



Simulations of dynamos in compressible flows

Lecture 1: General and numerical considerations

Maarit J. Korpi

`maarit.korpi@helsinki.fi`

Division of Geosciences and Astronomy

Department of Physics

University of Helsinki

Lecture 1: Role of compressibility

- ⑥ Role of compressibility: general remarks
- ⑥ Basic equations and Mach number
- ⑥ Smooth transitions to supersonic speeds
 - △ Expansion to vacuum
 - △ de Laval Nozzle
- ⑥ Shocks
 - △ Formation mechanisms
 - △ Rankine-Hugoniot jump conditions
- ⑥ Resolving shocks with finite-difference methods
 - △ Localised shock-capturing viscosities
 - △ Taming the wiggles
 - △ Examples with the PENCIL CODE

Incompressible fluid dynamics

- ⑥ Density is roughly constant, as of water
- ⑥ Governing equations: conservation of mass and momentum, and induction equation
- ⑥ Energy equation decouples from the above in the case of constant μ
- ⑥ Pressure obtained by solving the pressure Poisson equation

$$\nabla \cdot \mathbf{u} = 0,$$

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} - \nabla p + \mathbf{J} \times \mathbf{B} + \mu \nabla^2 \mathbf{u},$$

$$\nabla^2 p = f(\mu, \mathbf{u})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{J}).$$

Compressible fluid dynamics

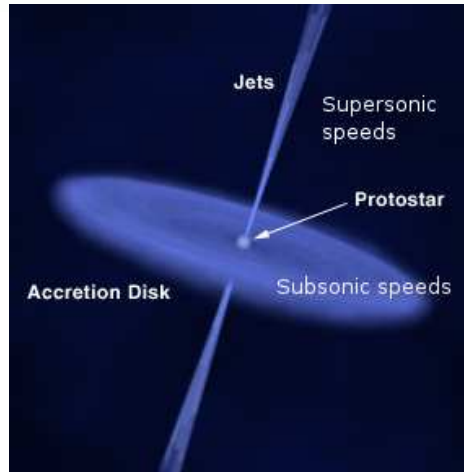
- ⑥ Non-constant densities
- ⑥ Incompressible eqs. supplemented with equation of state and energy conservation equation

$$\begin{aligned}\frac{D\rho}{Dt} &= -\rho \nabla \cdot \mathbf{u}, \\ \rho \frac{D\mathbf{u}}{Dt} &= \rho \mathbf{f} - \nabla p + \mathbf{J} \times \mathbf{B} + \mu \nabla^2 \mathbf{u} + \left(\xi + \frac{1}{3}\mu\right) \nabla \nabla \cdot \mathbf{u}, \\ p &= (\gamma - 1)\rho e, \quad \gamma = c_P/c_V, \\ \rho \frac{De}{Dt} &= -p \nabla \cdot \mathbf{u} + \nabla \cdot k \nabla T + \frac{\mu}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)^2 \\ &\quad + \xi (\nabla \cdot \mathbf{u})^2 + \rho \eta \mu_0 \mathbf{J}^2, \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \mu_0 \mathbf{J}).\end{aligned}$$

Mach number

- ⑥ Dimensionless number defined as $\text{Ma} = \frac{u}{c_s}$
 - △ u is the velocity of the fluid
 - △ Adiabatic sound speed $c_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_s$
- ⑥ $\text{Ma}^2 = \frac{u^2}{c_s^2} \approx \frac{\Delta \rho}{\rho}$
- ⑥ Measures the compressibility of the fluid
 - △ Subsonic flows: $\text{Ma} \ll 1$, incompressible equations can be used
 - △ Transonic flows: $\text{Ma} \approx 1$, compressible effects significant
 - △ Supersonic flows: $\text{Ma} > 1$, shock formation

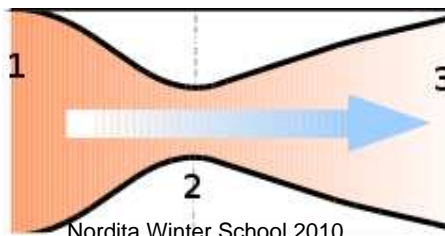
Compressible flows in nozzles



Increasing the cross section S of a pipe decelerates the flow ... for an incompressible fluid. This can be easily seen from the condition for conservation of mass flux

$$\rho u S = \text{cst}, \quad (1)$$

which holds in the steady state $\nabla \cdot (\rho \mathbf{u}) = 0$. What if the flow is compressible, i.e. S and ρ simultaneously change? One such example is De Laval nozzle, which is a hourglass-shaped pipe where density is decreasing in the direction of the flow.



