MEAN–FIELD DYNAMO THEORY

K.-H. Rädler
Astrophysical Institute Potsdam

OUTLINE

- INTRODUCTION
  Electrodynamics of conducting moving matter
  and the kinematic dynamo problem
- MEAN–FIELD ELECTRODYNAMICS
- MEAN–FIELD MAGNETOFLUIDDYNAMICS
- MEAN-FIELD ASPECTS OF THE KARLSRUHE DYNAMO

MEAN–FIELD ELECTRODYNAMICS

- Basic equations

Faraday's law Ampere's law $\cdots$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B}\,, \quad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{J}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Ohm's law

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B})$$

$\Rightarrow$ Induction equation

$$\boldsymbol{\nabla}\Big(\eta \boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{U} \times \boldsymbol{B}\Big) + \partial_t \boldsymbol{B} = \boldsymbol{0}\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\eta = 1/\mu\sigma$$

- Mean fields

$$F = \overline{F} + F'$$

Mean scalar field $\overline{F}$ defined by any averaging procedure which(exactly or approximately) satisfies
Reynolds' averaging rules,

(R1) $\quad \overline{F + G} = \overline{F} + \overline{G}$

(R2) $\quad \overline{\overline{F}} = \overline{F} \quad \rightleftharpoons \quad \overline{F'} = 0$

(R3) $\quad \overline{\overline{F}\,G} = \overline{F}\,\overline{G} \quad \rightleftharpoons \quad \overline{\overline{F}\,\overline{G}} = \overline{F}\,\overline{G}\,, \quad \overline{\overline{F}\,G'} = 0$

(R4) $\quad \overline{\partial F/\partial x} = \partial\overline{F}/\partial x\,, \quad \overline{\partial F/\partial t} = \partial\overline{F}/\partial t$

$\Rightarrow \quad \overline{F\,G} = \overline{F}\,\overline{G} + \overline{F'\,G'}$

Mean vector field $\overline{\boldsymbol{F}}$ defined by averaging its components with respect to the coordinate system chosen,

$$\boldsymbol{F} = e_i F_i\,, \quad \boldsymbol{F} = \overline{\boldsymbol{F}} + \boldsymbol{F}'\,, \quad \overline{\boldsymbol{F}} = e_i \overline{F}_i\,,$$

mean tensor fields analogously.

Notations $\overline{F}$ and $\langle F \rangle$ equivalent

- Examples of averages

  - Statistical or ensemble averages

$$\overline{F}(\boldsymbol{x},t) = \int F(\boldsymbol{x},t;q)g(q)\mathrm{d}q\,, \qquad \int g(q)\mathrm{d}q = 1$$

R1, R2, R3, R4 satisfied
Relation of averages to observable quantities???

  - Space averages

$$\overline{F}(\boldsymbol{x},t) = \int F(\boldsymbol{x}-\boldsymbol{\xi},t)g(\boldsymbol{\xi})\mathrm{d}^3\xi\,, \qquad \int g(\boldsymbol{\xi})\mathrm{d}^3\xi = 1$$

R1, R4 satisfied,
R2, R3 in general not,
can be justified as approximation if (length) scale separation

    - Special cases

$$\overline{F}(\boldsymbol{x},t) = \frac{1}{2L}\int_{-L}^{L} F(\boldsymbol{x}-\boldsymbol{\xi}'',t)\,\mathrm{d}\xi_3 \qquad \boldsymbol{\xi}'' = (0,0,\xi_3)$$

$$\overline{F}(\boldsymbol{x},t) = \frac{1}{(2L)^2}\int_{-L}^{L}\int_{-L}^{L} F(\boldsymbol{x}-\boldsymbol{\xi}',t)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \qquad \boldsymbol{\xi}' = (0,\xi_2,\xi_3)$$

In the limit $L \to \infty$ R1, R2, R3, R4 satisfied

- Examples of averages [2]

- Statistical or ensemble averages ...
- Space averages ...
  - Special cases ...
  - Azimuthal average (Braginsky's average)

$$\overline{F}(r,\vartheta,t) = \frac{1}{2\pi}\int_0^{2\pi} F(r,\vartheta,\varphi,t)\mathrm{d}\varphi$$

R1, R2, R3, R4 satisfied
Average axisymmetric !!

- Time averages

$$\overline{F}(\boldsymbol{x},t) = \int F(\boldsymbol{x},t-\tau)g(\tau)\mathrm{d}\tau\,, \quad \int g(\tau)\mathrm{d}\tau = 1$$

R1, R4 satisfied,
R2, R3 in general not,
can be justified as approximation if (time) scale separation

- Examples of averages [3]

- Statistical or ensemble averages ...
- Space averages ...
  - Special cases ...
  - Azimuthal average (Braginsky's average) ...
- Time averages ...
- Average defined by filtering of a spectrum

$$F(\boldsymbol{x}, t) = \int_{\infty} \hat{F}(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \mathrm{d}^3 k$$

$$\overline{F}(\boldsymbol{x}, t) = \int_{|\boldsymbol{k}|<K} \hat{F}(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \mathrm{d}^3 k$$

R1, R2, R4 satisfied,
R3 in general not,
can be justified as approximation
if length scale (i.e., wave number) separation

- Basic mean–field equations

$$\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}\,, \quad \cdots \quad \boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$$

Maxwell equations and Ohm's law for mean fields

$$\nabla \times \overline{\boldsymbol{E}} = -\partial_t \overline{\boldsymbol{B}}\,, \quad \nabla \times \overline{\boldsymbol{B}} = \mu \overline{\boldsymbol{J}}\,, \quad \nabla \cdot \overline{\boldsymbol{B}} = 0$$

$$\overline{\boldsymbol{J}} = \sigma(\overline{\boldsymbol{E}} + \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}})$$

Mean–field induction equation

$$\nabla \times \left(\eta \nabla \times \overline{\boldsymbol{B}} - \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} - \boldsymbol{\mathcal{E}}\right) + \partial_t \overline{\boldsymbol{B}} = \boldsymbol{0}\,, \quad \nabla \cdot \overline{\boldsymbol{B}} = 0$$

$\boldsymbol{\mathcal{E}}$ mean electromotive force due to fluctuations

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$$

- The mean electromotive force $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$

$$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + (\boldsymbol{u} \times \boldsymbol{b})') - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

$$(\boldsymbol{u} \times \boldsymbol{b})' = \boldsymbol{u} \times \boldsymbol{b} - \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

$\Rightarrow$ $\boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(\overline{B})}$, where $\boldsymbol{b}^{(0)}$ is a functional of $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$
and $\boldsymbol{b}^{(\overline{B})}$ a functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$,
which is linear in $\overline{\boldsymbol{B}}$

$\Rightarrow$ $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{B})}$, with $\boldsymbol{\mathcal{E}}^{(0)}$ independent of $\overline{\boldsymbol{B}}$
and $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ linear and homogeneous in $\overline{\boldsymbol{B}}$

$$\mathcal{E}_i^{(\overline{B})}(\boldsymbol{x}, t) = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{B}_j(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \mathrm{d}^3\xi \, \mathrm{d}\tau$$

K_{ij} depends on $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ only, vanishes for large $|\boldsymbol{\xi}|$ and τ.
$\boldsymbol{\mathcal{E}}^{(\overline{B})}$ at $(\boldsymbol{x}, t)$ depends on the behavior of $\overline{\boldsymbol{B}}$
in some surroundings of $(\boldsymbol{x}, t)$ only.

- The mean electromotive force [2]

...

$\Rightarrow$ $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{B})}$, with $\boldsymbol{\mathcal{E}}^{(0)}$ independent of $\overline{\boldsymbol{B}}$
and $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ linear and homogeneous in $\overline{\boldsymbol{B}}$

$$\mathcal{E}_i^{(\overline{B})}(\boldsymbol{x}, t) = \int_0^{\infty} \int_{\infty} K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{B}_j(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \mathrm{d}^3\xi \, \mathrm{d}\tau$$

On this level the mean–field induction equation
is a linear inhomogeneous integro–differential equation,

$$\nabla\Big(\eta\nabla \times \overline{\boldsymbol{B}} - \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} - \iint \boldsymbol{K}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \circ \overline{\boldsymbol{B}}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)\, \mathrm{d}^3\xi \mathrm{d}\tau\Big)$$

$$+\partial_t \overline{\boldsymbol{B}} = \nabla \times \boldsymbol{\mathcal{E}}^{(0)}$$

- The mean electromotive force [3]

Assume until further notice that $\boldsymbol{b}$ decays to zero if $\overline{\boldsymbol{B}} = \mathbf{0}$
(purely hydrodynamic "background turbulence",
no small–scale dynamo)

$\Rightarrow$ $\boldsymbol{\mathcal{E}}^{(0)}$ decays to zero

$$\mathcal{E}_i(\boldsymbol{x}, t) = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{B}_j(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \mathrm{d}^3\xi \, \mathrm{d}\tau$$

Assume weak variation of $\overline{\boldsymbol{B}}$ in space and time,

$$\overline{B}_j(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) = \overline{B}_j(\boldsymbol{x}, t) - \frac{\partial \overline{B}_j(\boldsymbol{x}, t)}{\partial x_k} \xi_k - \frac{\partial \overline{B}_j(\boldsymbol{x}, t)}{\partial t} \tau - \cdots$$

$$\Rightarrow \mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \frac{\partial \overline{B}_j}{\partial x_k} + c_{ij} \frac{\partial \overline{B}_j}{\partial t} + \cdots$$

$$a_{ij} = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \mathrm{d}^3\xi \mathrm{d}\tau$$

$$b_{ijk} = -\int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \xi_k \mathrm{d}^3\xi \mathrm{d}\tau$$

$$c_{ij} = -\int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \tau \mathrm{d}^3\xi \mathrm{d}\tau$$

- The mean electromotive force [4]

...

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k} + c_{ij}\frac{\partial \overline{B}_j}{\partial t} + \cdots$$

very often simply $$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k} \qquad (*)$$

Relation $(*)$ is an approximation,
which needs to be checked in any application !!!

It requires (length) scale separation !!!

It reduces the mean–field induction equation
to a second–order parabolic differential equation,

$$\boldsymbol{\nabla}\left(\eta\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} - \overline{\boldsymbol{U}}\times\overline{\boldsymbol{B}} - \mathrm{a}\circ\overline{\boldsymbol{B}} - \mathrm{b}\circ(\boldsymbol{\nabla}\overline{\boldsymbol{B}})\right) + \partial_t\overline{\boldsymbol{B}} = 0$$

- Definitions concerning turbulence

When speaking of "turbulence" we think until further notice simply of irregular motions and dot not refer to specific properties of, e.g., "developed turbulence".

