MEAN–FIELD DYNAMO THEORY

K.-H. Rädler
Astrophysical Institute Potsdam

OUTLINE

- INTRODUCTION
  Electrodynamics of conducting moving matter
  and the kinematic dynamo problem

- MEAN–FIELD ELECTRODYNAMICS

- MEAN–FIELD MAGNETOFLUIDDYNAMICS

- MEAN-FIELD ASPECTS OF THE KARLSRUHE DYNAMO

ELECTRODYNAMICS OF CONDUCTING MOVING MATTER

- Basic equations

(Pre-)Maxwell equations

Faraday's law $\qquad$ Ampere's law $\qquad$ ...

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Constitutive equations

... $\qquad$ Ohm's law

$$\boldsymbol{B} = \mu \boldsymbol{H}\,, \quad \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} + \boldsymbol{E}^{(\mathrm{ext})})$$

- Electromagnetic potentials

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

equivalent to

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}\,, \quad \boldsymbol{E} = -\partial_t \boldsymbol{A} + \boldsymbol{\nabla}\phi\,.$$

$\boldsymbol{A}$ and ϕ not uniquely defined.

$\boldsymbol{B}$ and $\boldsymbol{E}$ invariant under $\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi$ and $\phi \to \phi - \chi$ with arbitrary χ

Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$

Lorentz convention $\boldsymbol{\nabla} \cdot \boldsymbol{A} + \partial_t \phi = 0$

- Biot–Savart's law

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

equivalent to

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}\,, \quad \boldsymbol{E} = -\partial_t \boldsymbol{A} + \boldsymbol{\nabla}\phi$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J}$$

$$\Rightarrow \boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

Assume proper behavior of $\boldsymbol{J}$ at infinity

$$\Rightarrow \boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu}{4\pi} \int_\infty \frac{\boldsymbol{J}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}^3 x'$$

$$\Rightarrow \boldsymbol{B}(\boldsymbol{x}, t) = -\frac{\mu}{4\pi} \int_\infty \frac{(\boldsymbol{x} - \boldsymbol{x}') \times \boldsymbol{J}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \mathrm{d}^3 x'$$

- Biot–Savart's law

. . .

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J}$$

$$\Rightarrow \boldsymbol{\nabla}^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

$$\Rightarrow \boldsymbol{A}(\boldsymbol{x}, t) = \frac{\mu}{4\pi} \int_{\infty} \frac{\boldsymbol{J}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|} \mathrm{d}^3 x'$$

$$= \frac{\mu}{4\pi} \int_{\infty} \frac{\boldsymbol{J}(\boldsymbol{x} - \boldsymbol{x}', t)}{|\boldsymbol{x}'|} \mathrm{d}^3 x'$$

$$\Rightarrow \boldsymbol{B}(\boldsymbol{x}, t) = -\frac{\mu}{4\pi} \int_{\infty} \frac{(\boldsymbol{x} - \boldsymbol{x}') \times \boldsymbol{J}(\boldsymbol{x}', t)}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \mathrm{d}^3 x'$$

$$= \frac{\mu}{4\pi} \int_{\infty} \frac{\boldsymbol{\nabla} \times \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{x}', t)}{|\boldsymbol{x}'|} \mathrm{d}^3 x'$$

Integral does not change if $\boldsymbol{J} \rightarrow \boldsymbol{J} + \boldsymbol{\nabla}\varphi$!

- Induction equation

$$\boldsymbol{\nabla}\times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla}\times \boldsymbol{B} = \mu \boldsymbol{J}\,, \quad \boldsymbol{\nabla}\cdot \boldsymbol{B} = 0\,, \quad \boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} - \boldsymbol{U}\times\boldsymbol{B} - \boldsymbol{E}^{(\mathrm{ext})}$$

$$\Rightarrow \boldsymbol{\nabla}\left(\eta\boldsymbol{\nabla}\times\boldsymbol{B} - \boldsymbol{U}\times\boldsymbol{B}\right) + \partial_t\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{E}^{(\mathrm{ext})}\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$\eta = 1/\mu\sigma$ magnetic diffusivity

