Dynamos on galactic scales,

or

Dynamos around us.

Part III. Galactic dynamos

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"The argument in the past has frequently been a process of elimination: one observed certain phenomena, and one investigated what part of the phenomena could be explained; then the unexplained part was taken to show the effects of the magnetic field. It is clear in this case that, the larger one's ignorance, the stronger the magnetic field."

L. Woltjer, *Proc. IAU Symp. 31*, 479-485 (1967)

"As usual in astrophysics, the way out of a difficulty is to invoke the poorly understood magnetic field. ... One tends to ignore the field so long as one can get away with it."

D. Cox, in *The Interstellar Medium in Galaxies*, Kluwer, 181-200 (1990)

1. Necessity of dynamo action

- ❑ Can the magnetic fields observed be primordial?
- ❑ Do they need to be maintained by ongoing dynamo action?
- ❑ Dynamo action: conversion of kinetic energy into magnetic energy *with no electric currents at infinity*

1.1. Magnetic fields in a highly conducting turbulent medium

"If $R_m >> 1$, magnetic field decays only slowly and so does not necessarily need to be continuously maintained."

Wrong, if the system is turbulent:
energy is transferred along the spectrum and then dissipates in a time of order l_0/v_0, and this time is much shorter than the Ohmic decay time l_0^2/η if $R_m = l_0 v_0/\eta \gg 1$.

Conclusion: any (3D, MHD) magnetised, turbulent system must host a dynamo
(unless the magnetic field is driven by external currents or decays).

Even without turbulence, a sufficiently strong random magnetic field would drive random motions, and they will drain magnetic energy.

1.2. Magnetic field in a differentially rotating, turbulent disc

(A) The decay problem (Parker 1979)

Fully ionised plasma, the Ohmic decay of a large-scale magnetic field is very slow:

$$\eta = 10^7 (T/10^4)^{-3/2}\ \mathrm{cm^2/s},\quad v_0 = 10\ \mathrm{km/s},\quad h = 500\ \mathrm{pc}$$

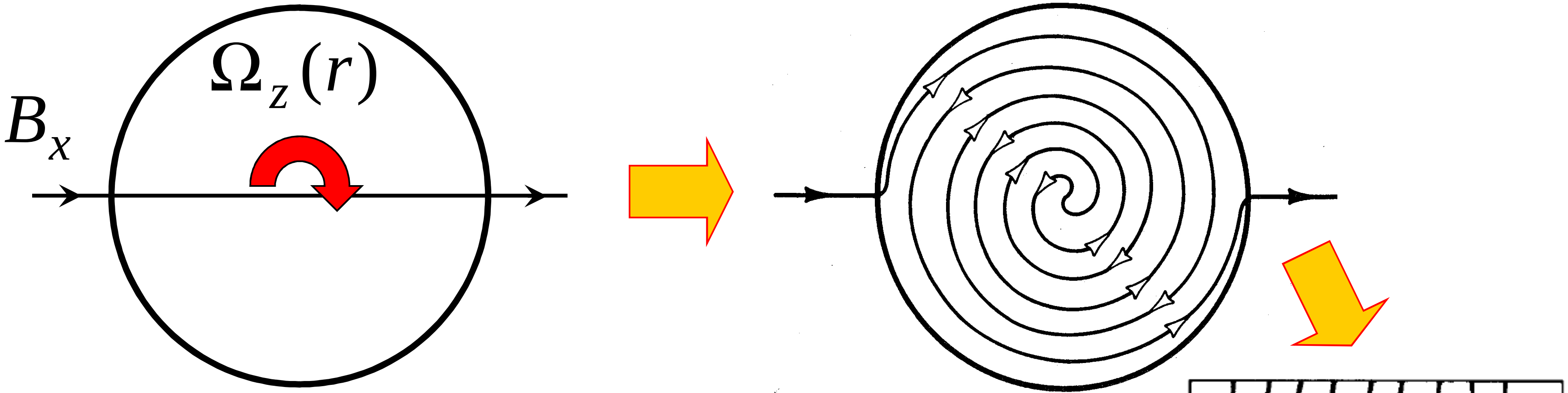
$$\Rightarrow\quad R_m \cong 10^{20}\ (!?),\qquad \tau_{\mathrm{decay}} = h^2/\eta \cong 10^{27}\ \mathrm{yr} \gg \text{Hubble time}$$