Turbulence is called

◇ homogeneous

if all quantities depending on $\boldsymbol{u}$ are invariant under translation of $\boldsymbol{u}$ $(\boldsymbol{u}(\boldsymbol{x},t) \longrightarrow \boldsymbol{u}(\boldsymbol{x}+\Delta\boldsymbol{x},t))$

◇ (statistically) steady

... under time shift in $\boldsymbol{u}$ $(\boldsymbol{u}(\boldsymbol{x},t) \longrightarrow \boldsymbol{u}(\boldsymbol{x},t+\Delta t))$

◇ axisymmetric w.r.t. a given axis

... under rotation of $\boldsymbol{u}$ about this axis

◇ isotropic

... under any rotation of $\boldsymbol{u}$ about any axis

◇ mirror–symmetric w.r.t. a given plane or a given point

... under reflexion of $\boldsymbol{u}$ about this plane or this point

- A simple (academic) example:

$\boldsymbol{u}$ homogeneous isotropic turbulence, $\overline{\boldsymbol{U}} = \mathbf{0}$

Assume $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Homogeneity $\longrightarrow$ $a_{ij}\,,\;\; b_{ijk}$ independent of position

Isotropy $\longrightarrow$ $a_{ij} = \alpha\,\delta_{ij},\;\; b_{ijk} = \beta\,\epsilon_{ijk}$

$$\Rightarrow \quad \boldsymbol{\mathcal{E}} = \alpha\,\overline{\boldsymbol{B}} - \beta\,\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}$$

mean–field version of Ohm's law

$$\overline{\boldsymbol{J}} = \sigma_{\mathrm{m}}(\overline{\boldsymbol{E}} + \alpha\,\overline{\boldsymbol{B}}) \qquad \sigma_{\mathrm{m}} = \sigma/(1+\mu\sigma\beta) = \sigma/(1+\beta/\eta)$$

mean–field conductivity

mean–field induction equation

$$\eta_{\mathrm{m}}\boldsymbol{\nabla}^2\overline{\boldsymbol{B}} + \alpha\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} = \partial_t\overline{\boldsymbol{B}} \qquad \eta_{\mathrm{m}} = \eta + \beta$$

mean–field diffusivity

α changes sign but β remains unchanged under reflexion of $\boldsymbol{u}$

$\Rightarrow$ $\alpha = 0$ for mirror–symmetric turbulence

- A simple example [2]

Concerning the behavior of α and β under reflexion of $\boldsymbol{u}$

Relation $\quad \boldsymbol{\mathcal{E}} = \alpha \overline{\boldsymbol{B}} - \beta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} \qquad (1)$

is a consequence of $\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})') - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = 0\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0\,. \qquad (2)$

Consider (1) at $\boldsymbol{x} = \boldsymbol{0}$ and write

$$\langle \boldsymbol{u}(\boldsymbol{x}, t) \times \boldsymbol{b}(\boldsymbol{x}, t) \rangle (\boldsymbol{0}, t) = \alpha[\boldsymbol{u}] \overline{\boldsymbol{B}}(\boldsymbol{0}, t) - \beta[\boldsymbol{u}] (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})(\boldsymbol{0}, t)\,. \qquad (3)$$

If (2) is satisfied with a set $(\boldsymbol{u}, \boldsymbol{b}, \overline{\boldsymbol{B}})$,
it is, too, with $(\boldsymbol{u}^{\mathsf{refl}}, \boldsymbol{b}^{\mathsf{refl}}, \overline{\boldsymbol{B}}^{\mathsf{refl}})$, where $\boldsymbol{u}^{\mathsf{refl}}(\boldsymbol{x}, t) = -\boldsymbol{u}(-\boldsymbol{x}, t)$ etc.

Therefore, in addition to (3), we have

$$\langle \boldsymbol{u}(-\boldsymbol{x}, t) \times \boldsymbol{b}(-\boldsymbol{x}, t) \rangle (\boldsymbol{0}, t) = -\alpha[\boldsymbol{u}^{\mathsf{refl}}] \overline{\boldsymbol{B}}(\boldsymbol{0}, t) - \beta[\boldsymbol{u}^{\mathsf{refl}}] (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})(\boldsymbol{0}, t)\,. \qquad (4)$$

Considering that $\langle \boldsymbol{u}(-\boldsymbol{x}, t) \times \boldsymbol{b}(-\boldsymbol{x}, t) \rangle (\boldsymbol{0}, t) = \langle \boldsymbol{u}(\boldsymbol{x}, t) \times \boldsymbol{b}(\boldsymbol{x}, t) \rangle (\boldsymbol{0}, t)$
the comparison of (3) and (4) delivers us

$$\alpha[\boldsymbol{u}^{\mathsf{refl}}] = -\alpha[\boldsymbol{u}]\,, \qquad \beta[\boldsymbol{u}^{\mathsf{refl}}] = \beta[\boldsymbol{u}]\,.$$

- A simple example [3]

◦ Approximations for the coefficients α and β

Second–order correlation approximation (SOCA)

= First–order smoothing approximation (FOSA)

$$\partial_t \boldsymbol{b} - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0 \qquad \Leftarrow \qquad (\boldsymbol{u} \times \boldsymbol{b})' \text{ canceled !}$$

sufficient condition: $|\boldsymbol{b}|/|\overline{\boldsymbol{B}}| \ll 1$

no growing solutions if $\overline{\boldsymbol{B}} = 0$

Assume infinitely extended homogeneous fluid

$$\boldsymbol{b}(\boldsymbol{x}, t) = \int_\infty G^{(\eta)}(\boldsymbol{x} - \boldsymbol{x}', t - t_0) \boldsymbol{b}(\boldsymbol{x}, t_0) \mathrm{d}^3 x' + \int_{t_0}^{t} \int_\infty G^{(\eta)}(\boldsymbol{x} - \boldsymbol{x}', t - t') \boldsymbol{\nabla}' \times (\boldsymbol{u}(\boldsymbol{x}', t') \times \overline{\boldsymbol{B}}(\boldsymbol{x}', t')) \mathrm{d}^3 x' \mathrm{d}t$$

$$\Rightarrow \quad \mathcal{E}_i = \int_0^\infty \int_\infty K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{B}_j(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \mathrm{d}^3 \xi \, \mathrm{d}\tau$$

$$K_{ij}(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = -\frac{1}{\xi} \frac{\partial G^{(\eta)}(\xi, \tau)}{\partial \xi} (\epsilon_{ijl} \xi_k + \epsilon_{jkl} \xi_i) \langle u_k(\boldsymbol{x}, t) \, u_l(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau) \rangle$$

(This result applies independent of homogeneity and isotropy of turbulence)

- A simple example [4] ○ Approximations for α and β [2]

Accept SOCA so that

$$\partial_t \boldsymbol{b} - \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \,. \qquad (*)$$

Let u_c, λ_c and τ_c be
characteristic magnitude, lengths and time scale of $\boldsymbol{u}$.

Define parameter $q = \lambda_c^2 / \eta \tau_c$ so that $|\partial_t \boldsymbol{b}| / |\eta \boldsymbol{\nabla}^2 \boldsymbol{b}| \approx q$.

Define Strouhal number $St = u_c \tau_c / \lambda_c$
and magnetic Reynolds number $Rm = u_c \lambda_c / \eta$.
If $St = 1$ then $q = Rm$.

Case $q \gg 1$, often labeled "high–conductivity limit"

$$(*) \quad \Rightarrow \quad \partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

Case $q \ll 1$, often labeled "low–conductivity limit"

$$(*) \quad \Rightarrow \quad \eta \boldsymbol{\nabla}^2 \boldsymbol{b} = - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}})$$

- A simple example [5] ○ Approximations for α and β [3]

Case $q \gg 1$ – "high–conductivity limit"

$$\partial_t \boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \, , \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

Sufficient (not necessary) condition of validity: $St \ll 1$.

No restriction concerning Rm !!!

Ignore, for simplicity, variation of $\overline{\boldsymbol{B}}$ with t.

$$\boldsymbol{b}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{x}, t_0) + \boldsymbol{\nabla} \times \Big(\int_0^{t-t_0} \boldsymbol{u}(\boldsymbol{x}, t-\tau) \, \mathrm{d}\tau \times \overline{\boldsymbol{B}}(\boldsymbol{x}) \Big)$$

Assume $t - t_0$ sufficiently large so that $\langle \boldsymbol{u}(\boldsymbol{x}, t) \times \boldsymbol{b}(\boldsymbol{x}, t_0) \rangle$ vanishes, let $t \to -\infty$.

$\Rightarrow \quad \mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \partial \overline{B}_j / \partial x_k \qquad \leftarrow \quad$ No other derivatives of $\overline{\boldsymbol{B}}$!

$$a_{ij} = (\epsilon_{ikl} \delta_{jm} + \epsilon_{ijk} \delta_{ml}) \int_0^\infty \langle u_k(\boldsymbol{x}, t) \partial u_l(\boldsymbol{x}, t-\tau) / \partial x_m \rangle \, \mathrm{d}\tau$$

$$b_{ijk} = \epsilon_{ijl} \int_0^\infty \langle u_l(\boldsymbol{x}, t) u_k(\boldsymbol{x}, t-\tau) \rangle \, \mathrm{d}\tau$$

(These results apply independent of homogeneity and isotropy of turbulence)

Homogeneous isotropic turbulence: $\langle \cdots \rangle$ independent of $\boldsymbol{x}$,

$a_{ij} = \alpha\delta_{ij}$ and $b_{ijk} = \beta\epsilon_{ijk}$.

$$\Rightarrow \quad \alpha = \tfrac{1}{3} a_{ii}\,, \qquad \beta = \tfrac{1}{6}\epsilon_{ijk} b_{ijk}$$

$$\Rightarrow \quad \alpha = -\tfrac{1}{3}\int_0^\infty \langle \boldsymbol{u}(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{x},t-\tau))\rangle\,\mathrm{d}\tau$$

$$= -\tfrac{1}{3}\langle \boldsymbol{u}(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{x},t))\rangle\,\tau_{\mathrm{c}}^{(\alpha)}$$

$\langle \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle$ kinetic helicity

$$\beta = \tfrac{1}{3}\int_0^\infty \langle \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t-\tau)\rangle\,\mathrm{d}\tau$$

$$= \tfrac{1}{3}\langle \boldsymbol{u}^2(\boldsymbol{x},t)\rangle\,\tau_{\mathrm{c}}^{(\beta)}$$

Remember: sufficient condition for applicability is $St \ll 1$!