Compressible flows in nozzles (2)

Let us neglect the effect of magnetic fields, external gravitational potential, and assume that the Reynolds number of the flow is large. The compressible Euler equation in component form reads

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho} \nabla p = 0$$

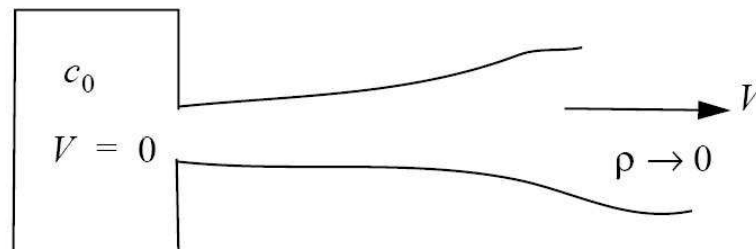
Taking the dot product with u_i , and assuming that the flow is time-independent, one obtains

$$\frac{d}{dt} \left(\frac{1}{2} u^2 \right) + \frac{1}{\rho} \frac{dp}{dt} = 0$$

Writing this in terms of the specific enthalpy $h = \frac{c_s^2}{\gamma - 1}$; $\frac{dh}{dt} = \frac{1}{\rho} \frac{dp}{dt}$, one obtains the compressible Bernoulli equation

$$\frac{1}{2} u^2 + h = cst = \frac{1}{2} u_0^2 + h_0.$$

Let us first consider expansion of gas from a container into a vacuum.

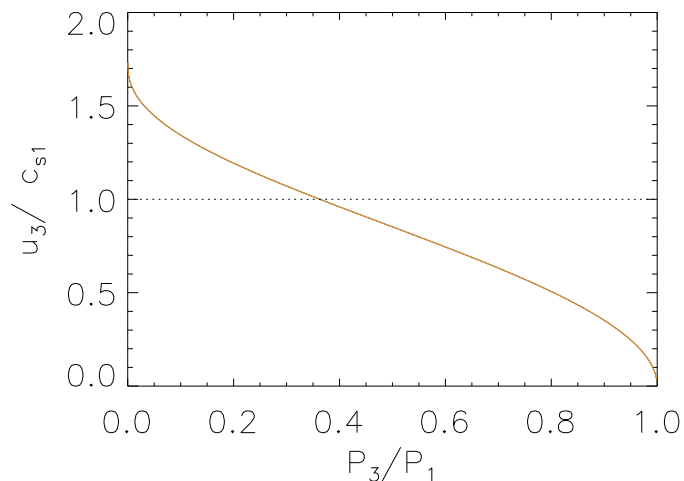


Compressible flows in nozzles (3)

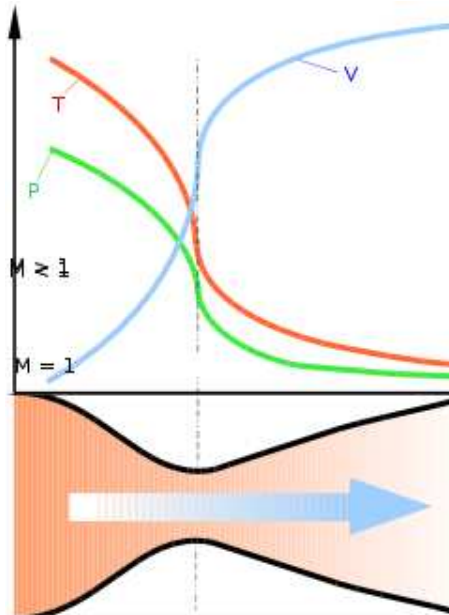
Far away from the container density tends to zero, as does the sound speed. The Bernoulli equation gives therefore $u = \sqrt{\frac{2}{\gamma-1}} c_0$, i.e. the velocity is supersonic with respect to the ambient sound speed.