Magnetic Reynolds number $Rm = \frac{U_c L_c}{\eta}$

$$|\boldsymbol{\nabla}\times(\boldsymbol{U}\times\boldsymbol{B})|/|\boldsymbol{\nabla}\times(\eta\boldsymbol{\nabla}\times\boldsymbol{B})| \approx Rm$$

- Magnetic energy

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla}\times\boldsymbol{B} = \mu\boldsymbol{J}\,, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,, \quad \boldsymbol{E} = \frac{\boldsymbol{J}}{\sigma} - \boldsymbol{U}\times\boldsymbol{B} - \boldsymbol{E}^{(\text{ext})}$$

Magnetic energy density $\boldsymbol{B}^2/2\mu$

$$\partial_t(\boldsymbol{B}^2/2\mu) = -\boldsymbol{J}\cdot\boldsymbol{E} - \boldsymbol{\nabla}\cdot\boldsymbol{S}\,, \qquad \boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}$$

$\boldsymbol{S}$ Pointing vector

Total magnetic energy $\int_\infty (\boldsymbol{B}^2/2\mu)\,\mathrm{d}v$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\infty \frac{\boldsymbol{B}^2}{2\mu}\mathrm{d}v = -\int_{\mathcal{V}} \frac{\boldsymbol{J}^2}{\sigma}\mathrm{d}v - \int_{\mathcal{V}} \boldsymbol{U}\cdot(\boldsymbol{J}\times\boldsymbol{B})\,\mathrm{d}v - \int_{\mathcal{V}} \boldsymbol{J}\cdot\boldsymbol{E}^{(\text{ext})}\mathrm{d}v$$

$\mathcal{V}$ finite fluid body, $\boldsymbol{J} = \boldsymbol{0}$ outside,

$\boldsymbol{S}$ vanishes at infinity stronger than $O(r^{-2})$

- Induction equation [2]

○ Conductor at rest $(Rm = 0)$, η constant

$$\eta \boldsymbol{\nabla}^2 \boldsymbol{B} - \partial_t \boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{E}^{(\mathrm{ext})}, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\Rightarrow \boldsymbol{B}(\boldsymbol{x}, t) = \int_\infty G^{(\eta)}(\boldsymbol{x} - \boldsymbol{x}', t - t_0)\, \boldsymbol{B}(\boldsymbol{x}', t_0)\, \mathrm{d}^3 x'$$
$$+ \int_{t_0}^{t} \int_\infty G^{(\eta)}(\boldsymbol{x} - \boldsymbol{x}', t - t')\, \boldsymbol{\nabla}' \times \boldsymbol{E}^{(\mathrm{ext})}(\boldsymbol{x}', t')\, \mathrm{d}^3 x'\, \mathrm{d}t'$$

$(t > t_0)$

with Green's function $G^{(\eta)}(\boldsymbol{x}, t) = (4\pi\eta t)^{-3/2} \exp(-x^2/4\eta t)$

$G^{(\eta)}(\boldsymbol{x}, t)$ satisfies

$$\eta \Delta G^{(\eta)} - \partial_t G^{(\eta)} = 0 \text{ for } t > 0$$

$$G^{(\eta)} \to \delta^3(\boldsymbol{x}) \text{ as } t \to +0$$

Heat conduction equation

$$\eta \Delta T - \partial_t T = -q$$

$$\Rightarrow \quad T(\boldsymbol{x},t) = \int_\infty G^{(\eta)}(\boldsymbol{x}-\boldsymbol{x}', t-t_0)\, T(\boldsymbol{x}', t_0)\, \mathrm{d}^3 x' + \int_{t_0}^{t} \int_\infty G^{(\eta)}(\boldsymbol{x}-\boldsymbol{x}', t-t')\, q(\boldsymbol{x}', t')\, \mathrm{d}^3 x' \, \mathrm{d}t'$$

$$\eta \Delta G^{(\eta)} - \partial_t G^{(\eta)} = 0 \ \text{ for } \ t > 0$$

$$G^{(\eta)} \to \delta^3(\boldsymbol{x}) \ \text{ as } \ t \to +0$$

- Magnetic flux

$$\Phi_\mathrm{m} = \int_\mathcal{S} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{s}$$

Assume that surface $\mathcal{S}$ moves with the fluid.