...

$$\beta = \tfrac{1}{3}\langle \boldsymbol{u}^2 \rangle \, \tau_{\mathrm{c}}^{(\beta)}$$

Recall: mean-field conductivity $\quad \sigma_{\mathrm{m}} = \dfrac{\sigma}{1 + \beta/\eta}$

Example: solar convection zone

Assume $\quad \sigma = 3 \cdot 10^3 \, \mathrm{S/m} \quad \rightarrow \quad \eta = 2.7 \cdot 10^2 \, \mathrm{m^2/s}$

$$\langle \boldsymbol{u}^2 \rangle^{1/2} = u_{\mathrm{c}} = 2 \cdot 10^2 \, \mathrm{m/s}$$

$$\lambda_{\mathrm{c}} = 5 \cdot 10^5 \, \mathrm{m} \,, \quad \tau_{\mathrm{c}}^{(\beta)} = \tau_{\mathrm{c}} = 3 \cdot 10^2 \, \mathrm{s}$$

$$\Rightarrow \quad q \approx 3 \cdot 10^6 \,, \quad St \approx 10^{-1} \quad (\rightarrow \; Rm = 3 \cdot 10^5 \,)$$

$$\beta/\eta \approx 1.5 \cdot 10^4 \,, \quad \sigma_{\mathrm{m}}/\sigma \approx \eta/\beta \approx 7 \cdot 10^{-5}$$

- A simple example [8]   ○ Approximations for α and β [6]

Case $q \ll 1$ – "low–conductivity limit"

$$\eta \boldsymbol{\nabla}^2 \boldsymbol{b} = -\boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}), \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0$$

Sufficient (not necessary) condition of validity: $Rm \ll 1$

$$\boldsymbol{\nabla} \times \boldsymbol{b} = \frac{1}{\eta} \boldsymbol{u} \times \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \cdots$$

Remember "Biot–Savart trick"

$$\boldsymbol{b}(\boldsymbol{x}, t) = \frac{1}{4\pi\eta} \int_\infty \boldsymbol{\xi} \times (\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\xi}, t) \times \overline{\boldsymbol{B}}(\boldsymbol{x} + \boldsymbol{\xi}, t)) \frac{\mathrm{d}^3 \xi}{\xi^3}$$

$$\mathcal{E}_i = \frac{\epsilon_{ilm} \delta_{jn} + \epsilon_{ijl} \delta_{mn}}{4\pi\eta} \int_\infty \langle u_l(\boldsymbol{x}, t) u_m(\boldsymbol{x} + \boldsymbol{\xi}, t) \rangle \, \xi_n \overline{B}_j(\boldsymbol{x} + \boldsymbol{\xi}, t) \frac{\mathrm{d}^3 \xi}{\xi}$$

$$\mathcal{E}_i = \frac{\epsilon_{ilm}\delta_{jn} + \epsilon_{ijl}\delta_{mn}}{4\pi\eta} \int_\infty \langle u_l(\boldsymbol{x},t) u_m(\boldsymbol{x}+\boldsymbol{\xi},t)\rangle \, \xi_n \overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t) \frac{\mathrm{d}^3\xi}{\xi}$$

Expansion

$$\overline{B}_j(\boldsymbol{x}+\boldsymbol{\xi},t) = \overline{B}_j(\boldsymbol{x},t) + \xi_k \frac{\partial \overline{B}_j(\boldsymbol{x},t)}{\partial x_k}$$

Higher derivatives of $\overline{\boldsymbol{B}}$ neglected!

$$\Rightarrow \qquad \mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$$

$$a_{ij} = \frac{\epsilon_{ilm}\delta_{jn} + \epsilon_{ijl}\delta_{mn}}{4\pi\eta} \int_\infty \langle u_l(\boldsymbol{x},t) u_m(\boldsymbol{x}+\boldsymbol{\xi},t)\rangle \, \xi_n \frac{\mathrm{d}^3\xi}{\xi^3}$$

$$b_{ijk} = \frac{\epsilon_{ilm}\delta_{jn} + \epsilon_{ijl}\delta_{mn}}{4\pi\eta} \int_\infty \langle u_l(\boldsymbol{x},t) u_m(\boldsymbol{x}+\boldsymbol{\xi},t)\rangle \, \xi_n \xi_k \frac{\mathrm{d}^3\xi}{\xi^3}$$

(These results apply independent of homogeneity and isotropy of turbulence)

Homogeneous isotropic turbulence: $\langle \cdots \rangle$ independent of $\boldsymbol{x}$,

$$\alpha = \tfrac{1}{3} a_{ii} \text{ and } \beta = \tfrac{1}{6} \epsilon_{ijk} b_{ijk}$$

$$\Rightarrow \quad \alpha = -\frac{1}{12\pi\eta} \int_\infty \langle \boldsymbol{u}(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t)) \rangle \frac{\mathrm{d}^3\xi}{\xi}$$

$$= -\frac{1}{12\pi\eta} \int_\infty \langle \boldsymbol{u}(\boldsymbol{x},t) \cdot (\boldsymbol{\xi} \times \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t)) \rangle \frac{\mathrm{d}^3\xi}{\xi^3}$$

$$\beta = \frac{1}{36\pi\eta} \int_\infty \langle \boldsymbol{u}(\boldsymbol{x},t) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t) \rangle \frac{\mathrm{d}^3\xi}{\xi^3}$$

Another (equivalent) representation of α and β

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi} + \boldsymbol{\nabla}\phi\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0\,, \quad \langle \boldsymbol{\psi} \rangle = \boldsymbol{0}\,, \quad \langle \phi \rangle = 0$$

$$\Rightarrow \alpha = -\frac{1}{3\eta} \langle \boldsymbol{u} \cdot \boldsymbol{\psi} \rangle = -\frac{1}{3\eta} \langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle \qquad \leftarrow \text{independent of } \phi$$

$$\beta = \frac{1}{3\eta} (\langle \psi^2 \rangle - \langle \phi^2 \rangle) \qquad > 0 \text{ at least in imcompressible case}$$

Remember: sufficient condition for applicability is $Rm \ll 1$

$$\boldsymbol{\nabla}^2\psi = -\boldsymbol{\nabla}\times\boldsymbol{u}$$

$$\psi(\boldsymbol{x},t) = \frac{1}{4\pi}\int_\infty \frac{\boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{\xi},t)}{|\boldsymbol{\xi}-\boldsymbol{x}|}\mathrm{d}^3\xi$$

$$= \frac{1}{4\pi}\int_\infty \boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t)\frac{\mathrm{d}^3\xi}{\xi}$$

- A simple example [11] ◦ Approximations for α and β [9]

⋄ Arbitrary q

$$\alpha = -\tfrac{1}{3}\int_0^\infty \int_\infty G(\xi,\tau)\langle \boldsymbol{u}(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t-\tau))\rangle\,\mathrm{d}^3\xi\,\mathrm{d}\tau$$

$$= -\tfrac{1}{3}\int_0^\infty \int_\infty \frac{\partial G(\xi,\tau)}{\partial \xi}\langle \boldsymbol{u}(\boldsymbol{x},t)\times\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\rangle\cdot\frac{\boldsymbol{\xi}}{\xi}\,\mathrm{d}^3\xi\,\mathrm{d}\tau$$

$$\beta = -\tfrac{1}{9}\int_0^\infty \int_\infty \frac{\partial G(\xi,\tau)}{\partial \xi}\langle \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t-\tau)\rangle\,\xi\,\mathrm{d}^3\xi\,\mathrm{d}\tau$$

Statements like "α is the kinetic helicity" are very questionable!

◇ Derivations in Fourier space

$$F(\boldsymbol{x},t)=\iint \widehat{F}(\boldsymbol{k},\omega)\,\exp(\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t))\,\mathrm{d}^3k\,\mathrm{d}\omega$$

Consider homogeneous statistically steady turbulence,

$$Q_{ij}(\boldsymbol{\xi},\tau)=\langle u_i(\boldsymbol{x},t)\,u_j(\boldsymbol{x}+\boldsymbol{\xi},t+\tau)\rangle\,.$$

Fourier transform $\widehat{Q}_{ij}(\boldsymbol{k},\omega)$ has to satisfy Bochner's theorem,

$$\widehat{Q}_{ij}(\boldsymbol{k},\omega)X_iX_j^*\geq 0 \quad \text{for any (complex) } \boldsymbol{X}(\boldsymbol{k},\omega)\,.$$

- A simple example [13]

◦ Dynamo action of homogeneous isotropic turbulence

Recall $\quad \eta_{\mathrm{m}} \boldsymbol{\nabla}^2 \overline{\boldsymbol{B}} + \alpha \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \partial_t \overline{\boldsymbol{B}} = \mathbf{0}\,, \quad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{B}} = 0 \qquad (*)$

Look for solutions of the form $\quad \overline{\boldsymbol{B}} = \Re\Big(\hat{\boldsymbol{B}} \exp(\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{x} + pt))\Big) \qquad (**)$

$$\Rightarrow \qquad (\eta_{\mathrm{m}} k^2 + p)\hat{\boldsymbol{B}} - \mathrm{i}\alpha \boldsymbol{k} \times \hat{\boldsymbol{B}} = \mathbf{0}\,, \quad \boldsymbol{k} \cdot \hat{\boldsymbol{B}} = 0$$

Chose, in a given Cartesian coordinate system, $\boldsymbol{k} = (0, 0, k)\,,\ k > 0$

$$\Rightarrow \qquad (\eta_{\mathrm{m}} k^2 + p)\hat{B}_x + \mathrm{i}\alpha k \hat{B}_y = 0\,,\ \mathrm{i}\alpha k \hat{B}_x - (\eta_{\mathrm{m}} k^2 + p)\hat{B}_y = 0\,,\ \hat{B}_z = 0$$

Non–trivial solutions $(**)$ of $(*)$ possible if $(\eta_{\mathrm{m}} k^2 + p)^2 - (\alpha k)^2 = 0$,

i.e. $\qquad p = -\eta_{\mathrm{m}} k^2 \pm |\alpha| k$

Non–decaying solutions of $(*)$ possible if

$$|\alpha| \geq \eta_{\mathrm{m}} k$$

- A simple example [14] ○ Dynamo action [1]

○ Fluid sphere with homogeneous isotropic turbulence surrounded by free space

$\eta_{\mathrm{m}} \nabla^2 \overline{\boldsymbol{B}} + \alpha \nabla \times \overline{\boldsymbol{B}} - \partial_t \overline{\boldsymbol{B}} = \mathbf{0}\,, \quad \nabla \cdot \overline{\boldsymbol{B}} = 0$ in $\mathcal{V}$

$\nabla \times \overline{\boldsymbol{B}} = \mathbf{0}\,, \quad \nabla \cdot \overline{\boldsymbol{B}} = 0 \quad (\rightarrow\ \overline{\boldsymbol{B}} = \nabla \Phi\,, \quad \Delta \Phi = 0\ )$ in $\mathcal{V}'$

$[\overline{\boldsymbol{B}}] = \mathbf{0}$ across $\partial \mathcal{V}$

$\overline{\boldsymbol{B}} = \mathrm{O}(r^{-3})$ as $r \to \infty$

Dimensionless parameter $R_\alpha = |\alpha| R / \eta_{\mathrm{m}}$

Ansatz $\overline{\boldsymbol{B}} = \hat{\boldsymbol{B}}(\boldsymbol{x}) \exp(pt) \ \Rightarrow$ eigenvalue problem for $p = p(R_\alpha)$

Non–decaying solutions $(p \geq 0)$ for $R_\alpha \geq 4.49$

Most easily excitable mode of dipole type

Voigtmann 1968

Conflict with Bondi-Gold theorem !