A similar situation occurs for de Laval nozzle at 2 (minimal cross-section) versus 3 (exit). Between 1 and 2 both the cross-section S and density decrease, leading to acceleration of the flow according to the mass flux conservation. If the flow can be accelerated to transonic speeds ($u_2 \approx c_s$), then the flow at 3 will be supersonic. The flow speed at exit can be obtained using the Bernoulli equation:

$$u_3 = c_{s1} \sqrt{\frac{2}{\gamma-1} \left[1 - \left(\frac{P_3}{P_1} \right)^{\frac{\gamma-1}{\gamma}} \right]}.$$



Compressible flows in nozzles (4)



In De Laval nozzle the transition to supersonic speeds occurs in a smooth manner (no shock formation is involved). Practical applications of De Laval nozzle include the rocket engine, and in astrophysical contexts the formation of supersonic outflows and jets e.g. from accretion disks around protostellar objects. Effects of rotation and magnetic field play a very important role especially in the confinement of the jets.

Shocks



- ⑥ Shocks are discontinuities in the flow properties (density, pressure, temperature, entropy, velocity)
- ⑥ very thin: thickness a few to ten times the mean-free path of the gas
- ⑥ Form in compressible flows with high Mach number
- ⑥ Shocks do not interact with their surroundings, as ordinary waves travel slower than the shock front itself
- ⑥ Formation mechanisms include
 - △ Nonlinear steepening of ordinary waves (ocean waves, solar chromosphere and corona)
 - △ Explosive events (supernova explosions)
 - △ Collisions (magnetospheric bow shock)

Jump conditions (1)

In shocks kinetic energy is converted into thermal energy in such a way that the total energy is conserved. Let us assume that the Reynolds number of the flow is very high, due to which we can use the inviscid approximation (neglect the viscous terms). In one dimension the relevant equations in conservative form read

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) &= 0, \\ \frac{\partial}{\partial t} (\rho u_x) + \frac{\partial}{\partial x} (\rho u_x^2 + p) &= 0, \\ \frac{\partial}{\partial t} \left(\frac{1}{2} \rho u_x^2 + \rho e \right) + \frac{\partial}{\partial x} \left[u_x \left(\frac{1}{2} \rho u_x^2 + \rho e + p \right) \right] &= 0,\end{aligned}$$

In a reference frame comoving with the shock front, $\frac{\partial}{\partial t} = 0$, we get simple relations

$$\begin{aligned}\rho u_x &= \text{cst} = J, \\ \rho u_x^2 + p &= \text{cst} = I, \\ u_x \left(\frac{1}{2} \rho u_x^2 + \rho e + p \right) &= \text{cst} = JE.\end{aligned}$$

Jump conditions (2)

Therefore we can write for upstream (unshocked; 1) and downstream (shocked; 2) of the shock

$$\begin{aligned}\rho_1 u_1 &= \rho_2 u_2 \Leftrightarrow \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2, \\ u_1 \left(\frac{1}{2} \rho_1 u_1^2 + \rho_1 e_1 + p_1 \right) &= u_2 \left(\frac{1}{2} \rho_2 u_2^2 + \rho_2 e_2 + p_2 \right),\end{aligned}$$