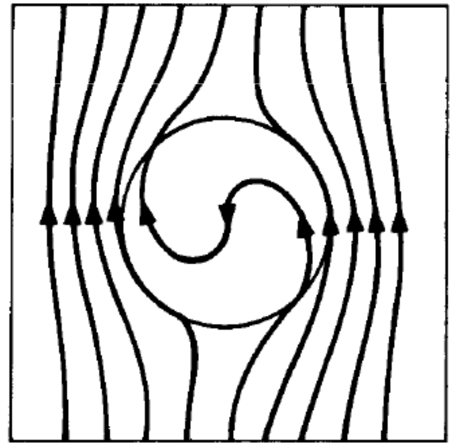
However, turbulent diffusion destroys the large-scale magnetic field much faster:

$$\beta = \tfrac{1}{3} l_0 v_0 \simeq 10^{26}\,\frac{\mathrm{cm}^2}{\mathrm{s}} \Rightarrow \tau_{\mathrm{decay}} = h^2/\beta \cong 5\times10^8\ \mathrm{yr} = \frac{\text{galactic lifetime}}{20}$$

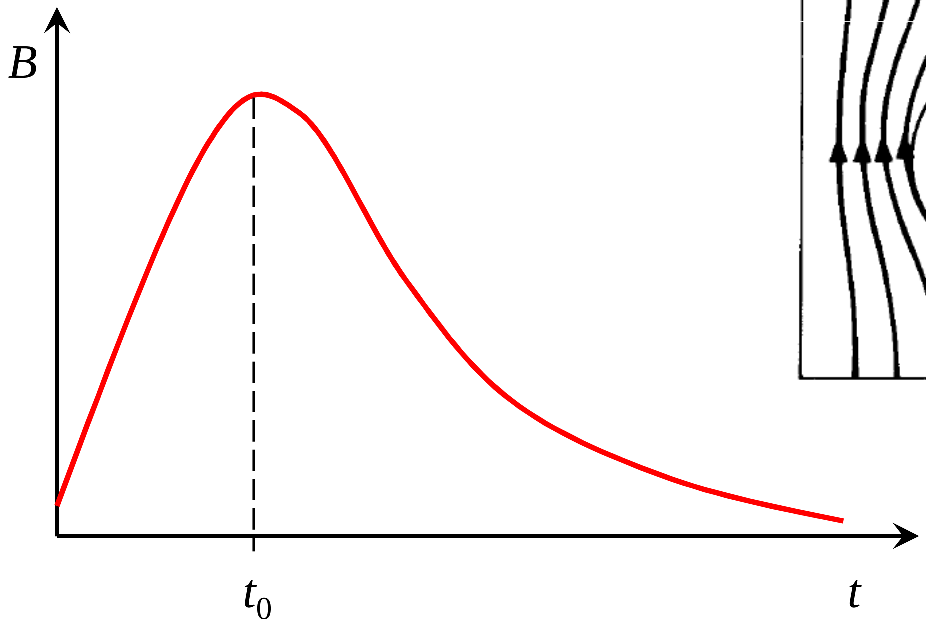
Without dynamo action, *turbulent magnetic diffusion* destroys a *large-scale* magnetic field in a fraction of the galactic lifetime.