- Other simple examples:

$\boldsymbol{u}$ axisymmetric turbulence, $\overline{\boldsymbol{U}} = \mathbf{0}$

E.g., inhomogeneous turbulence showing intensity gradient ($\nabla\langle\boldsymbol{u}^2\rangle \neq \mathbf{0}$)

homogeneous turbulence subject to Coriolis force (angular velocity $\boldsymbol{\Omega}$)

homogeneous turbulence influenced by a mean magnetic field ($\overline{\boldsymbol{B}}$)

Assume again $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Axis of symmetry defined by unit vector $\boldsymbol{e}$

$$\begin{aligned} a_{ij} &= a_1\,\delta_{ij} + a_2\,\epsilon_{ijl}\,e_l + a_3\,e_i\,e_j \\ b_{ijk} &= b_1\,\epsilon_{ijk} + b_2\,\delta_{ij}\,e_k + b_3\,\delta_{ik}\,e_j + b_4\,\delta_{jk}\,e_i \\ &\quad + b_5\,\epsilon_{ijl}\,e_k\,e_l + b_6\,\epsilon_{ikl}\,e_j\,e_l + b_7\,\epsilon_{jkl}\,e_i\,e_l + b_8\,e_i\,e_j\,e_k\,. \end{aligned}$$

Without loss of generality $b_4 = 0$ and $b_5 = b_6$

$$(\epsilon_{ijl}e_k + \epsilon_{jkl}e_i + \epsilon_{kil}e_j)e_l = \epsilon_{ijk}$$

- Other simple examples [2]

...

Change of notation

$a_1 \longrightarrow -\alpha_1\,,\ a_2 \longrightarrow -\alpha_1\,,\ a_3 \longrightarrow -\alpha_2$

$b_1, b_2, b_3, b_5, b_7, b_8 \ \longrightarrow$ combinations of $\beta_1, \beta_1, \beta_1, \kappa_1, \kappa_2, \kappa_3$

$\Rightarrow$

$$\begin{aligned}\boldsymbol{\mathcal{E}} \;=\; & -\alpha_1\,\overline{\boldsymbol{B}} - \alpha_2\,(\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{e} - \gamma\,\boldsymbol{e}\times\overline{\boldsymbol{B}} \\ & -\beta_1\,\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} - \beta_2\,(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}))\boldsymbol{e} - \delta\,\boldsymbol{e}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \\ & -\kappa_1\,\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})} - \kappa_2\,\boldsymbol{e}\times(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}) - \kappa_3\,(\boldsymbol{e}\cdot(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}))\,\boldsymbol{e}\end{aligned}$$

$$(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}_{ij} = \tfrac{1}{2}(\partial\overline{B}_i/\partial x_j + \partial\overline{B}_j/\partial x_i)$$

- Other simple examples [3]

$\boldsymbol{\mathcal{E}}$ invariant under reflexion of $\boldsymbol{u}$ at planes containing $\boldsymbol{e}$

E.g. $\boldsymbol{e} = (0, 0, 1)$, reflexion at $x = 0$

$$F_x^{\mathsf{refl}}(x, y, z) = -F_x(-x, y, z)$$
$$F_y^{\mathsf{refl}}(x, y, z) = F_y(-x, y, z)$$
$$F_z^{\mathsf{refl}}(x, y, z) = F_x(-x, y, z)$$

$\Rightarrow$ $\alpha_1 = \alpha_2 = \delta = \kappa_1 = \kappa_3 = 0$

$\Rightarrow$
$$\boldsymbol{\mathcal{E}} = -\gamma\, \boldsymbol{e} \times \overline{\boldsymbol{B}} \\ -\beta_1 \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \beta_2 (\boldsymbol{e} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\boldsymbol{e} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathsf{s})})$$

- Other simple examples [4]

$\boldsymbol{\mathcal{E}}$ invariant under reflexion of $\boldsymbol{u}$ at planes perpendicular to $\boldsymbol{e}$

E.g. $\boldsymbol{e} = (0, 0, 1)$, reflexion at $z = 0$

$$F_x^{\mathrm{refl}}(x, y, z) = F_x(x, y, -z)$$
$$F_y^{\mathrm{refl}}(x, y, z) = F_y(x, y, -z)$$
$$F_z^{\mathrm{refl}}(x, y, z) = -F_x(x, y, -z)$$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \gamma = 0$

$$\Rightarrow \quad \boldsymbol{\mathcal{E}} \;=\; -\beta_1\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \beta_2\, (\boldsymbol{e} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\boldsymbol{e} - \delta\, \boldsymbol{e} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$$
$$-\kappa_1\, \boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}) - \kappa_3\, (\boldsymbol{e} \cdot (\boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}))\, \boldsymbol{e})$$

- Other simple examples [5]

◦ Inhomogeneous turbulence

$$\begin{aligned}\boldsymbol{\mathcal{E}} \;&=\; -\gamma\, \boldsymbol{e} \times \overline{\boldsymbol{B}} \\ &\quad -\beta_1\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \beta_2\, (\boldsymbol{e} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\boldsymbol{e} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})})\end{aligned}$$

$$\Rightarrow\; \overline{\boldsymbol{J}} = \boldsymbol{\sigma}_{\mathrm{m}} \cdot \Big(\overline{\boldsymbol{E}} - \gamma\, \boldsymbol{e} \times \overline{\boldsymbol{B}} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})})\Big)$$

$$\sigma_{\mathrm{m}\,ij} = \sigma\,((1 + \mu\sigma\beta_1)\delta_{ij} + \mu\sigma\beta_2 e_i e_j)^{-1}$$

anisotropic mean–field conductivity!

$$\begin{aligned}\overline{\boldsymbol{J}}_{\|} \;&=\; \sigma_{\mathrm{m}\|}\overline{\boldsymbol{E}}_{\|} \\ \overline{\boldsymbol{J}}_{\perp} \;&=\; \sigma_{\mathrm{m}\perp}\Big(\overline{\boldsymbol{E}}_{\perp} - \gamma\, \boldsymbol{e} \times \overline{\boldsymbol{B}}_{\perp} \\ &\qquad\qquad -\tfrac{1}{2}\kappa_2\, \boldsymbol{e} \times ((\boldsymbol{e} \cdot \boldsymbol{\nabla}_{\|})\overline{\boldsymbol{B}}_{\perp} + \boldsymbol{\nabla}_{\perp}(\boldsymbol{e} \cdot \overline{\boldsymbol{B}}_{\|}))\Big)\end{aligned}$$

$$\sigma_{\mathrm{m}\,\|} = \sigma/(1 + \mu\sigma(\beta_1 + \beta_2))\,, \quad \sigma_{\mathrm{m}\,\perp} = \sigma/(1 + \mu\sigma\beta_1)$$

- Other simple examples [6] ∘ Inhomogeneous turbulence [2]

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= -\gamma\, \boldsymbol{e} \times \overline{\boldsymbol{B}} \qquad \Leftarrow \text{“}\gamma \text{ effect”} \\ &\quad -\beta_1\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \beta_2\, (\boldsymbol{e} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\boldsymbol{e} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\boldsymbol{\nabla} \overline{\boldsymbol{B}})^{(\mathrm{s})})\end{aligned}$$

- ◇ transport of mean magnetic flux in the absence of mean motion
- ◇ “turbulent diamagnetism” (Rädler 1966)
- ◇ “topological pumping” (Drobyshevski et al. 1980)

- Other simple examples [7] ∘ Inhomogeneous turbulence [3]

$$\gamma = \tfrac{1}{2}\epsilon_{ijk} a_{ij} e_k$$

Accept again SOCA

$$\gamma \boldsymbol{e} \ = \ \frac{1}{2}\int_0^\infty \langle (\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\,\boldsymbol{u}(\boldsymbol{x},t-\tau) + \boldsymbol{u}(\boldsymbol{x},t)(\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x},t-\tau))\rangle\,\mathrm{d}\tau$$

for $q \to \infty$

$$\gamma \boldsymbol{e} \ = \ \frac{1}{8\pi\eta}\int_\infty \langle (\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\,\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t) + \boldsymbol{u}(\boldsymbol{x},t)(\boldsymbol{\nabla}\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t))\rangle\,\frac{\mathrm{d}^3\xi}{\xi}$$

for $q \to 0$

Assume in addition incompressible fluid

$$\gamma \boldsymbol{e} = \frac{1}{2}\boldsymbol{\nabla}\int_0^\infty \langle (\boldsymbol{e}\cdot\boldsymbol{u}(\boldsymbol{x},t))(\boldsymbol{e}\cdot\boldsymbol{u}(\boldsymbol{x},t-\tau))\rangle\,\mathrm{d}\tau \qquad \text{for } q \to \infty$$

$$\gamma \boldsymbol{e} = \frac{1}{8\pi\eta}\boldsymbol{\nabla}\int_\infty \langle (\boldsymbol{e}\cdot\boldsymbol{u}(\boldsymbol{x},t))(\boldsymbol{e}\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t))\rangle\,\frac{\mathrm{d}^3\xi}{\xi} \qquad \text{for } q \to 0$$

- Other simple examples [8]

◦ Homogeneous turbulence subject to Coriolis force

Angular velocity $\boldsymbol{\Omega} = \Omega\, \boldsymbol{e}$

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} \;=\;& -\beta_1\, \nabla \times \overline{\boldsymbol{B}} - \beta_2\, (\boldsymbol{e} \cdot (\nabla \times \overline{\boldsymbol{B}}))\boldsymbol{e} \\
& -\delta\, \boldsymbol{e} \times (\nabla \times \overline{\boldsymbol{B}}) \\
& -\kappa_1\, \boldsymbol{e} \cdot (\nabla \overline{\boldsymbol{B}})^{(\mathrm{s})} - \kappa_2\, \boldsymbol{e} \times (\boldsymbol{e} \cdot (\nabla \overline{\boldsymbol{B}})^{(\mathrm{s})}) \\
& -\kappa_3\, (\boldsymbol{e} \cdot (\boldsymbol{e} \cdot (\nabla \overline{\boldsymbol{B}})^{(\mathrm{s})}))\, \boldsymbol{e}) \\
=\;& -\beta_1\, \nabla \times \overline{\boldsymbol{B}} - \tilde{\beta}_2\, (\boldsymbol{\Omega} \cdot (\nabla \times \overline{\boldsymbol{B}}))\boldsymbol{\Omega} \\
& -\tilde{\delta}\, \boldsymbol{\Omega} \times (\nabla \times \overline{\boldsymbol{B}}) \qquad \Leftarrow \quad \text{“}\boldsymbol{\Omega} \times \overline{\boldsymbol{J}} \text{ effect”} \\
& - \cdots
\end{aligned}
$$

together with differential rotation capable of dynamo action

$$\overline{\boldsymbol{J}} = \boldsymbol{\sigma}_{\mathrm{m}} \cdot (\overline{\boldsymbol{E}} - \mu\tilde{\delta}\, \boldsymbol{\Omega} \times \overline{\boldsymbol{J}} - \cdots)$$