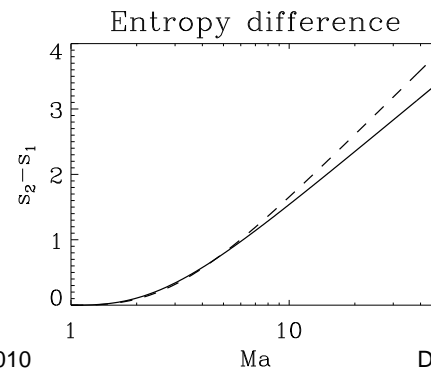
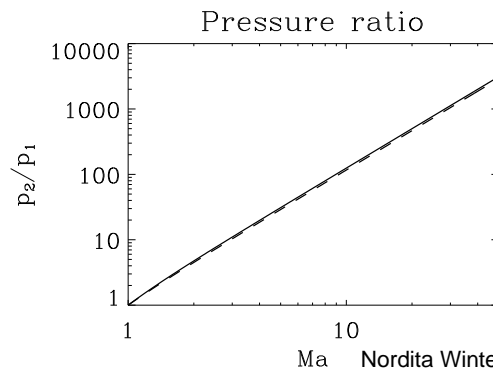
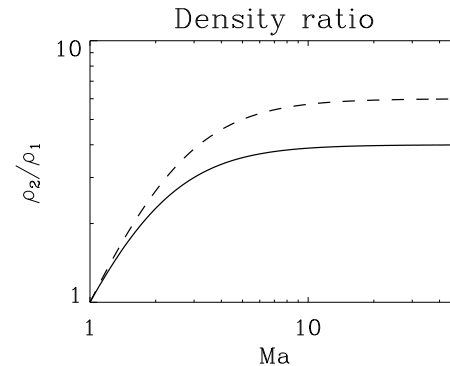
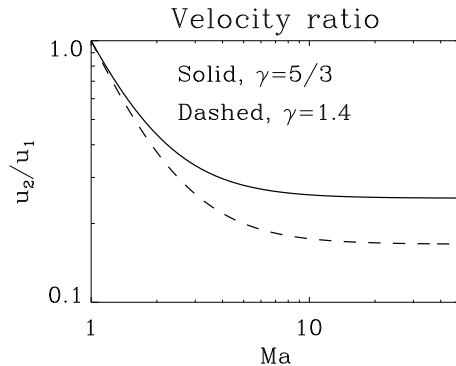
Using the equation of state for an ideal gas $p = \rho e (\gamma - 1)$ to eliminate the internal energy e , we can rather easily obtain the Rankine-Hugoniot jump conditions for inviscid shocks

$$\begin{aligned}\frac{\rho_2}{\rho_1} &= \frac{\frac{p_2}{p_1} (\gamma + 1) + (\gamma - 1)}{(\gamma + 1) + \frac{p_2}{p_1} (\gamma - 1)} = \frac{u_1}{u_2} \\ \frac{p_2}{p_1} &= \frac{\frac{\rho_2}{\rho_1} (\gamma + 1) - (\gamma - 1)}{(\gamma + 1) - \frac{\rho_2}{\rho_1} (\gamma - 1)} \\ \frac{u_2}{u_1} &= \frac{2\gamma}{\gamma + 1} \left(1 + \frac{1}{\gamma Ma^2} \right) - 1\end{aligned}$$

Jump conditions (3)

For infinitely strong shock, $Ma \rightarrow \infty$, and the ratios of the quantities read

$$\frac{u_2}{u_1} = \frac{2\gamma}{\gamma + 1} - 1 = \frac{\gamma - 1}{\gamma + 1}$$
$$\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}$$
$$\frac{p_2}{p_1} \rightarrow \infty$$



Resolving shocks numerically

- ⑥ Many MHD solvers for compressible flows (spectral methods, ZEUS, Nirvana, Pluto, Flash, PENCIL,...)
- ⑥ Huge density and temperature ranges to be covered
 - △ Non-conservative formulations (logarithmic variables) often used
 - △ Often more accurate than conservative formulation
 - △ Conservation properties used to control the accuracy of the solution
- ⑥ To model discontinuities, many schemes need to employ artificial viscosities
 - △ Artificial pressure at regions of compression to force shock to be resolved by the finite grid
 - △ Hyperdiffusive operators/upwinding to tame wiggles

Shock viscosity in FD-schemes (1)

To guarantee numerical stability of the solution, the mesh Reynolds number, defined as

$$Re_{\text{mesh}} = \max |\mathbf{u}| \max (\delta x, \delta y, \delta z) / \nu,$$

where ν is the constant kinematic (shear) viscosity (earlier denoted with μ) of the fluid, should be small enough. The exact value is problem-dependent; for unshocked flows, $Re_{\text{mesh}} \approx 5$, but for shocked flows it can be much less. In practise this means that stabilising the flow with constant kinematic viscosity requires very high viscosity in the shocked region, causing too much diffusion in the calmer parts. A way to go around this problem is to use a localised shock viscosity, effective only in the regions undergoing compression, i.e. $\nabla \cdot \mathbf{u} < 0$, as

$$\xi_{\text{shock}} = c_{\text{shock}} \Delta x^2 |\nabla \cdot \mathbf{u}|,$$

where c_{shock} is a constant of the order of unity. Such a viscosity, called as the von Neumann-Richtmeyer viscosity, can be thought of as an artificial pressure spreading the shock over resolved scales on the grid. This viscosity is then used as the (usually neglected) bulk viscosity ξ .

