MEAN–FIELD DYNAMO THEORY

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OUTLINE

- INTRODUCTION
  Electrodynamics of conducting moving matter
  and the kinematic dynamo problem

- MEAN–FIELD ELECTRODYNAMICS

- MEAN–FIELD MAGNETOFLUIDDYNAMICS

- MEAN-FIELD ASPECTS OF THE KARLSRUHE DYNAMO

MEAN–FIELD MAGNETOFLUIDDYNAMICS

For the sake of simplicity incompressible homogeneous fluid

- Basic equations

Induction equation

$$\partial_t \boldsymbol{B} = -\boldsymbol{\nabla}\Big(\eta\boldsymbol{\nabla} \times \boldsymbol{B} - \boldsymbol{U} \times \boldsymbol{B}\Big), \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Momentum balance

$$\varrho(\partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \boldsymbol{\nabla})\boldsymbol{U}) = -\boldsymbol{\nabla} P + \varrho\nu\boldsymbol{\nabla}^2\boldsymbol{U} - 2\boldsymbol{\Omega} \times \boldsymbol{U} + \frac{1}{\mu}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{F},$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{U} = 0$$

Induction equation

$$\partial_t \overline{\boldsymbol{B}} = -\boldsymbol{\nabla}\left(\eta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}} - \overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \mathcal{E}\right), \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Momentum balance

$$\varrho(\partial_t \overline{\boldsymbol{U}} + (\overline{\boldsymbol{U}} \cdot \boldsymbol{\nabla})\overline{\boldsymbol{U}})$$

$$= -\boldsymbol{\nabla}\overline{P} + \varrho\nu\boldsymbol{\nabla}^2\overline{\boldsymbol{U}} - 2\boldsymbol{\Omega} \times \overline{\boldsymbol{U}} + \frac{1}{\mu}(\boldsymbol{\nabla} \times \overline{\boldsymbol{B}}) \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{F}} + \mathcal{F},$$

$\mathcal{E}$ and $\mathcal{F}$ $\quad\quad \boldsymbol{\nabla} \cdot \overline{\boldsymbol{U}} = 0$

mean electromotive and ponderomotive forces due to fluctuations

$$\mathcal{E} = \overline{\boldsymbol{u} \times \boldsymbol{b}}$$

$$\mathcal{F} = -\varrho\overline{(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}} + \frac{1}{\mu}\overline{(\boldsymbol{\nabla} \times \boldsymbol{b}) \times \boldsymbol{b}}$$

- Mean electromotive force $\boldsymbol{\mathcal{E}}$
  and mean ponderomotive force $\boldsymbol{\mathcal{F}}$

$$\partial_t \boldsymbol{b} - \nabla \times (\overline{\boldsymbol{U}} \times \boldsymbol{b} + (\boldsymbol{u} \times \boldsymbol{b})') - \eta \nabla^2 \boldsymbol{b} = \nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}) \,, \qquad \nabla \cdot \boldsymbol{b} = 0 \qquad (*1)$$

$$(\boldsymbol{u} \times \boldsymbol{b})' = \boldsymbol{u} \times \boldsymbol{b} - \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$$

$$\partial_t \boldsymbol{u} + (\overline{\boldsymbol{U}} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\overline{\boldsymbol{U}} + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u})' = -\frac{1}{\varrho}\nabla p + \nu \nabla^2 \boldsymbol{u} - 2\boldsymbol{\Omega} \times \boldsymbol{u}$$

$$+\frac{1}{\mu\varrho}((\nabla \times \overline{\boldsymbol{B}}) \times \boldsymbol{b} + (\nabla \times \boldsymbol{b}) \times \overline{\boldsymbol{B}} + ((\nabla \times \boldsymbol{b})\boldsymbol{b})') + \boldsymbol{f} \,, \quad \nabla \cdot \boldsymbol{u} = 0 \qquad (*2)$$

$$((\boldsymbol{u} \cdot \nabla)\boldsymbol{u})' = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \langle (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \rangle \qquad p = P - \overline{P}$$

$$((\nabla \times \boldsymbol{b}) \times \boldsymbol{b})' = (\nabla \times \boldsymbol{b}) \times \boldsymbol{b} - \langle (\nabla \times \boldsymbol{b}) \times \boldsymbol{b} \rangle \qquad \boldsymbol{f} = \boldsymbol{F} - \overline{\boldsymbol{F}}$$

$\Rightarrow$ $\boldsymbol{b}$ and $\boldsymbol{u}$ are functionals of $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{U}}$ and $\boldsymbol{f}$

$\Rightarrow$ $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ are functionals of $\boldsymbol{b}$, $\boldsymbol{u}$, $\overline{\boldsymbol{B}}$, $\overline{\boldsymbol{U}}$ and $\boldsymbol{f}$

They are not necessarily linear in $\overline{\boldsymbol{B}}$ or $\overline{\boldsymbol{U}}$.

