.
- Cannot simply add and subtract a mass term since this would violate gauge invariance.
- Must use effective progators and vertices that are encoded in the HTL correction term



Hard-thermal-loop perturbation theory

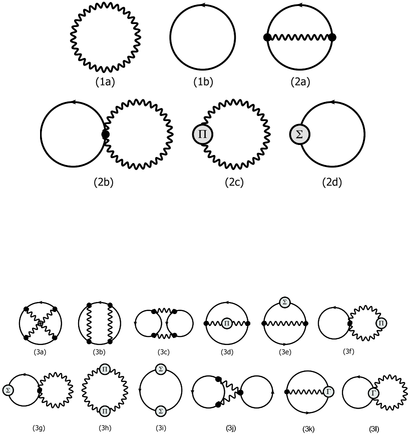
- HTL perturbation theory is a reorganization of the perturbative series for gauge theories which is similar in spirit to SPT.

$$\mathcal{L}_{\rm HTLpt} = \left(\mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm HTL}\right)\Big|_{g\to\sqrt{\delta}g} ,$$

$$\mathcal{L}_{\rm HTL} = -\frac{1}{2}(1-\delta)m_D^2 {\rm Tr}\left(G_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\right\rangle_y G^\mu{}_\beta\right) +(1-\delta)\, im_f^2\bar{\psi}\gamma^\mu\left\langle\frac{y^\mu}{y\cdot D}\right\rangle_y \psi .$$

- HTLpt is defined by expanding in powers of δ.

Feynman diagrams



Technicalities

- Calculations gauge invariant by construction and renormalization is required. Nontrivial cancellations.
- Double expansions in g^2, m_D/T and m_f^2
- Mass prescriptions
  - Weak-coupling expansion

$$m_D^2 = \frac{1}{3}N_f e^2 T^2\left[1 - \frac{e^2}{24\pi^2}\left(4\gamma + 7 + 4\log\frac{\hat{\mu}}{2} + 8\log 2\right) + ...\right]$$

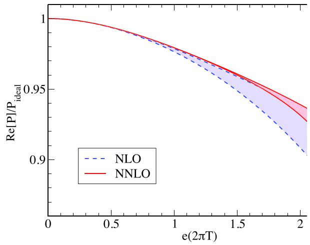
  - Variational mass

$$m^2 = \frac{d\mathcal{F}}{m^2}$$

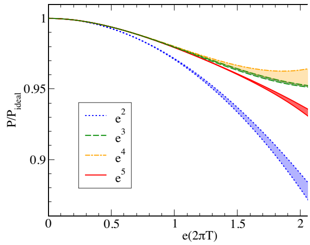
NNLO calculation in QED

$$\Omega_{\rm NNLO} = -\frac{\pi^2 T^4}{45}\left\{1+\frac{7}{4}N_f-\frac{15}{4}\hat{m}_D^3 \right.$$
$$+N_f\frac{\alpha}{\pi}\left[-\frac{25}{8}+\frac{15}{2}\hat{m}_D+15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_D^3-90\,\hat{m}_D\hat{m}_f^2\right]$$
$$+N_f\left(\frac{\alpha}{\pi}\right)^2\left[\frac{15}{64}(35-32\log 2)-\frac{45}{2}\hat{m}_D\right]$$
$$+N_f^2\left(\frac{\alpha}{\pi}\right)^2\left[\frac{25}{12}\left(\log\frac{\hat{\mu}}{2}+\frac{1}{20}+\frac{3}{5}\gamma-\frac{66}{25}\log 2+\frac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\right)\right.$$
$$\left.\left.+\frac{5}{4}\frac{1}{\hat{m}_D}-15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_D+30\frac{\hat{m}_f^2}{\hat{m}_D}\right]\right\}.$$

Comparision



HTLpt [a]



Weak-coupling expansion [a]

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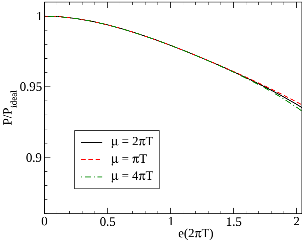
[a] JOA, N. Su, and M. Strickland PRD **80**, 085015 (2009).

[a] B. Kastening and C. X. Zhai, PRD **52**, 7232, (1995).

Comparision

2PI effective action vs HTLpt [7]



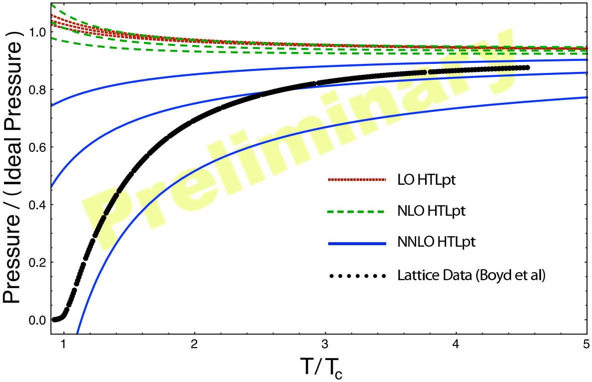
[7] JOA, N. Su, and M. Strickland PRD **80**, 085015 (2009). JOA and M. Strickland, PRD **71**, 025011 (2005). S. Borsanyi and U. Reinosa, PLB**661**, 88 (2008)

QCD - the goodies

$$\Omega_{\rm NNLO} = \mathcal{F}_{\rm ideal}\left\{1-\frac{15}{4}\hat{m}_D^3+\frac{N_c\alpha_s}{3\pi}\left[-\frac{15}{4}+\frac{45}{2}\hat{m}_D-\frac{135}{2}\hat{m}_D^2-\frac{495}{4}\left(\log\frac{\hat{\mu}}{2}+\frac{5}{22}+\gamma_E\right)\hat{m}_D^3\right]\right.$$
$$+\left(\frac{N_c\alpha_s}{3\pi}\right)^2\left[\frac{45}{4\hat{m}_D}-\frac{165}{8}\left(\log\frac{\hat{\mu}}{2}-\frac{72}{11}\log\hat{m}-\frac{84}{55}-\frac{6}{11}\gamma_E-\frac{74}{11}\frac{\zeta'(-1)}{\zeta(-1)}+\frac{19}{11}\frac{\zeta'(-3)}{\zeta(-3)}\right)\right.$$
$$\left.\left.+\frac{1485}{4}\left(\log\frac{\hat{\mu}}{2}-\frac{79}{44}+\gamma_E+\log 2-\frac{\pi^2}{11}\right)\hat{m}_D\right]\right\}.$$

- Variational mass has complex solution.
- Weak-coupling expansion of Debye mass involves magnetic mass and is IR-divergent.
- Use m_E^2 from dimensional reduction. Gauge invariant and well defined to all orders.

QCD - the goodies



Summary and Outlook

- Poor convergence of perturbation theory is a generic problem in quantum mechanics as well as scalar and gauge theories at finite temperature
- VPT, SPT, and HTLpt can be used to improve the convergence of perturbative calculations.
- NNLO calculations of QCD underway. Inclusion of magnetic mass effects?
- HTL perturbation theory can be used to calculate dynamic quantities systematically in a gauge-invariant manner.