Shock viscosity in FD-schemes (2)

The stress tensor then reads

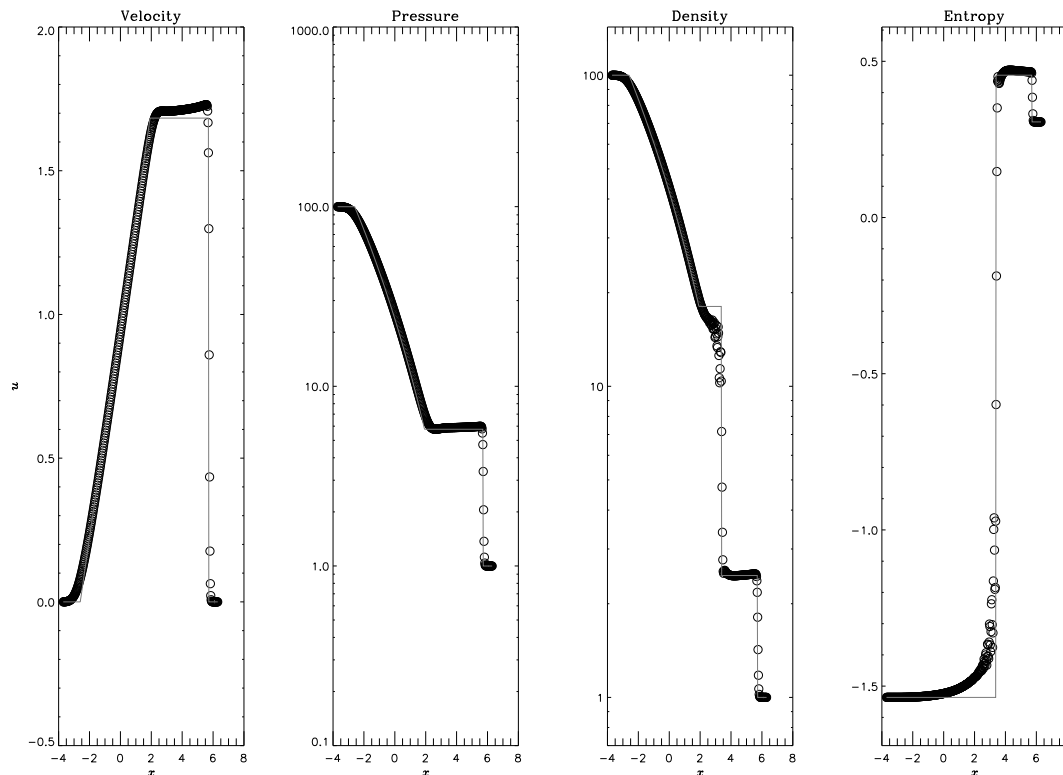
$$\tau = 2\rho\nu S_{ij} + \rho\xi_{\text{shock}}\delta_{ij}\nabla\cdot\mathbf{u}$$

and the viscous heating going into the energy equation is

$$\rho^{-1}\Gamma_{\text{visc}} = 2\nu S^2 + \xi_{\text{shock}}(\nabla\cdot\mathbf{u})^2$$