Contributions independent of $\overline{\boldsymbol{B}}$ or $\overline{\boldsymbol{U}}$ cannot be excluded.

- Mean electromotive force $\boldsymbol{\mathcal{E}}$

Relax the assumption that $\boldsymbol{b}$ decays to zero if $\overline{\boldsymbol{B}} = 0$,
i.e., admit mhd turbulence.

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(\overline{B})}, \quad \boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(\overline{B})}$$

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{(0)} + \boldsymbol{\mathcal{E}}^{(\overline{B})}$$

$$\boldsymbol{\mathcal{E}}^{(0)} = \underline{\langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(0)} \rangle}$$

$$\begin{aligned}
\boldsymbol{\mathcal{E}}^{(\overline{B})} &= \langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(\overline{B})} \rangle + \langle \boldsymbol{u}^{(\overline{B})} \times \boldsymbol{b}^{(0)} \rangle + \langle \boldsymbol{u}^{(\overline{B})} \times \boldsymbol{b}^{(\overline{B})} \rangle \\
&= \underline{\langle \boldsymbol{u} \times \boldsymbol{b}^{(\overline{B})} \rangle} + \underline{\langle \boldsymbol{u}^{(\overline{B})} \times \boldsymbol{b}^{(0)} \rangle} \\
&= \langle \boldsymbol{u}^{(0)} \times \boldsymbol{b}^{(\overline{B})} \rangle + \langle \boldsymbol{u}^{(\overline{B})} \times \boldsymbol{b} \rangle
\end{aligned}$$

- $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ – a simple (academic) example:

$(\boldsymbol{u}, \boldsymbol{b})$ homogeneous isotropic mhd turbulence, $\overline{\boldsymbol{B}}$ small, $\overline{\boldsymbol{U}} = \boldsymbol{0}$

$$\Rightarrow \quad \boldsymbol{\mathcal{E}} = \alpha \overline{\boldsymbol{B}} - \beta \boldsymbol{\nabla} \times \overline{\boldsymbol{B}}$$

Adopt equations $(*1)$ and $(*2)$ for $\boldsymbol{b}$ and $\boldsymbol{u}$ (with $\boldsymbol{\Omega} = \boldsymbol{0}$).

Expand $\boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)} + \cdots$, $\boldsymbol{b} = \boldsymbol{b}^{(0)} + \boldsymbol{b}^{(1)} + \cdots$,

with (1) for "first order in $\overline{\boldsymbol{B}}$".

Introduce SOCA in equation for $\boldsymbol{u}^{(1)}$
and analogous approximation in equation for $\boldsymbol{b}^{(1)}$.

Straightforward calculation delivers $\quad \alpha = \alpha^{(u)} - \alpha^{(b)}$, $\quad \beta = \beta^{(u)}$

$$\alpha^{(u)} = -\frac{1}{3} \int_{\infty} \int_{0}^{\infty} G^{(\eta)}(\xi, \tau) \left\langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}^{(0)}(\boldsymbol{x} + \boldsymbol{\xi})) \right\rangle \mathrm{d}^3\xi \mathrm{d}\tau$$

$$\alpha^{(b)} = -\frac{1}{3\mu\varrho} \int_{\infty} \int_{0}^{\infty} G^{(\nu)}(\xi, \tau) \left\langle \boldsymbol{b}^{(0)}(\boldsymbol{x}, t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}^{(0)}(\boldsymbol{x} + \boldsymbol{\xi})) \right\rangle \mathrm{d}^3\xi \mathrm{d}\tau$$

$$\beta^{(u)} = \frac{1}{3} \int_{\infty} \int_{0}^{\infty} G^{(\eta)}(\xi, \tau) \left\langle \boldsymbol{u}^{(0)}(\boldsymbol{x}, t) \cdot \boldsymbol{b}^{(0)}(\boldsymbol{x} + \boldsymbol{\xi}) \right\rangle \mathrm{d}^3\xi \mathrm{d}\tau$$

- $\boldsymbol{\mathcal{E}}^{(\overline{B})}$ – a simple (academic) example [2]

$$\alpha = \alpha^{(k)} - \alpha^{(m)}\,, \qquad \beta = \beta^{(k)}$$

High–conductivity and low-viscosity limit
$q = \lambda_{\mathrm{c}}^2 / \eta \tau_{\mathrm{c}} \to \infty\,,\ p = \lambda_{\mathrm{c}}^2 / \nu \tau_{\mathrm{c}} \to \infty$