$\boldsymbol{\sigma}_{\mathrm{m}}$ mean–field conductivity tensor

Note that $\boldsymbol{\Omega} \times (\nabla \times \overline{\boldsymbol{B}}) = -(\boldsymbol{\Omega} \cdot \nabla)\overline{\boldsymbol{B}} + \nabla(\boldsymbol{\Omega} \cdot \overline{\boldsymbol{B}})$

- Other simple examples [9] ◦ Coriolis force

$$\delta = \tfrac{1}{4}(b_{jji} - b_{jij})e_i$$

Accept again SOCA

$$\delta = -\tfrac{1}{4}\int_0^\infty \langle \boldsymbol{u}(\boldsymbol{x},t) \times \boldsymbol{u}(\boldsymbol{x},t-\tau)\rangle \cdot \boldsymbol{e}\,\mathrm{d}\tau \qquad \text{for } q \to \infty \qquad (*)$$

$$\delta = \frac{1}{16\pi\eta}\int_\infty \langle \boldsymbol{\xi} \times \boldsymbol{u}(\boldsymbol{x},t)\,(\boldsymbol{\xi}\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t))\rangle \cdot \boldsymbol{e}\,\frac{\mathrm{d}^3\xi}{\xi^3} \qquad \text{for } q \to 0$$

For statistically steady turbulence $(*)$ can be written as

$$\delta = \tfrac{1}{8}\int_0^\infty \langle \boldsymbol{u}(\boldsymbol{x},t) \times (\boldsymbol{u}(\boldsymbol{x},t+\tau) - \boldsymbol{u}(\boldsymbol{x},t-\tau))\rangle \cdot \boldsymbol{e}\,\mathrm{d}\tau \qquad \text{for } q \to \infty$$

In this case only the part of $\langle \boldsymbol{u}(\boldsymbol{x},t) \times \boldsymbol{u}(\boldsymbol{x},t+\tau)\rangle$

which is antisymmetric in τ contributes to δ !

- Other simple examples [10]

◦ Turbulence under the influence of a mean magnetic field

$$\overline{\boldsymbol{B}} = \overline{B}\, \boldsymbol{e}$$

General form of $\boldsymbol{\mathcal{E}}$ plus requirement $\boldsymbol{\mathcal{E}} \to -\boldsymbol{\mathcal{E}}$ as $\overline{\boldsymbol{B}} \to -\overline{\boldsymbol{B}}$ leads to

$$\boldsymbol{\mathcal{E}} = \big(\alpha - \tilde{\alpha}\,(\overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\big)\, \overline{\boldsymbol{B}} - \gamma\big(\tfrac{1}{2}\boldsymbol{\nabla}\overline{\boldsymbol{B}}^2 + (\overline{\boldsymbol{B}} \cdot \boldsymbol{\nabla})\overline{\boldsymbol{B}}\big) \times \overline{\boldsymbol{B}} - \beta\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}\,. \qquad (*)$$

Identity

$$\big(\tfrac{1}{2}\boldsymbol{\nabla}\overline{\boldsymbol{B}}^2 - (\overline{\boldsymbol{B}} \cdot \boldsymbol{\nabla})\overline{\boldsymbol{B}}\big) \times \overline{\boldsymbol{B}} + (\overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\, \overline{\boldsymbol{B}} - \overline{\boldsymbol{B}}^2 \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} = \boldsymbol{0}$$

allows to bring $(*)$ in the simpler form

$$\boldsymbol{\mathcal{E}} = \big(\alpha - \tilde{\alpha}\,(\overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\big)\, \overline{\boldsymbol{B}} - \gamma(\boldsymbol{\nabla}\overline{\boldsymbol{B}}^2) \times \overline{\boldsymbol{B}} - \beta\, \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}\,,$$

$\alpha, \tilde{\alpha}, \ldots$ may depend on $|\overline{\boldsymbol{B}}|$.

- Other simple examples [11] ◦ Mean magnetic field

...

$$\boldsymbol{\mathcal{E}} = \Big(\alpha - \tilde{\alpha}\,(\overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\Big)\,\overline{\boldsymbol{B}} - \gamma(\boldsymbol{\nabla}\overline{\boldsymbol{B}}^2) \times \overline{\boldsymbol{B}} - \beta\,\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}$$

The representation

$$\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ij}(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})_j$$

applies with

$$a_{ij} = \Big(\alpha - \tilde{\alpha}(\overline{\boldsymbol{B}} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}))\Big)\delta_{ij} + \gamma\epsilon_{ijk}\boldsymbol{\nabla}_k\overline{B}^2\,, \quad b_{ij} = -\beta\delta_{ij}$$

and also with

$$a_{ij} = \alpha\delta_{ij} + \gamma\epsilon_{ijk}\boldsymbol{\nabla}_k\overline{B}^2\,, \quad b_{ij} = -\tilde{\alpha}\overline{B}_i\overline{B}_j - \beta\delta_{ij}\,.$$

- More general cases

Assume $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

$$a_{ij} = -\alpha_{ij} + \epsilon_{ijk}\gamma_k\,, \qquad \alpha_{ij} = \alpha_{ji}$$

$$\frac{\partial\overline{B}_j}{\partial x_k} = (\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}_{jk} - \tfrac{1}{2}\epsilon_{jkl}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_l$$

$$b_{ijk}\frac{\partial\overline{B}_j}{\partial x_k} = -b_{ij}(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_j - \kappa_{ijk}(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}_{jk}$$

$$b_{ij} = \beta_{ij} + \epsilon_{ijk}\delta_k\,, \qquad \kappa_{ijk} = \kappa_{ikj}$$

$\Rightarrow$

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} - \boldsymbol{\gamma}\times\overline{\boldsymbol{B}} - \boldsymbol{\beta}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \boldsymbol{\delta}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \boldsymbol{\kappa}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}$$

$$\overline{\boldsymbol{J}} = \boldsymbol{\sigma}_{\mathrm{m}}\cdot(\overline{\boldsymbol{E}} + (\overline{\boldsymbol{U}} - \gamma)\times\overline{\boldsymbol{B}} - \boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} - \boldsymbol{\delta}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - \boldsymbol{\kappa}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})})$$

$$\sigma_{\mathrm{m\,ij}} = \sigma\,(\delta_{ij} + \mu\sigma\beta_{ij})^{-1}$$

- More general cases [2]

...

$$\mathcal{E} = -\boldsymbol{\alpha} \cdot \overline{\boldsymbol{B}}$$ anisotropic α effect

$$-\boldsymbol{\gamma} \times \overline{\boldsymbol{B}}$$ γ effect, pumping

$$-\boldsymbol{\beta} \cdot (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$$ anisotropic magnetic diffusivity

$$-\boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$$ $\boldsymbol{\delta} \times \overline{\boldsymbol{J}}$ effect

$$-\boldsymbol{\kappa} \cdot (\boldsymbol{\nabla} \overline{\boldsymbol{B}})^{(\mathrm{s})}$$ ...

$$\overline{\boldsymbol{J}} = \boldsymbol{\sigma}_{\mathrm{m}} \cdot \Big(\overline{\boldsymbol{E}}$$ mean-field conductivity

$$+(\overline{\boldsymbol{U}} - \boldsymbol{\gamma}) \times \overline{\boldsymbol{B}}$$ "effective velocity"

$$-\boldsymbol{\alpha} \cdot \overline{\boldsymbol{B}}$$ anisotropic α effect

$$-\boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\boldsymbol{B}})$$ $\boldsymbol{\delta} \times \overline{\boldsymbol{J}}$ effect

$$-\boldsymbol{\kappa} \cdot (\boldsymbol{\nabla} \overline{\boldsymbol{B}})^{(\mathrm{s})}\Big)$$ ...

- More general cases [3]

. . .

$$
\begin{aligned}
\alpha_{ij} &= -\tfrac{1}{2}(a_{ij} + a_{ji}) \\
\gamma_i &= \tfrac{1}{2}\epsilon_{ijk}a_{jk} \\
\beta_{ij} &= \tfrac{1}{4}(\epsilon_{ikl}b_{jkl} + \epsilon_{jkl}b_{ikl}) \\
\delta_i &= \tfrac{1}{4}(b_{jji} - b_{jij}) \\
\kappa_{ijk} &= -\tfrac{1}{2}(b_{ijk} + b_{ikj})
\end{aligned}
$$

- Structure of $\boldsymbol{\mathcal{E}}$, from symmetry arguments

Adopt the concept of

scalars — pseudoscalars

polar — axial vectors $(\boldsymbol{E}, \boldsymbol{J}, \boldsymbol{U} - -\boldsymbol{B}, \boldsymbol{H})$

true — pseudo quantities (scalars, vectors, tensors)

$$\mathcal{E}_i = a_{ij}\,\overline{B}_j + b_{ijk}\,\partial\overline{B}_j/\partial x_k$$

$$\boldsymbol{\mathcal{E}} = -\boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} - \boldsymbol{\gamma}\times\overline{\boldsymbol{B}} - \boldsymbol{\beta}\cdot(\nabla\times\overline{\boldsymbol{B}}) - \boldsymbol{\delta}\times(\nabla\times\overline{\boldsymbol{B}}) - \boldsymbol{\kappa}\cdot(\nabla\overline{\boldsymbol{B}})^{(\mathrm{s})}$$

Vectorial and tensorial construction elements for $a_{ij}, b_{ijk}, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\kappa}$:

$$\delta_{ij},\;\; \boldsymbol{g}\;(\text{e.g.} = \nabla\langle \boldsymbol{u}^2\rangle),\;\; \boldsymbol{D} = (\nabla\overline{\boldsymbol{U}})^{(\mathrm{s})},\;\; \cdots$$

$$\epsilon_{ijk},\;\; \boldsymbol{\Omega},\;\; \boldsymbol{W} = \nabla\times\overline{\boldsymbol{U}},\;\; \cdots$$

- Structure of $\mathcal{E}$ [2]

◦ Inhomogeneous turbulence on a rigidly rotating body

Vectorial and tensorial construction elements for $\mathcal{E}$:

isotropic tensors, $\boldsymbol{g}$, $\boldsymbol{\Omega}$

$$\begin{aligned}\alpha_{ij} \;=\; & \underline{\alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega})\delta_{ij}} + \alpha_2(g_i\Omega_j + g_i\Omega_j) \\ & +\alpha_3(\boldsymbol{g}\cdot\boldsymbol{\Omega})g_i g_j + \alpha_4(\boldsymbol{g}\cdot\boldsymbol{\Omega})\Omega_i\Omega_j \\ & +\alpha_5(\epsilon_{ilm}g_j + \epsilon_{jlm}g_i)g_l\Omega_m + \alpha_6(\epsilon_{ilm}\Omega_j + \epsilon_{jlm}\Omega_i)g_l\Omega_m\end{aligned}$$