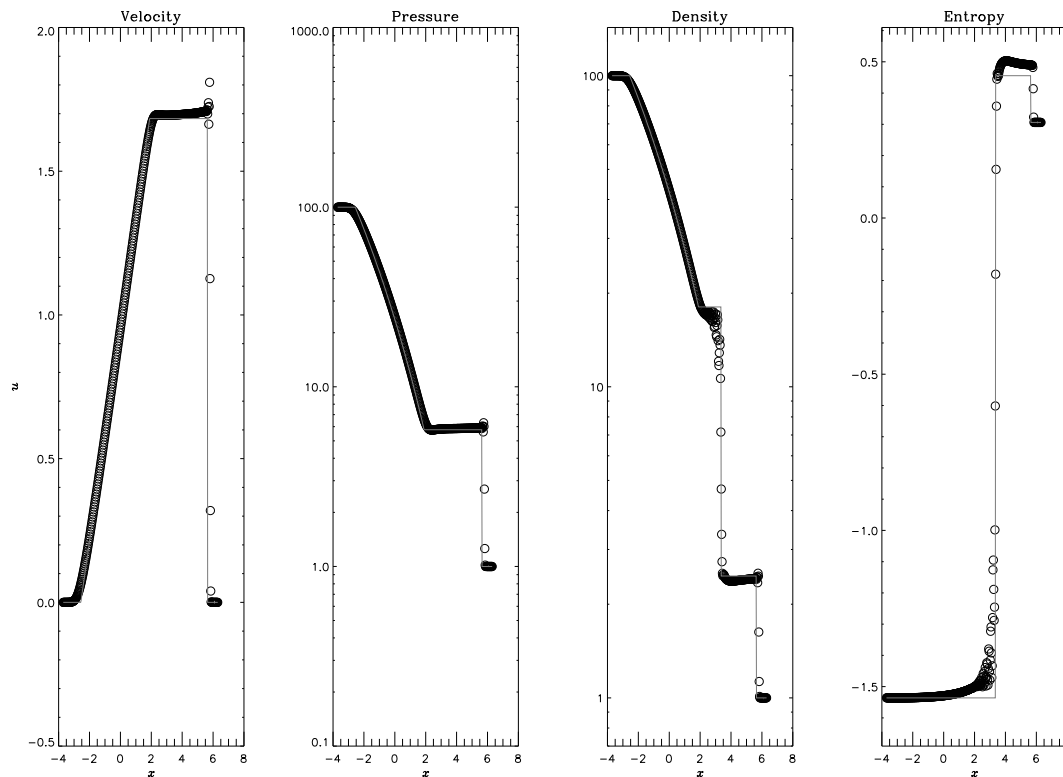
Examples with the PENCIL CODE (1)

Let us consider a shock produced by a jump in density, pressure and entropy (the so-called Riemann shock tube problem) such that $p/p_0 = 100$. Using the maximal velocity from an exact analytic solution and $Re_{\text{mesh}}=1$ gives a required constant kinematic viscosity of $\nu \approx 0.04$ for the selected one-dimensional grid with 400 meshpoints and the spatial extent of $L_x=10$.



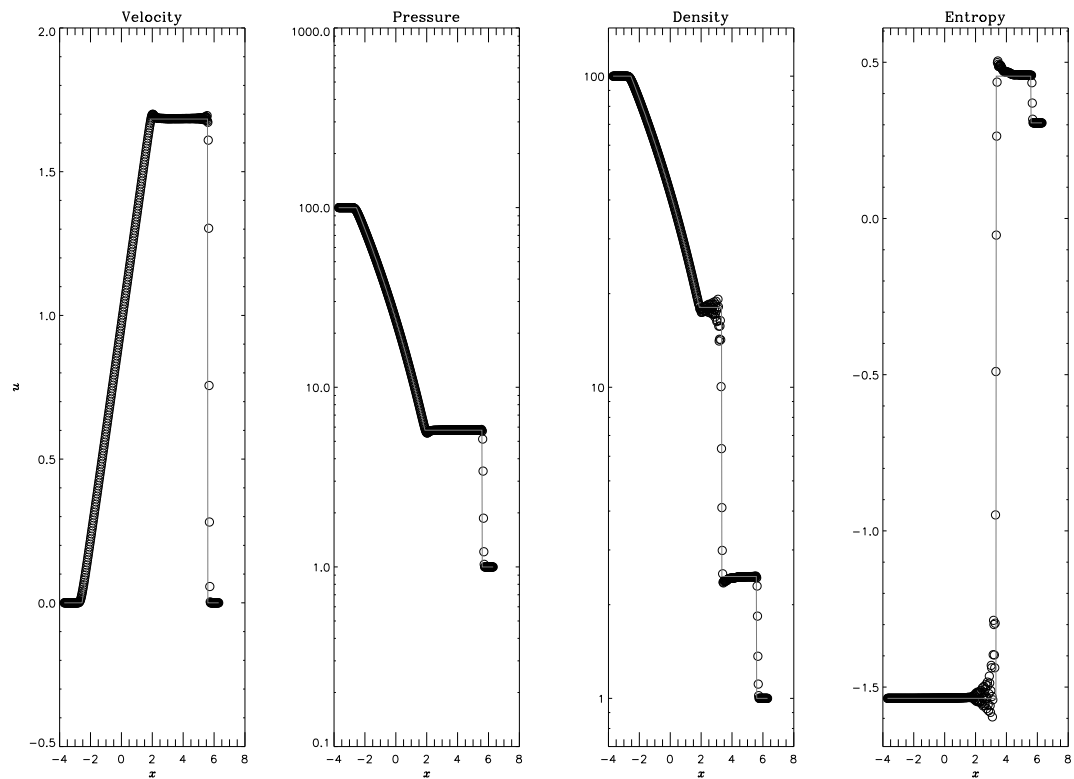
Examples with the PENCIL CODE (2)