(B) The wrap-up problem (Parker 1979)





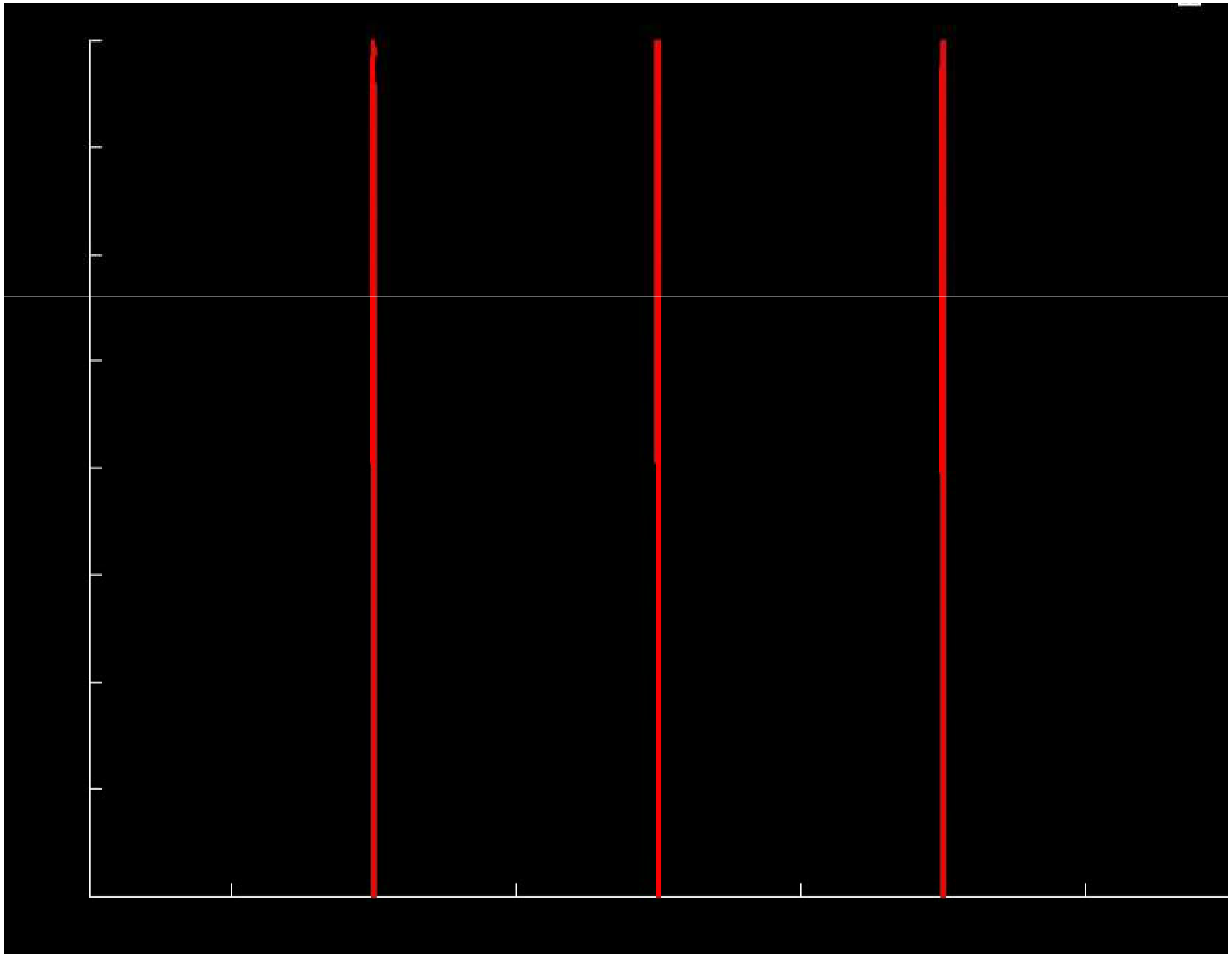
Expulsion of magnetic field from a region with closed streamlines



(Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids,* CUP, 1978, §3.7)