Then

$$\frac{\mathrm{d}\Phi_\mathrm{m}}{\mathrm{d}t} = \int_\mathcal{S} \Big(\partial_t \boldsymbol{B} - \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B})\Big) \cdot \mathrm{d}\boldsymbol{s}$$

$$= -\int_\mathcal{S} \boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{s} = -\int_{\partial\mathcal{S}} (\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l}$$

Assume $\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B})$, i.e. $\boldsymbol{E}^{(\mathrm{ext})} = \boldsymbol{0}$

$$\Rightarrow \frac{\mathrm{d}\Phi_\mathrm{m}}{\mathrm{d}t} = -\int_{\partial\mathcal{S}} \frac{\boldsymbol{J}}{\sigma} \cdot \mathrm{d}\boldsymbol{l}$$

Flux of vector field $\boldsymbol{F}$

through a surface $\mathcal{S}$ moving with the velocity $\boldsymbol{v}$

$$\Phi_F = \int_{\mathcal{S}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s}$$

$$\frac{\mathrm{d}\Phi_F}{\mathrm{d}t} = \int_{\mathcal{S}} \Big(\partial_t \boldsymbol{F} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{F}) + \boldsymbol{v}\,(\boldsymbol{\nabla} \cdot \boldsymbol{F})\Big) \cdot \mathrm{d}\boldsymbol{s}$$

- Induction equation [3]
  - High-conductivity limit ($Rm \to \infty$), $\boldsymbol{E}^{(\mathrm{ext})} = \boldsymbol{0}$

$$\partial_t \boldsymbol{B} - \boldsymbol{\nabla} \times (\boldsymbol{U} \times \boldsymbol{B}) = \boldsymbol{0}$$

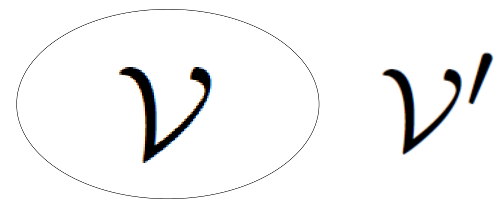
equivalent to

$$\frac{\mathrm{d}\Phi_{\mathrm{m}}}{\mathrm{d}t} = 0 \text{ for any material surface } \mathcal{S}$$

$\Rightarrow$ Concept of frozen–in magnetic flux
or frozen–in magnetic field lines

- Bondi and Gold theorem

  Consider finite fluid body
  surrounded by free space
  $\mathcal{V}$, $\mathcal{V}'$ simply connected



In $\mathcal{V}$: $\quad \boldsymbol{B} = \nabla\Phi\,, \quad \Delta\Phi = 0$

$$\Delta\Phi = 0 \;\Rightarrow\; \Phi = -\sum_{n\geq 1, |m|\leq n} \frac{c_n^m}{r^{n+1}} Y_n^m(\vartheta, \varphi)$$

$$\sum_{|m|\leq 1} \frac{c_1^m}{r^2} Y_1^m(\vartheta, \varphi) = \frac{\mu}{4\pi} \frac{\boldsymbol{m}\cdot\boldsymbol{r}}{r^3}$$

$$\boldsymbol{m} = \frac{1}{2}\int_{\mathcal{V}} \boldsymbol{r} \times \boldsymbol{J}\, \mathrm{d}v$$ magnetic dipole moment

In the case of an ideally conducting fluid all c_n^m bounded.

Bondi and Gold 1950, Rädler 1982

- Representation of a magnetic field by poloidal and toroidal parts

Spherical coordinate system (r, ϑ, φ)

- Axisymmetric vector field $\boldsymbol{F}$

$$\boldsymbol{F} = (F_r, F_\vartheta, F_\varphi)\,, \quad \partial F_r/\partial\varphi = \partial F_\vartheta/\partial\varphi = \partial F_\varphi/\partial\varphi = 0$$

Split $\boldsymbol{F}$ according to $\boldsymbol{F} = \boldsymbol{F}^{\mathsf{P}} + \boldsymbol{F}^{\mathsf{T}}$

with $\boldsymbol{F}^{\mathsf{P}} = (F_r, F_\vartheta, 0)\,, \quad \boldsymbol{F}^{\mathsf{T}} = (0, 0, F_\varphi)$