With too low viscosity, $\nu = 0.02$, the solution starts significantly deviating from the analytic solution: the shock front expands slightly too fast, the entropy is unusually high, wiggles start appearing.



Examples with the PENCIL CODE (3)

Example with shock viscosity $c_{\text{shock}} = 3$, and $\nu = 0.002$. The shock is well-resolved, but the wiggling is considerably stronger than with ordinary viscosity.



Taming the wiggles: upwinding

Centered difference operators replaced by upwind operators, e.g. for 6th order operators with a 5th order one with the farthest point downwind removed (for $u > 0$)

$$f'_0 = \frac{-f_{-3} + 9f_{-2} - 45f_{-1} + 45f_1 - 9f_2 + f_3}{60\delta x} - \frac{\delta x^6 f^{(7)}}{140} = D^{(\text{cent},6)} + \mathcal{O}(\delta x^6)$$

$$f'_0 = \frac{-2f_{-3} + 15f_{-2} - 60f_{-1} + 20f_0 + 30f_1 - 3f_2}{60\delta x} - \frac{\delta x^5 f^{(6)}}{60} = D^{(\text{up},5)} + \mathcal{O}(\delta x^5)$$

The difference between central and upwind derivative is

$$[D^{(\text{up},5)} - D^{(\text{cent},6)}]f_0 = \frac{-f_{-3} + 6f_{-2} - 15f_{-1} + 20f_0 - 15f_1 + 6f_2 - f_3}{60\delta x} = -\frac{\delta x^5}{60} f_0^{(6)}.$$