$$\begin{aligned}
\alpha^{(u)} &= -\frac{1}{3}\int_0^\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}^{(0)}(\boldsymbol{x}, t-\tau))\rangle \,\mathrm{d}\tau \\
&\qquad = -\frac{1}{3}\langle \boldsymbol{u}^{(0)} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}^{(0)})\rangle\, \tau^{(\alpha\, u)} \\
\alpha^{(b)} &= -\frac{1}{3\mu\varrho}\int_0^\infty \langle \boldsymbol{b}^{(0)}(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}^{(0)}(\boldsymbol{x}, t-\tau))\mathrm{d}\tau \\
&\qquad = -\frac{1}{3\mu\varrho}\langle \boldsymbol{b}^{(0)} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}^{(0)})\rangle\, \tau^{(\alpha\, b)} \\
\beta^{(u)} &= \frac{1}{3}\int_0^\infty \langle \boldsymbol{u}^{(0)}(\boldsymbol{x},t) \cdot \boldsymbol{u}^{(0)}(\boldsymbol{x}, t-\tau))\rangle \,\mathrm{d}\tau = \frac{1}{3}\langle \boldsymbol{u}^{(0)2}\rangle\, \tau^{(\beta)}
\end{aligned}$$

- $\boldsymbol{\mathcal{E}}^{(0)}$ – an example

Originally homogeneous isotropic mhd turbulence,
influenced by mean motion ($\overline{\boldsymbol{U}}$) and Coriolis force ($\boldsymbol{\Omega}$)

$$\boldsymbol{\mathcal{E}}^{(0)} = c_w \, \boldsymbol{\nabla} \times \overline{\boldsymbol{U}} + c_\Omega \, \boldsymbol{\Omega}$$

Yoshizawa 1990

In second–order correlation approximation

$$c_W = \frac{1}{3} \int_0^\infty \int_\infty \left(G^{(\eta)}(\boldsymbol{\xi}, \tau) + \frac{1}{2} G^{(\nu)}(\boldsymbol{\xi}, \tau) \right) \langle \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{b}(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau) \rangle \, \mathrm{d}^3\xi \mathrm{d}\tau$$

$$c_\Omega = -\frac{2}{3} \int_0^\infty \int_\infty G^{(\nu)}(\boldsymbol{\xi}, \tau) \, \langle \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{b}(\boldsymbol{x} + \boldsymbol{\xi}, t + \tau) \rangle \, \mathrm{d}^3\xi \mathrm{d}\tau$$

$\langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ cross helicty

- A relation for $\partial_t \boldsymbol{\mathcal{E}}$

$$\partial_t \boldsymbol{\mathcal{E}} = \langle \partial_t \boldsymbol{u} \times \boldsymbol{b} \rangle + \langle \boldsymbol{u} \times \partial_t \boldsymbol{b} \rangle$$

For the sake of simplicity $\overline{\boldsymbol{U}} = \boldsymbol{\Omega} = \boldsymbol{0}$

$$\partial_t \boldsymbol{\mathcal{E}} = \boldsymbol{X} + \boldsymbol{Y} + \boldsymbol{Z}$$

$$X_i = \tilde{a}_{ij} \overline{B}_j + \tilde{b}_{ijk} \frac{\partial \overline{B}_j}{\partial x_k}$$

$$\tilde{a}_{ij} = \epsilon_{ilm} \Big(\langle u_l \frac{\partial u_m}{\partial x_j} \rangle - \frac{1}{\mu \varrho} \langle b_l \frac{\partial b_m}{\partial x_j} \rangle \Big) , \qquad \tilde{b}_{ijk} = \cdots$$

$$\boldsymbol{Y} : \nu \langle (\boldsymbol{\nabla}^2 \boldsymbol{u}) \times \boldsymbol{b} \rangle , \; \eta \langle \boldsymbol{u} \times (\boldsymbol{\nabla}^2 \boldsymbol{b}) \rangle , \; \langle \tilde{\boldsymbol{f}} \times \boldsymbol{b} \rangle$$

$\boldsymbol{Z}$: terms of third order in $\boldsymbol{u}$ and $\boldsymbol{b}$

Sometimes ansatz $\boldsymbol{Y} + \boldsymbol{Z} = -\boldsymbol{\mathcal{E}} / \tau_*$ used

- A relation for $\partial_t \boldsymbol{\mathcal{E}}$ [2]

. . .

Sometimes ansatz $\boldsymbol{Y} + \boldsymbol{Z} = -\boldsymbol{\mathcal{E}}/\tau_*$ used

$$\Rightarrow \quad \mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k}$$

with $a_{ij} = \tilde{a}_{ij}\tau_*$ and $b_{ijk} = \tilde{b}_{ijk}\tau_*$

In the isotropic case

$$\alpha = -\tfrac{1}{3}\Big(\langle \boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle - \frac{1}{\mu\varrho}\langle \boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\rangle\Big)\,\tau_*\,, \qquad \beta = \tfrac{1}{3}\langle \boldsymbol{u}^2\rangle\,\tau_*$$