The action of differential rotation on magnetic field: flux expulsion from a region with closed streamlines

$$\Omega_z = e^{-r^2} \;, \qquad \mathbf{B}\big|_{t=0} = (0,1,0) \;.$$

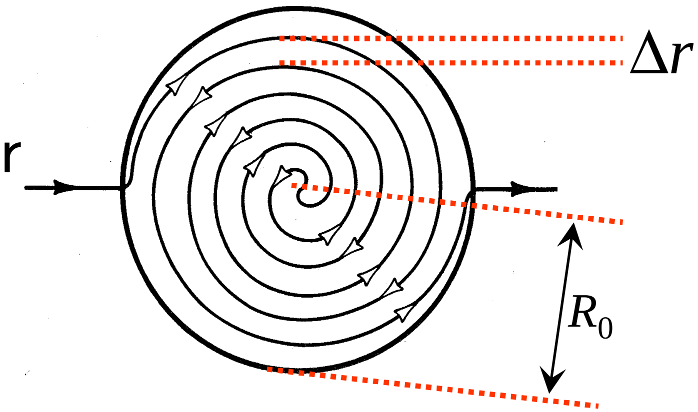


A. Baggaley

$G = |r\, d\Omega/dr|$: rotational shear rate,

$C_\omega = G R_0^2/\beta$: turbulent Reynolds number

β: turbulent magnetic diffusivity.



Initial growth:

$$\Delta r \simeq \frac{R_0}{Gt}, \quad p = \arctan\frac{B_r}{B_\phi} \simeq -\frac{1}{Gt} \text{ (magnetic pitch angle).}$$

End of the growth phase, $t = t_0$: *amplification time = diffusion time*,

$$\frac{1}{G} = \frac{[\Delta r(t_0)]^2}{\beta} \quad \Rightarrow \quad t_0 \simeq \frac{C_\omega^{1/2}}{G}, \quad p(t_0) \simeq -C_\omega^{-1/2}.$$

$$B_{\max} \simeq B_0 G t_0 \simeq B_0 C_\omega^{1/2}, \quad \Delta r(t_0) \simeq \frac{R_0}{C_\omega^{1/2}}.$$

Galactic discs:

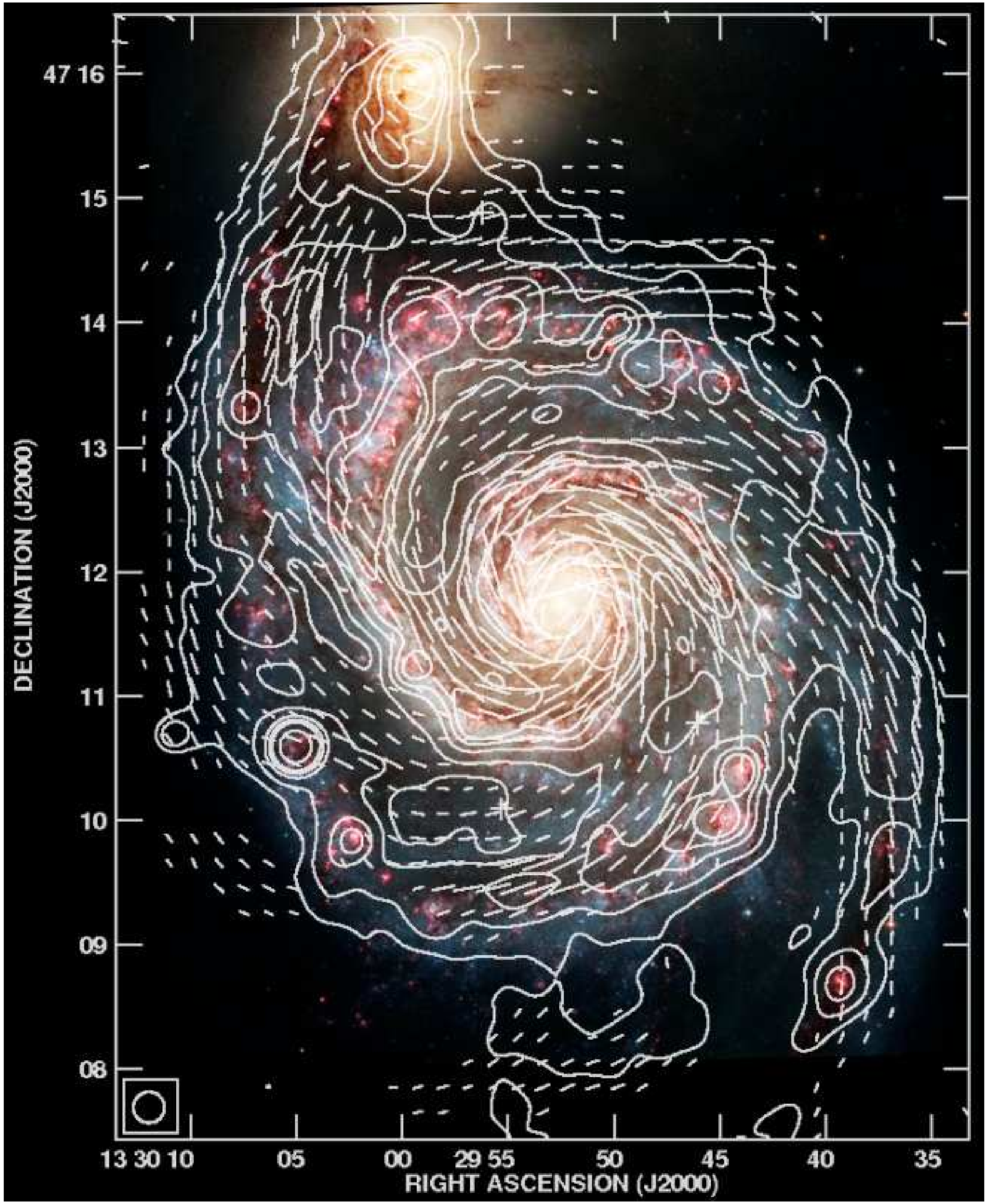
$$G = V_0/R_0, \quad \beta = \tfrac{1}{3} l_0 v_0 \quad \Rightarrow \quad C_\omega = 3 \frac{V_0}{v_0} \frac{R_0}{l_0} \simeq 6000,$$

$$p \simeq -C_\omega^{-1/2} \simeq -1^\circ, \qquad \Delta r \simeq R_0 C_\omega^{-1/2} \simeq 100\,\text{pc},$$

$$B_{\max} = B_0 C_\omega^{1/2} \simeq 0.1\,\mu\text{G} \quad \text{for } B_0 = 10^{-9}\,\text{G}.$$

$p \simeq -1^\circ$, $\Delta r \simeq 100$ pc, $B \simeq 0.1\,\mu$G:
a configuration very different from that observed,
$p \simeq -15^\circ$, $\Delta r > 1$ kpc, $B \simeq 3\,\mu$G.

Conclusion: to avoid twisting by differential rotation,
the *large-scale* galactic magnetic field
has to be supported (by a dynamo action).