$\boldsymbol{\nabla} \times \boldsymbol{F}^{\mathsf{P}}$ toroidal, $\quad \boldsymbol{\nabla} \times \boldsymbol{F}^{\mathsf{T}}$ poloidal

- Representation of a magnetic field by poloidal and toroidal parts [2]

  - Arbitrary vector field $\boldsymbol{F}$

$$\boldsymbol{F} = \boldsymbol{r} \times \boldsymbol{\nabla} U + \boldsymbol{r}\, V + \boldsymbol{\nabla} W$$

Put again $\boldsymbol{F} = \boldsymbol{F}^{\mathsf{P}} + \boldsymbol{F}^{\mathsf{T}}$,

now with $\boldsymbol{F}^{\mathsf{P}} = \boldsymbol{r}\, V + \boldsymbol{\nabla} W\,,\ \ \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{r} \times \boldsymbol{\nabla} U,$ $\Leftarrow$ unique definition!

i.e. $$\boldsymbol{F}^{\mathsf{P}} = \Big(rV + \frac{\partial W}{\partial r}, \frac{1}{r}\frac{\partial W}{\partial \vartheta}, \frac{1}{r \sin\vartheta}\frac{\partial W}{\partial \varphi}\Big),$$

$$\boldsymbol{F}^{\mathsf{T}} = \Big(0, -\frac{1}{\sin\vartheta}\frac{\partial U}{\partial \varphi}, \frac{\partial U}{\partial \vartheta}\Big).$$

Again $\boldsymbol{\nabla} \times \boldsymbol{F}^{\mathsf{P}}$ toroidal, $\boldsymbol{\nabla} \times \boldsymbol{F}^{\mathsf{T}}$ poloidal.

- Representation of a magnetic field by poloidal and toroidal parts [3]

  ◦ Magnetic field $\boldsymbol{B}$ (satisfying $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$)

$$\boldsymbol{B}=\boldsymbol{B}^{\mathsf{P}}+\boldsymbol{B}^{\mathsf{T}}$$

$$\boldsymbol{B}^{\mathsf{P}}=-\boldsymbol{\nabla}\times(\boldsymbol{r}\times\boldsymbol{\nabla}S)=\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times(\boldsymbol{r}S))$$

$$\boldsymbol{B}^{\mathsf{T}}=-\boldsymbol{r}\times\boldsymbol{\nabla}T=\boldsymbol{\nabla}\times(\boldsymbol{r}T))$$

$$\boldsymbol{B}^{\mathsf{P}}=\Big(-\frac{1}{r}\Omega S,-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial S}{\partial\vartheta}),\frac{1}{r\sin\vartheta}\frac{\partial}{\partial r}(r\frac{\partial S}{\partial\varphi})\Big)$$

$$\boldsymbol{B}^{\mathsf{T}}=\Big(0,\frac{1}{\sin\vartheta}\frac{\partial T}{\partial\varphi},-\frac{\partial T}{\partial\vartheta}\Big)$$

$$\Omega S=\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta\frac{\partial S}{\partial\vartheta})+\frac{1}{\sin^2\vartheta}\frac{\partial^2 S}{\partial\varphi^2}$$

$\boldsymbol{\nabla}\times\boldsymbol{B}=\boldsymbol{0}$ outside a conducting sphere $\Rightarrow$ $\boldsymbol{B}^{\mathsf{P}}=\boldsymbol{\nabla}\Phi$ but $\boldsymbol{B}^{\mathsf{T}}=\boldsymbol{0}$

- Symmetry properties of the basic equations

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \nabla \times \boldsymbol{H} = \boldsymbol{J}\,, \quad \nabla \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{B} = \mu \boldsymbol{H}\,, \quad \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}) \qquad \mu\,,\ \sigma \text{ constants}$$