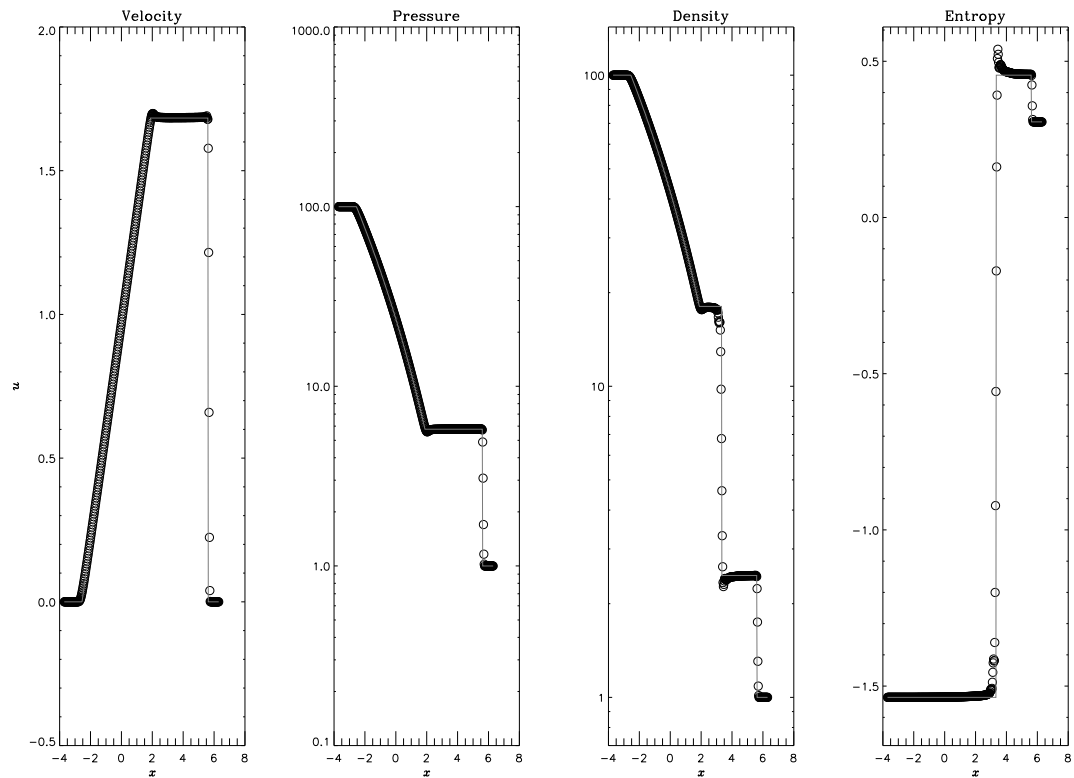
The 5th-order upwinding operator therefore effectively acts as a hyperdiffusive one

$$-u f'_{(\text{up},5\text{th})} = -u f'_{(\text{centr},6\text{th})} + \frac{|u| \delta x^5}{60} f^{(6)}.$$

The advantage over adopting constant hyperdiffusion is that in the upwinding scheme hyperdiffusion is only applied where advection occurs (not everywhere).

Examples with the PENCIL CODE (4)

Example with shock viscosity $c_{\text{shock}} = 3$, $\nu = 0.002$ and using upwinding operators for density, entropy and velocity. Wiggling has been significantly reduced.



Taming the wiggles: hyperdiffusion

Usual Laplacian viscosity $\nu \nabla^2 \mathbf{u}$ is equivalent to a multiplication by k^2 in Fourier space, where k is the wavenumber. Also the thermal conduction term in the energy equation for constant heat conduction κ

$$\rho \frac{De}{Dt} = \dots + \kappa c_V \nabla^2 e + \dots$$

is of this form. Sometimes wiggles can be damped solely by adding the thermal conduction term. One can also use high-order filtering by replacing the Laplacian with a higher order operator equivalent of multiplication with k^n in Fourier space. For a 6th order accurate finite differences the highest order operator is ∇^6 . For the continuity equation, for example, one adds a term

$$\frac{\partial \rho}{\partial t} = \dots + c_{\text{hyp}} \nabla^6 \rho,$$

where the coefficient can be estimated to be

$$c_{\text{hyp}} = \left(\frac{\delta x}{\pi} \right)^4.$$

Taming the wiggles: hyperdiffusion

For energy and magnetic equations one replaces the Laplacians with the hyperdiffusive operator. The above formulations conserve mass, energy and magnetic flux. For hyperdiffusion of the velocity, conservative formulation is quite complex. For $\nu = \text{cst}$

$$f_{\text{visc}}^{(\text{hyper})} = \nu_{\text{hyp}} \left(\nabla^6 \mathbf{u} + \frac{1}{3} \nabla^4 (\nabla (\nabla \cdot \mathbf{u})) + 2S_{\text{hyp}} \cdot \nabla \ln \rho \right) \quad (2)$$

Some caution must be taken when dynamo action is investigated in a closed system utilizing hyperdiffusive operators. In such a system, magnetic helicity is conserved, and the usage of hyperdiffusion has been reported to result in prolonged saturation times and enhanced saturation amplitudes of the magnetic field.

Examples with the PENCIL CODE (5)

Example with shock viscosity $c_{\text{shock}} = 3$, $\nu = 0.002$ and constant thermal conduction with $\chi = 3 \cdot 10^{-4}$. The wiggles are quite efficiently removed as with upwinding derivatives.