All coefficients are scalars, no pseudoscalars,

may depend on $\boldsymbol{g}^2$, $\boldsymbol{\Omega}^2$ and $(\boldsymbol{g}\cdot\boldsymbol{\Omega})^2$

Steenbeck, Krause and Rädler 1966 (SOCA, linearity in $\boldsymbol{g}$ and $\boldsymbol{\Omega}$):

$$\alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega}) = \kappa\,\langle\boldsymbol{u}^2\rangle(\lambda_\mathrm{c}^2\tau_\mathrm{c}/\eta)\,\boldsymbol{\Omega}\cdot\boldsymbol{\nabla}\log(\varrho\sqrt{\langle\boldsymbol{u}^2\rangle})\,, \qquad \kappa\approx 1$$

In SOCA trace($\boldsymbol{\alpha}$) (NOT $\alpha_1(\boldsymbol{g}\cdot\boldsymbol{\Omega})$!!!)

determined by $\langle\boldsymbol{u}(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\xi},t+\tau))\rangle$

- Structure of $\mathcal{E}$ [3] ◦ Inhomogeneous turbulence on a rigidly rotating body [2]

. . .

$$\gamma_i = \underline{\gamma_1 g_i} + \gamma_2 (\boldsymbol{g}\cdot\boldsymbol{\Omega})\Omega_i + \gamma_3 \epsilon_{ilm} g_l \Omega_m$$

$$\begin{aligned}\beta_{ij} = {} & \beta_1 \delta_{ij} + \beta_2 (\boldsymbol{g}\cdot\boldsymbol{\Omega})(g_i\Omega_j + g_i\Omega_j) + \beta_3 g_i g_j + \beta_4 \Omega_i \Omega_j \\ & + \beta_5 (\boldsymbol{g}\cdot\boldsymbol{\Omega})(\epsilon_{ilm} g_j + \epsilon_{jlm} g_i) g_l \Omega_m \\ & + \beta_6 (\boldsymbol{g}\cdot\boldsymbol{\Omega})(\epsilon_{ilm} \Omega_j + \epsilon_{jlm} \Omega_i) g_l \Omega_m\end{aligned}$$

$$\delta_i = \delta_1 (\boldsymbol{g}\cdot\boldsymbol{\Omega}) g_i + \underline{\delta_2 \Omega_i} + \delta_3 (\boldsymbol{g}\cdot\boldsymbol{\Omega}) \epsilon_{ilm} g_l \Omega_m$$

All coefficients are scalars, no pseudoscalars, ...

- Structure of $\mathcal{E}$ [4] ∘ Inhomogeneous turbulence on a rigidly rotating body [3]

$$
\begin{aligned}
\kappa_{ijk} \;=\;& \kappa_1(\boldsymbol{g}\cdot\boldsymbol{\Omega})(\delta_{ij}g_k+\delta_{ik}g_j)+\kappa_2(\delta_{ij}\Omega_k+\delta_{ik}\Omega_j)\\
&+\kappa_3(\epsilon_{ijl}g_k+\epsilon_{ikl}g_j)g_l+\kappa_4(\boldsymbol{g}\cdot\boldsymbol{\Omega})(\epsilon_{ijl}g_k+\epsilon_{ikl}g_j)\Omega_l\\
&+\kappa_5(\boldsymbol{g}\cdot\boldsymbol{\Omega})(\epsilon_{ijl}\Omega_k+\epsilon_{ikl}\Omega_j)g_l+\kappa_6(\epsilon_{ijl}\Omega_k+\epsilon_{ikl}\Omega_j)\Omega_l\\
&+\kappa_7(\boldsymbol{g}\cdot\boldsymbol{\Omega})g_ig_jg_k+\kappa_8 g_i(g_j\Omega_k+g_k\Omega_j)\\
&+\kappa_9\Omega_ig_jg_k+\kappa_{10}(\boldsymbol{g}\cdot\boldsymbol{\Omega})g_i\Omega_j\Omega_k\\
&+\kappa_{11}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\Omega_i(g_j\Omega_k+g_k\Omega_j)+\kappa_{12}\Omega_i\Omega_j\Omega_k\\
&+\kappa_{13}(\boldsymbol{g}\cdot\boldsymbol{\Omega})g_i(g_j\epsilon_{klm}+g_k\epsilon_{jlm})g_l\Omega_m\\
&+\kappa_{14}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\epsilon_{ilm}g_jg_kg_l\Omega_m\\
&+\kappa_{15}g_i(\Omega_j\epsilon_{klm}+\Omega_k\epsilon_{jlm})g_l\Omega_m\\
&+\kappa_{16}\Omega_i(g_j\epsilon_{klm}+g_k\epsilon_{jlm})g_l\Omega_m\\
&+\kappa_{17}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\Omega_i(\Omega_j\epsilon_{klm}+\Omega_k\epsilon_{jlm})g_l\Omega_m\\
&+\kappa_{18}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\epsilon_{ilm}\Omega_j\Omega_kg_l\Omega_m
\end{aligned}
$$

All coefficients are scalars, no pseudoscalars, ...

- Structure of $\mathcal{E}$ [5]

◦ Inhomogeneous turbulence on a rotating body
in the presence of another mean motion

Vectorial and tensorial construction elements for $\mathcal{E}$:
isotropic tensors, $\boldsymbol{g}$, $\boldsymbol{\Omega}$, $\boldsymbol{W}$, $\boldsymbol{D}$

For simplicity, however, linearity of $\mathcal{E}$ in $\boldsymbol{g}$, $\boldsymbol{\Omega}$, $\boldsymbol{W}$, $\boldsymbol{D}$

$$\begin{aligned}\alpha_{ij} \;=\;& \underline{\alpha_1^{(\Omega)}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\delta_{ij}} + \alpha_2^{(\Omega)}(g_i\Omega_j + g_j\Omega_i)\\ &+\underline{\alpha_1^{(W)}(\boldsymbol{g}\cdot\boldsymbol{W})\delta_{ij}} + \alpha_2^{(W)}(g_iW_j + g_jW_i)\\ &+\alpha^{(D)}(\epsilon_{ilm}D_{lj} + \epsilon_{jlm}D_{li})g_m\end{aligned}$$

$$\gamma_i \;=\; \underline{\gamma^{(0)}g_i} + \gamma^{(\Omega)}\epsilon_{ilm}g_l\Omega_m + \gamma^{(W)}\epsilon_{ilm}g_lW_m + \gamma^{(D)}g_lD_{il}$$

Coefficients are scalars independent of $\boldsymbol{g}$, $\boldsymbol{\Omega}$, $\boldsymbol{W}$ and $\boldsymbol{D}$

- Structure of $\mathcal{E}$ [6] ∘ Inhomogeneous turbulence on a rotating body [2]

...

$$\beta_{ij} = \beta^{(0)}\delta_{ij} + \underline{\beta^{(D)} D_{ij}}$$

$$\delta_i = \underline{\delta^{(\Omega)}\Omega_i + \delta^{(W)} W_i}$$

$$\begin{aligned} \kappa_{ijk} = \; & \kappa^{(\Omega)}(\delta_{ij}\Omega_k + \delta_{ik}\Omega_j) + \kappa^{(W)}(\delta_{ij}W_k + \delta_{ik}W_j) \\ & + \kappa^{(D)}(\epsilon_{ijl}D_{kl} + \epsilon_{ikl}D_{jl}) \end{aligned}$$

Coefficients are scalars independent of $\boldsymbol{g}$, $\boldsymbol{\Omega}$, $\boldsymbol{W}$ and $\boldsymbol{D}$

- Structure of $\mathcal{E}$ [7] ○ Inhomogeneous turbulence on a rotating body [3]

. . .

$$
\begin{aligned}
\mathcal{E} \;=\; & -\alpha_1^{(\Omega)}(\boldsymbol{g}\cdot\boldsymbol{\Omega})\overline{\boldsymbol{B}} - \alpha_2^{(\Omega)}((\boldsymbol{\Omega}\cdot\overline{\boldsymbol{B}})\boldsymbol{g} + (\boldsymbol{g}\cdot\overline{\boldsymbol{B}})\boldsymbol{\Omega}) \\
& -\alpha_1^{(W)}(\boldsymbol{g}\cdot\boldsymbol{W})\overline{\boldsymbol{B}} - \alpha_2^{(W)}((\boldsymbol{W}\cdot\overline{\boldsymbol{B}})\boldsymbol{g} + (\boldsymbol{g}\cdot\overline{\boldsymbol{B}})\boldsymbol{W}) \\
& -\alpha^{(D)}\widehat{\boldsymbol{\alpha}}(\boldsymbol{g},\boldsymbol{D})\cdot\overline{\boldsymbol{B}} \\
& -(\gamma^{(0)}\boldsymbol{g} + \gamma^{(\Omega)}\boldsymbol{g}\times\boldsymbol{\Omega} + \gamma^{(W)}\boldsymbol{g}\times\boldsymbol{W} + \gamma^{(D)}\boldsymbol{g}\cdot\boldsymbol{D})\times\overline{\boldsymbol{B}} \\
& -\beta^{(0)}\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} - \beta^{(D)}\boldsymbol{D}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \\
& -(\delta^{(\Omega)}\boldsymbol{\Omega} + \delta^{(W)}\boldsymbol{W})\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \\
& -(\kappa^{(\Omega)}\boldsymbol{\Omega} + \kappa^{(W)}\boldsymbol{W})\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(s)} - \kappa^{(D)}\,\widehat{\boldsymbol{\kappa}}(\boldsymbol{D})\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(s)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\alpha}_{ij} &= (\epsilon_{ilm}D_{lj} + \epsilon_{jlm}D_{li})g_m \\
\hat{\kappa}_{ijk} &= \epsilon_{ijl}D_{kl} + \epsilon_{ikl}D_{jl}
\end{aligned}
$$

- Calculating the mean electromotive force

◦ Second–order and higher–order correlation approximation

$$\partial_t \boldsymbol{b} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} - \eta \nabla \times \boldsymbol{b}) = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}} + (\boldsymbol{u} \times \boldsymbol{b})') , \ \nabla \cdot \boldsymbol{b} = 0$$

Second–order correlation approximation: $(\boldsymbol{u} \times \boldsymbol{b})'$ canceled

nth order correlation approximation: $(\boldsymbol{u} \times \boldsymbol{b})'$ expressed by $(n-1)$th order results

Convergency proof by Krause 1968

- Calculating e.m.f. [2]
  - Test–field method

Schrinner et al. 2005 … 2007

Recall that

◇ $\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$ is calculated on the basis of

$$\partial_t \boldsymbol{b} - \boldsymbol{\nabla} \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + \boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle) - \eta \boldsymbol{\nabla} \times \boldsymbol{b}) = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{b} = 0,$$

◇ $\boldsymbol{\mathcal{E}}$ is functional of $\boldsymbol{u}$, $\overline{\boldsymbol{U}}$ and $\overline{\boldsymbol{B}}$, which is linear in $\overline{\boldsymbol{B}}$.