Define translation $\boldsymbol{F}^{(\mathrm{tr})}(\boldsymbol{x}, t) = \boldsymbol{F}(\boldsymbol{x} + \Delta \boldsymbol{x}, t)$

time shift $\boldsymbol{F}^{(\mathrm{ts})}(\boldsymbol{x}, t) = \boldsymbol{F}(\boldsymbol{x}, t + \Delta t)$

rotation $\boldsymbol{F}^{(\mathrm{rot})}(\boldsymbol{x}, t) = \boldsymbol{D}^{-1}\boldsymbol{F}(\boldsymbol{D}\boldsymbol{x}, t)\,,\ \det(\boldsymbol{D}) = 1$

reflexion $\boldsymbol{F}^{(\mathrm{refl})}(\boldsymbol{x}, t) = \boldsymbol{D}^{-1}\boldsymbol{F}(\boldsymbol{D}\boldsymbol{x}, t)\,,\ \det(\boldsymbol{D}) = -1$, e.g. $D_{ij} = -\delta_{ij}$

If above equations satisfied with $(\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{U})$

then also with $(\boldsymbol{E}^{(\mathrm{tr})}, \boldsymbol{B}^{(\mathrm{tr})}, \boldsymbol{H}^{(\mathrm{tr})}, \boldsymbol{J}^{(\mathrm{tr})}, \boldsymbol{U}^{(\mathrm{tr})})$

$(\boldsymbol{E}^{(\mathrm{ts})}, \boldsymbol{B}^{(\mathrm{ts})}, \boldsymbol{H}^{(\mathrm{ts})}, \boldsymbol{J}^{(\mathrm{ts})}, \boldsymbol{U}^{(\mathrm{ts})})$

$(\boldsymbol{E}^{(\mathrm{rot})}, \boldsymbol{B}^{(\mathrm{rot})}, \boldsymbol{H}^{(\mathrm{rot})}, \boldsymbol{J}^{(\mathrm{rot})}, \boldsymbol{U}^{(\mathrm{rot})})$

$(\boldsymbol{E}^{(\mathrm{refl})}, -\boldsymbol{B}^{(\mathrm{refl})}, -\boldsymbol{H}^{(\mathrm{refl})}, \boldsymbol{J}^{(\mathrm{refl})}, \boldsymbol{U}^{(\mathrm{refl})})$.

THE KINEMATIC DYNAMO PROBLEM

- Basic equations

Consider (simply connected) conducting body

surrounded by free space

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,,\ \boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J}\,,\ \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \text{everywhere}$$

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B}) \ \text{ in } \mathcal{V}\,, \qquad \boldsymbol{J} = \boldsymbol{0} \ \text{ in } \mathcal{V}'$$

$$\boldsymbol{B} = \mathrm{O}(a^{-3}) \text{ as } a \to \infty \qquad a \text{ distance from conducting body}$$

or (equivalent) ...

- Basic equations

...

$$\nabla \times (\eta \nabla \times \boldsymbol{B} - \boldsymbol{U} \times \boldsymbol{B}) + \partial_t \boldsymbol{B} = 0 \,, \quad \nabla \cdot \boldsymbol{B} = 0 \ \text{ in } \mathcal{V}$$

$$\nabla \times \boldsymbol{B} = 0 \,, \quad \nabla \cdot \boldsymbol{B} = 0 \ \ (\Rightarrow \ \boldsymbol{B} = \nabla \Phi \,, \ \ \Delta \Phi = 0 \,) \text{ in } \mathcal{V}'$$

$$[\boldsymbol{B}] = 0 \ \text{ across } \partial \mathcal{V}$$

$$\boldsymbol{B} = \mathrm{O}(a^{-3}) \text{ as } a \to \infty$$

DYNAMO: $\boldsymbol{B} \not\longrightarrow 0 \quad \text{as} \ \ t \to \infty$

- Anti–dynamo theorems

No dynamo

◇ with a magnetic field which is symmetric about any axis

Cowling 1934, Braginsky 1964, ...

◇ with a magnetic field which depends on two cartesian coordinates only in a fluid of infinite extend in the direction of the third cooordinate

No dynamo

◇ due to a non–radial flow in a spherical fluid body with steady spherically symmetric distribution of the magnetic diffusivity

Elsasser 1946, Bullard and Gellman 1954, Ivers and James 1988, ...

◇ due to a planar flow in an infinitely extended fluid volume in which the magnetic diffusivity varies at most in the direction perpendicular to the flow planes

No spherical dynamos with purely poloidal or purely toroidal magnetic fields

Kaiser 1994, 1995