M51 t λ6 cm

I-contours + B-vectors

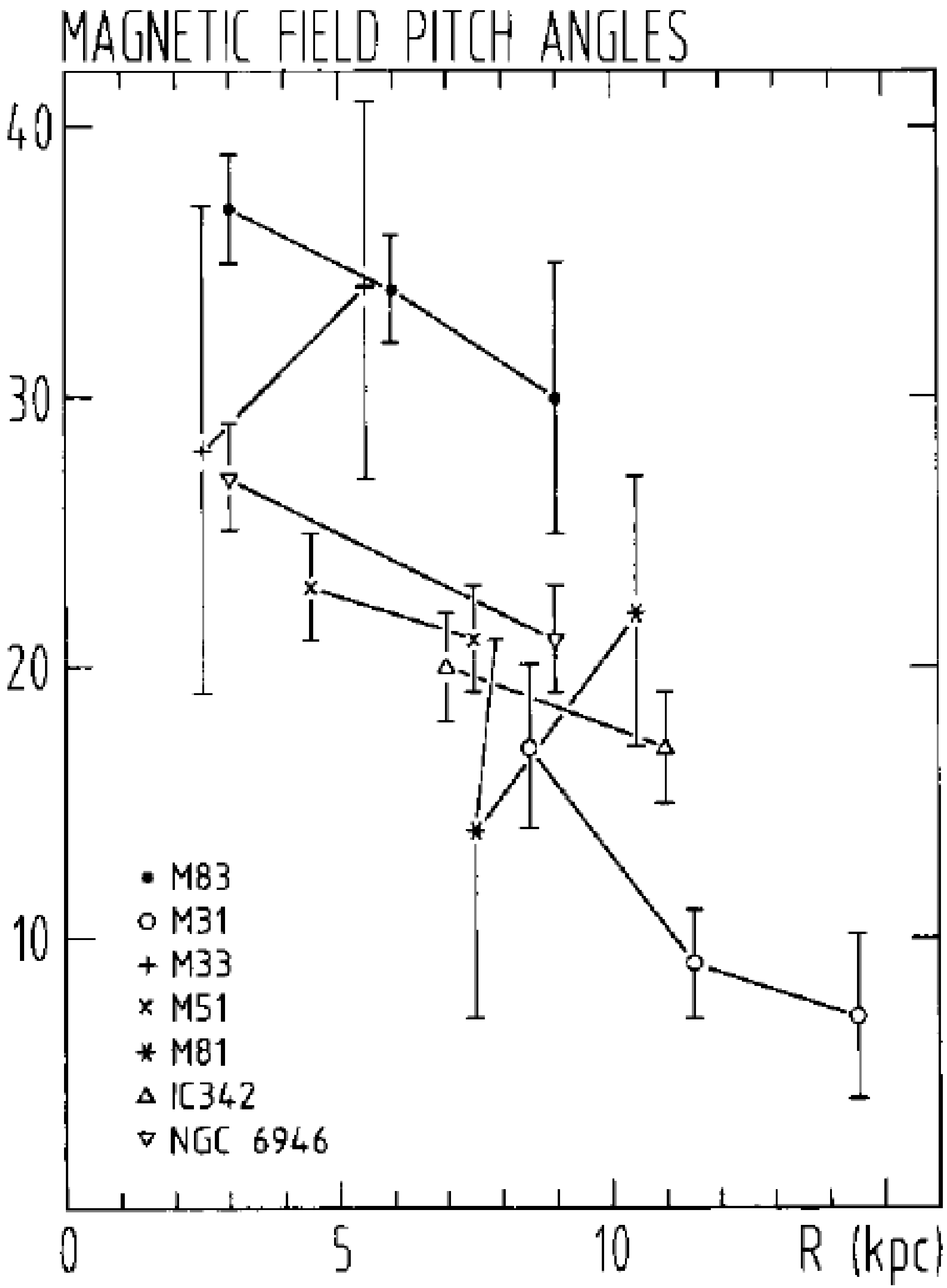
(Effelsberg+VLA;

A. Fletcher & R. Beck)

Magnetic pitch angle:

$p \approx -20°$

$-p$



$$p = \arctan \frac{B_r}{B_\phi}$$

Beck et al., ARAA, 1996

2. Disc dynamos and dynamo control parameters

$\vec{B} =$ large-scale magnetic field

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \vec{B} + \vec{V} \times \vec{B}) + \beta \nabla^2 \vec{B}$$