Assume that $\mathcal{E}_i = a_{ij} \overline{B}_j + b_{ijk} \partial \overline{B}_j / \partial x_k \,. \quad (*)$

◇ a_{ij} and b_{ijk} are functionals of $\boldsymbol{u}$ and $\overline{\boldsymbol{U}}$ only, independent of $\overline{\boldsymbol{B}}$.

Specify $\overline{\boldsymbol{B}}$ to be a "test–field" $\overline{\boldsymbol{B}}^{(n)}$,
denote the corresponding $\boldsymbol{b}$ and $\boldsymbol{\mathcal{E}}$ by $\boldsymbol{b}^{(n)}$ and $\boldsymbol{\mathcal{E}}^{(n)}$.
Then $\mathcal{E}_i^{(n)} = a_{ij} \overline{B}_j^{(n)} + b_{ijk} \partial \overline{B}_j^{(n)} / \partial x_k \,. \quad (**)$

Calculate $\boldsymbol{\mathcal{E}}^{(n)}$ for a number of $\overline{\boldsymbol{B}}^{(n)}$ $(n = 1, 2, \ldots)$
and solve the (sufficiently large) system of equations $(**)$ for a_{ij} and b_{ijk}.

• Calculating e.m.f. [3] ∘ Test–field method [2]

Some comments

◇ The test–fields should be linearly independent
and all higher than first-order spatial derivatives should be small

◇ The test–fields need not to satisfy any boundary conditions,
and they need not to be solenoidal

◇ The test–field method works independent on whether $\boldsymbol{u}$ or $\overline{\boldsymbol{U}}$
depend on $\overline{\boldsymbol{B}}$,
is therefore suitable for investigating magnetic quenching

◇ The method described can be extended to more complex ansatzes
for $\boldsymbol{\mathcal{E}}$, e.g. with $\boldsymbol{\mathcal{E}}^{(0)}$ or higher–order derivatives of $\overline{\boldsymbol{B}}$

- Calculating e.m.f. [4]
  - Some results for mean–field coefficients

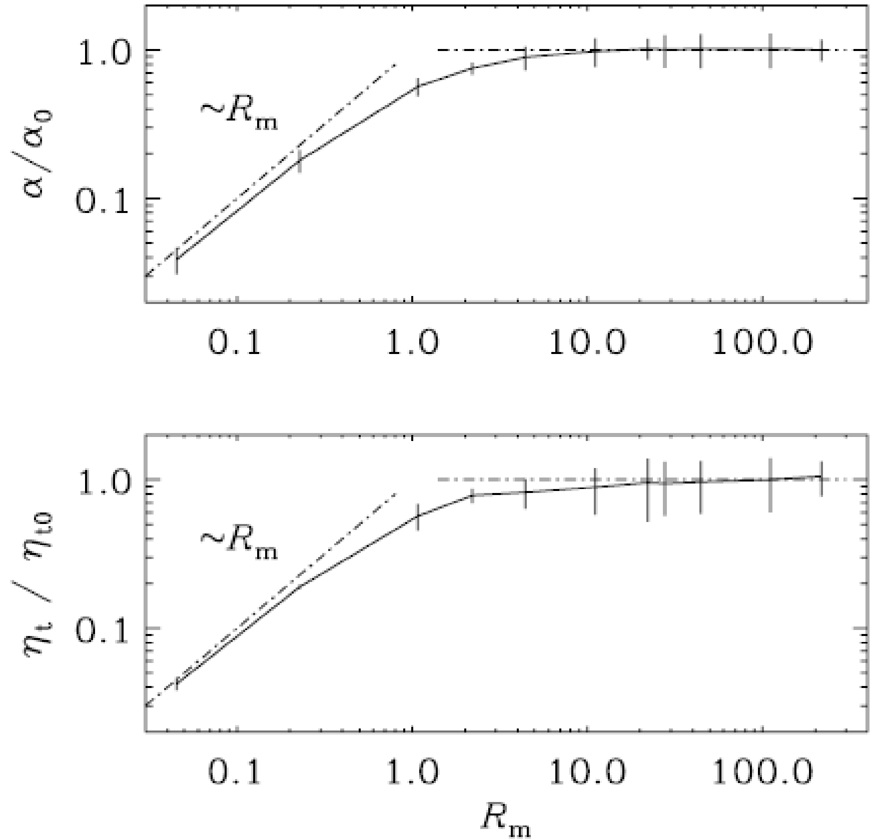


Figure 6. Dependence of the normalized values of α and η_t on R_m for Re = 2. 2. The vertical bars denote twice the error estimated by averaging over subsections of the full time-series (see the text). The run with R_m = 220 (Re = 2.2) was done at a resolution of 512^3 meshpoints.

Sur et al. 2008

- Kinematic mean–field dynamo models

◦ Basic equations

Consider (simply connected) conducting body

surrounded by free space

$$\nabla \times \overline{\boldsymbol{E}} = -\partial_t \overline{\boldsymbol{B}}\,, \ \nabla \times \overline{\boldsymbol{B}} = \mu \overline{\boldsymbol{J}}\,, \ \nabla \cdot \overline{\boldsymbol{B}} = 0 \quad \text{everywhere}$$

$$\overline{\boldsymbol{J}} = \sigma(\overline{\boldsymbol{E}} + \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}) \ \text{ in } \mathcal{V}\,, \quad \overline{\boldsymbol{J}} = \boldsymbol{0} \ \text{ in } \mathcal{V}'$$

$$\overline{\boldsymbol{B}} = \mathrm{O}(a^{-3}) \text{ as } a \to \infty \qquad a \text{ distance from conducting body}$$

or (equivalent) $\cdots$

. . .

$$\nabla \times (\eta \nabla \times \overline{B} - \overline{U} \times \overline{B} - \mathcal{E}) + \partial_t \overline{B} = 0\,, \quad \nabla \cdot \overline{B} = 0 \ \text{ in } \mathcal{V}$$

$$\nabla \times \overline{B} = 0\,, \quad \nabla \cdot \overline{B} = 0 \ \left(\Rightarrow \ \overline{B} = \nabla \Phi\,, \quad \Delta \Phi = 0 \right) \text{ in } \mathcal{V}'$$

$$[\overline{B}] = 0 \ \text{ across } \partial \mathcal{V}$$

$$\overline{B} = \mathrm{O}(a^{-3}) \text{ as } a \to \infty$$

MEAN–FIELD DYNAMO: $\overline{B} \not\longrightarrow 0$ as $t \to \infty$

• Kinematic mean–field dynamo models [3] ◦ Basic equations [3]

◇ Magnetic energy stored in the mean field

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\infty \frac{\overline{\boldsymbol{B}}^2}{2\mu}\mathrm{d}v = -\int_{\partial\mathcal{V}} \frac{\overline{\boldsymbol{J}}^2}{\sigma}\,\mathrm{d}v - \int_{\partial\mathcal{V}} \overline{\boldsymbol{U}}\cdot(\overline{\boldsymbol{J}}\times\overline{\boldsymbol{B}})\,\mathrm{d}v + \int_{\partial\mathcal{V}} \overline{\boldsymbol{J}}\cdot\boldsymbol{\mathcal{E}}\,\mathrm{d}v$$

MEAN–FIELD DYNAMO: $\frac{\mathrm{d}}{\mathrm{d}t}\int_\infty \frac{\overline{\boldsymbol{B}}^2}{2\mu}\mathrm{d}v \geq 0$

◇ Existence of a mean–field dynamo
implies the existence of a dynamo in the original sense.

◇ Mean fields are not subject Cowling's theorem (unless $\boldsymbol{\mathcal{E}}\cdot\overline{\boldsymbol{B}} = 0$).

- Kinematic mean–field dynamo models [4]

◦ "Traditional" assumptions

Shape of the fluid body and distribution of magnetic diffusivity

symmetric about rotation axis

symmetric about equatorial plane

steady

All mean quantities depending on $\boldsymbol{U}(=\overline{\boldsymbol{U}}+\boldsymbol{u})$ are invariant under

rotation of $\boldsymbol{U}$ about rotation axis

reflexion of $\boldsymbol{U}$ about equatorial plane

time shift in $\boldsymbol{U}$

Some consequences

◇ $\overline{\boldsymbol{U}}=\overline{\boldsymbol{U}}^{(\mathrm{mer})}+\overline{\boldsymbol{U}}^{(\mathrm{rot})}$

$\overline{\boldsymbol{U}}^{(\mathrm{mer})}$ and $\overline{\boldsymbol{U}}^{(\mathrm{rot})}$ symmetric about rotation axis and equatorial plane, steady

◇ $\langle\boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle$ symmetric about rotation axis,

antisymmetric about equatorial plane, steady

- Kinematic mean–field dynamo models [5] ◦ "Traditional" assumptions [2

◇ The solutions $\overline{B}$ of the basic equations

are superpositions of modes of the form

$$\overline{B} = \Re\Big(\hat{B}\,\mathrm{exp}(\mathrm{i}m\phi + (\lambda + \mathrm{i}\omega)t)\Big)$$

$\hat{B}$ – symmetric about rotation axis

– antisymmetric or symmetric about equatorial plane

– steady (*Am* or *Sm* modes)

m integer, λ and ω real

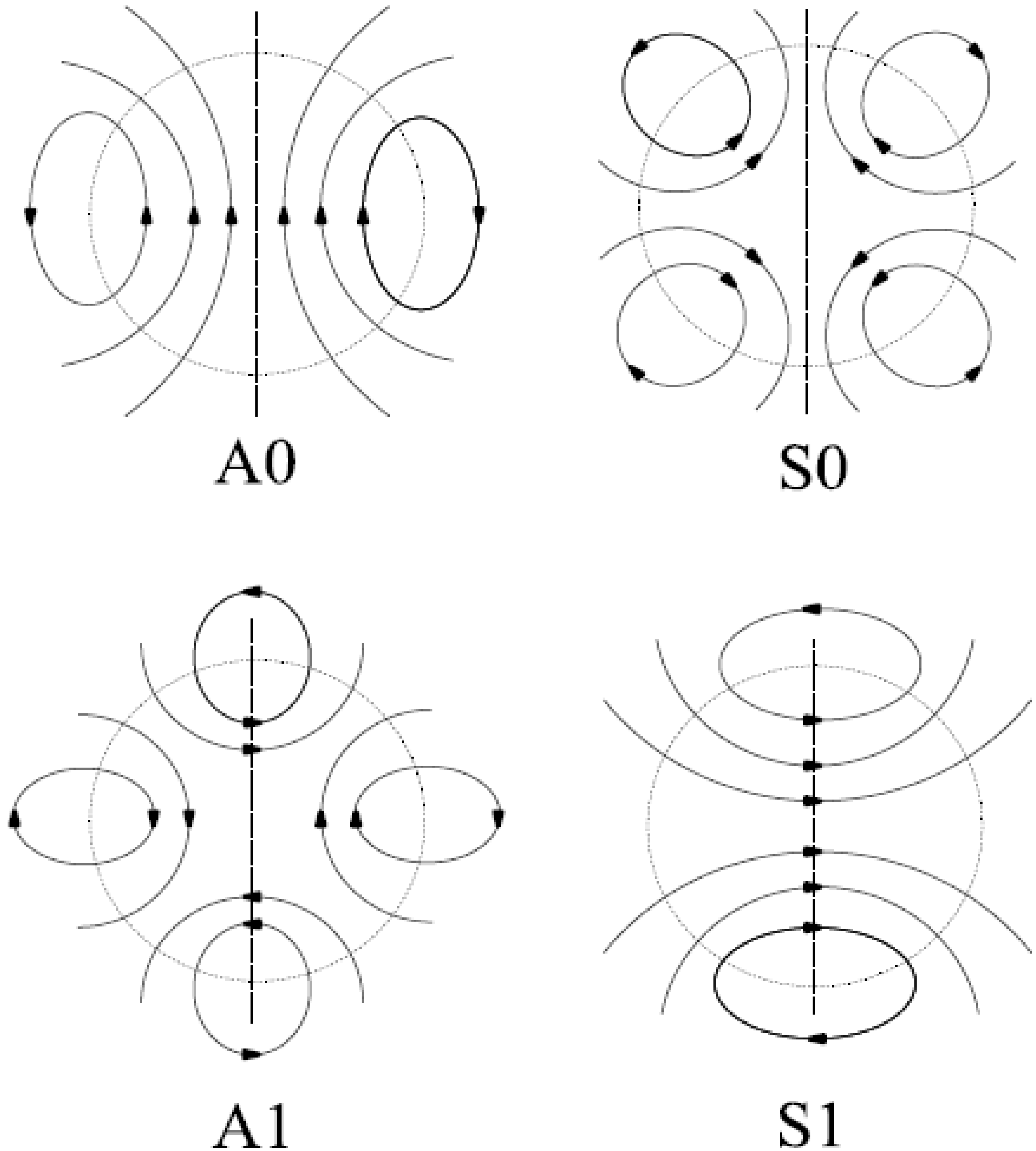
DYNAMO: $\lambda \geq 0$

For $\omega = 0$ monotonic, for $\omega \neq 0$ oscillatory time dependence.

If $\omega \neq 0$ axisymmetric ($m = 0$) modes are intrinsically oscillatory,

non–axisymmetric ($m \neq 0$) modes are waves

traveling in azimuthal direction.


A0
S0
A1
S1

- Kinematic mean–field dynamo models [6]

◦ Basic dynamo mechanisms

In all cases investigated so far

interplay between the poloidal and toroidal parts

of the mean magnetic field

◇ α^2 mechanism

works with α–effect alone

preferably non–oscillatory magnetic fields

axisymmetric and non–axisymmetric fields

Steenbeck & Krause 1969, ...

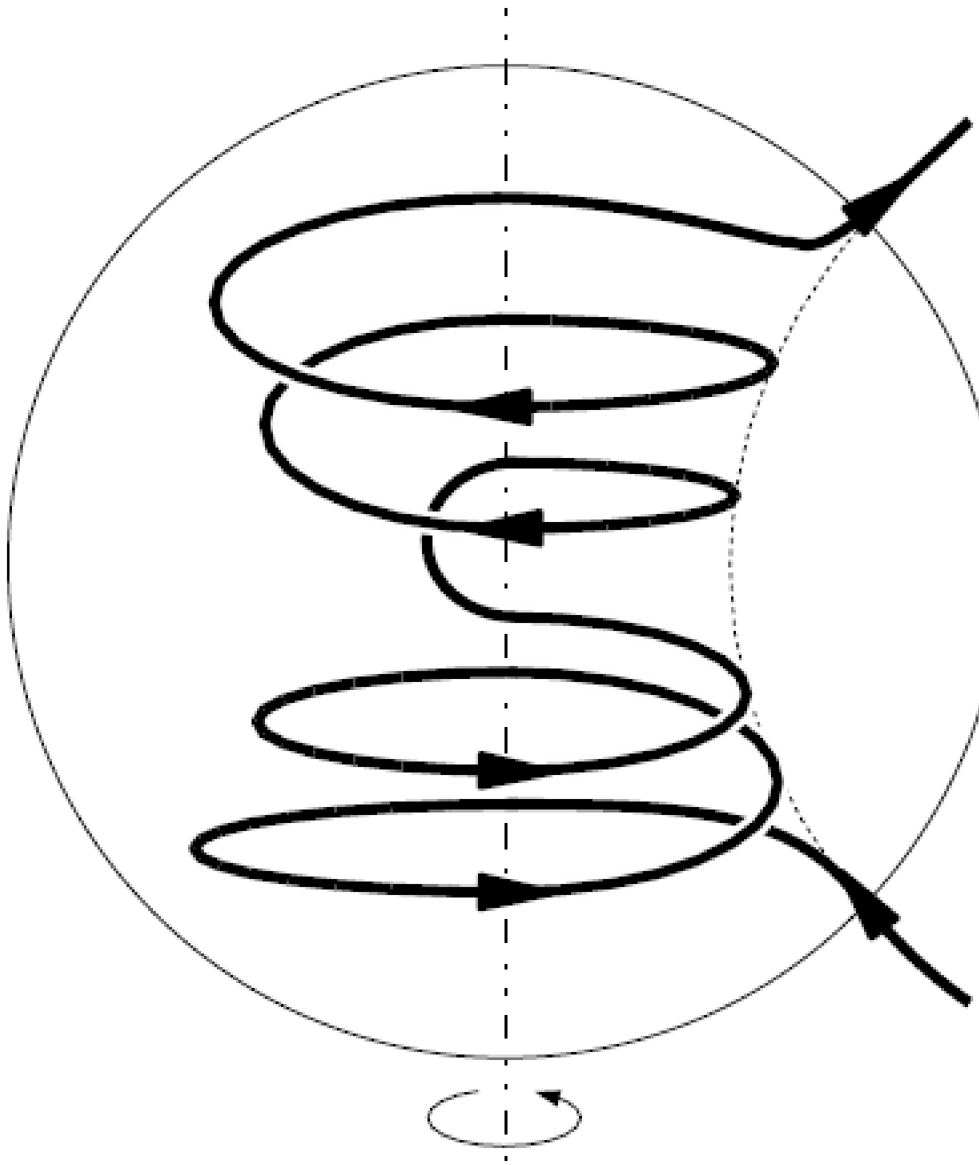
◇ $\alpha\omega$ mechanism

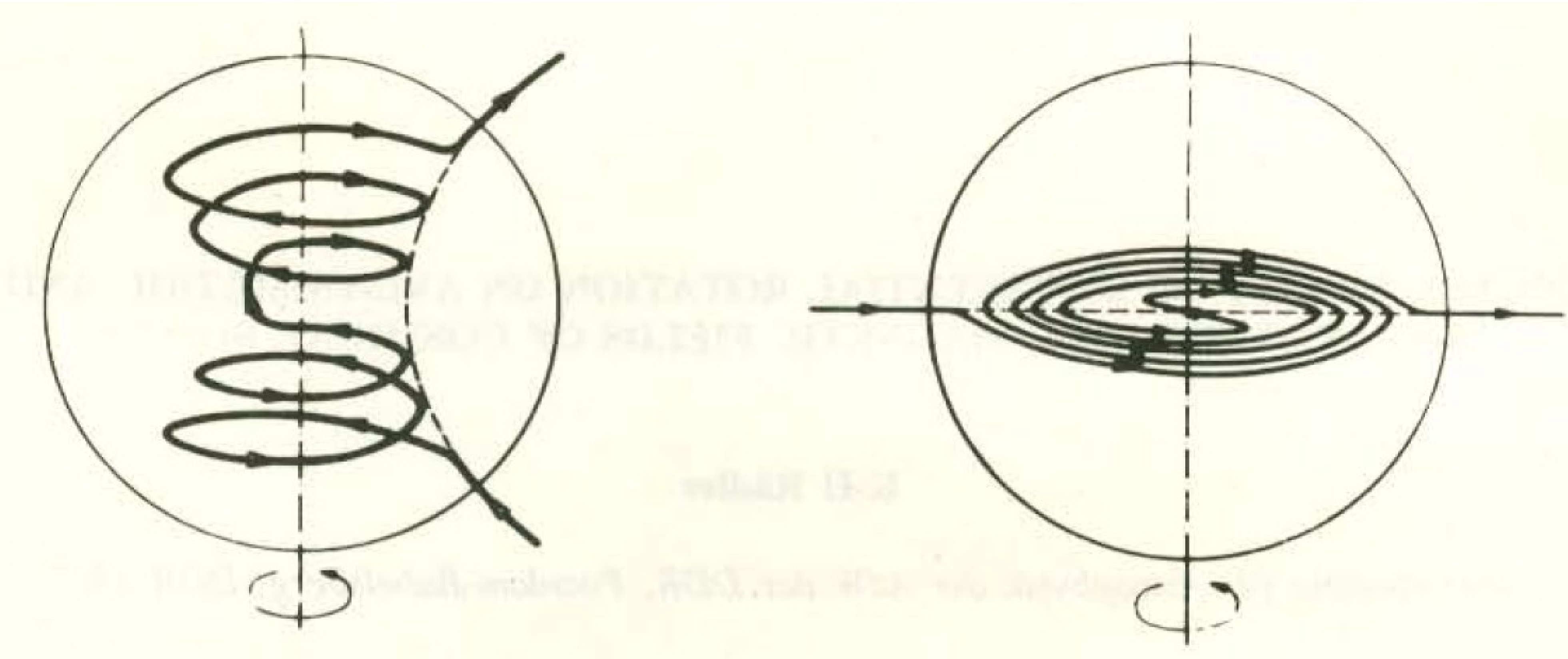
works with α–effect and differential rotation

oscillatory and non–oscillatory magnetic fields

preferably axisymmetric fields

Steenbeck & Krause 1969, ...





◇ $\delta\omega$ mechanism

works, e.g., with $\boldsymbol{\Omega} \times \boldsymbol{J}$–effect (no α–effect!!!) and differential rotation

non–oscillatory magnetic fields

preferably axisymmetric fields

Rädler 1969, ...