Cylindrical coordinates (r, ϕ, z), $\quad \vec{e}_z \parallel \vec{\Omega}$,

$$\vec{V} = (0, r\Omega, 0), \quad \Omega = \Omega(r).$$

Thin disc: $|z| \le h, \quad r \le R, \quad R >> h,$

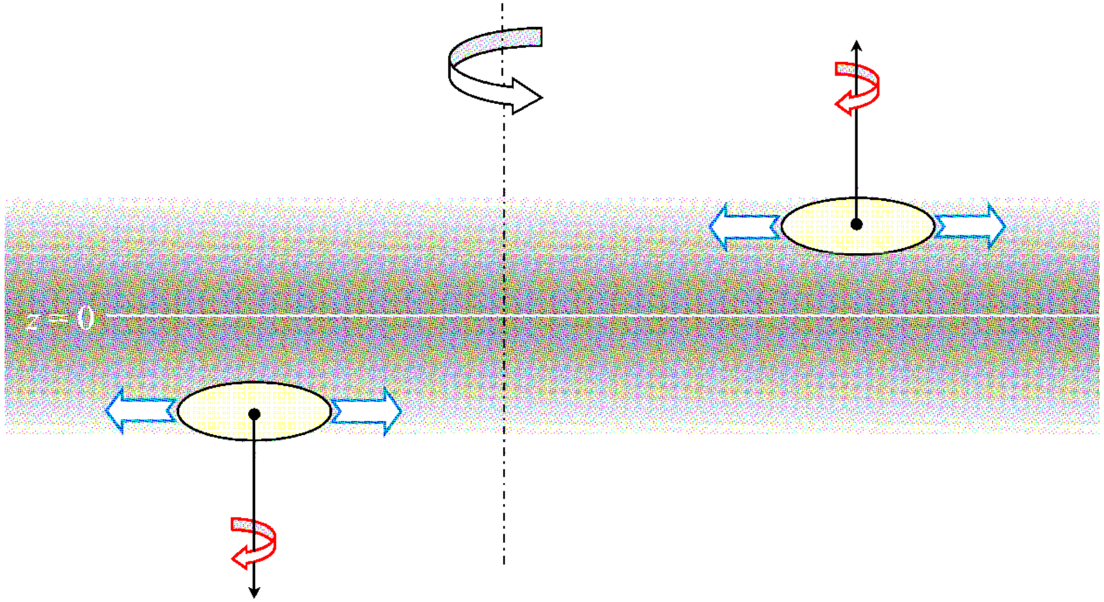
$$\partial/\partial z \gg \partial/\partial r,\ \partial/r\partial\phi \Rightarrow \nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial z^2}, \cdots$$

Axial symmetry: $\partial/\partial\phi = 0.$

The physical picture of the galactic dynamo

(A) Helicity of interstellar turbulence:

consequence of angular momentum conservation in a rotating, stratified layer

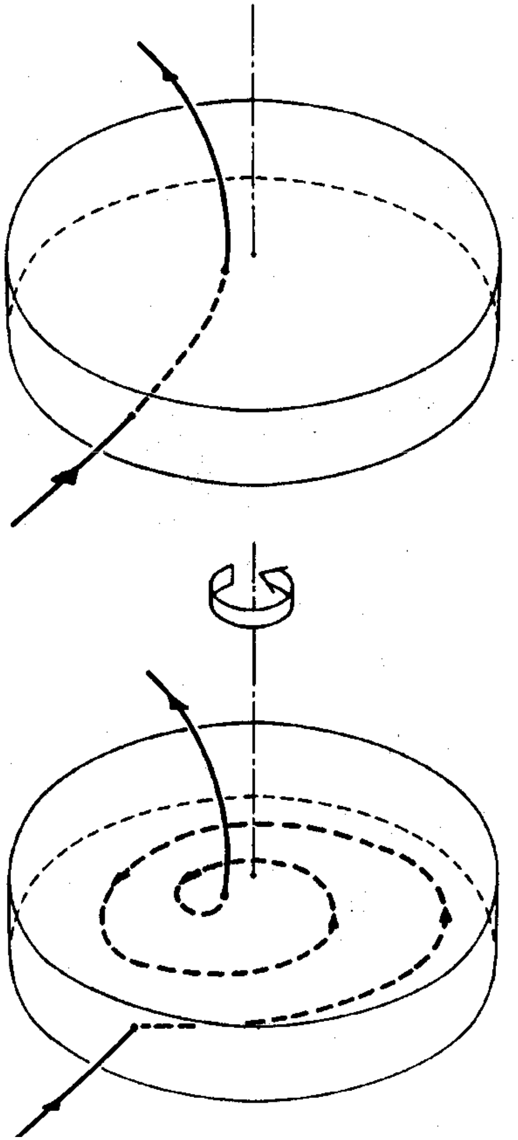


$$\alpha \simeq \frac{l_0^2 \Omega}{h}$$

(F. Krause, 1967)

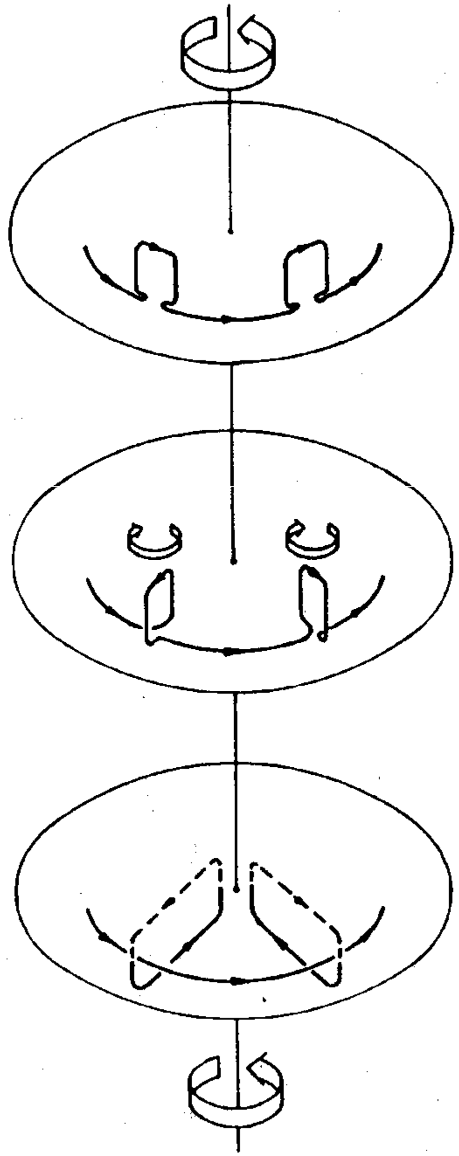
(B) Differential rotation:

B_ϕ produced from B_r



(C) Helical turbulence:

B_r produced from B_ϕ



2.1. Basic equations

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\alpha \vec{B} + \vec{V} \times \vec{B}) + \beta \nabla^2 \vec{B}$$

$$\mathbf{B}(t, \mathbf{r}) = \tilde{\mathbf{B}}(t, r, z) e^{im\phi} \; , \qquad \epsilon = \frac{h_0}{R_0} \ll 1 \; .$$

$$\left(\frac{\partial}{\partial t} + imR_\omega\Omega + \frac{\epsilon^2 m^2}{r^2}\right)\tilde{B}_r = -R_\alpha \frac{\partial}{\partial z}(\alpha \tilde{B}_\phi) + \frac{\partial^2 \tilde{B}_r}{\partial z^2} + \epsilon^2 \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_r)\right] + im\epsilon R_\alpha \frac{\alpha}{r}\tilde{B}_z - \frac{2im\epsilon^2}{r^2}\tilde{B}_\phi \,,$$

$$\left(\frac{\partial}{\partial t} + imR_\omega\Omega + \frac{\epsilon^2 m^2}{r^2}\right)\tilde{B}_\phi = R_\omega G\tilde{B}_r + R_\alpha \frac{\partial}{\partial z}(\alpha \tilde{B}_r) + \frac{\partial^2 \tilde{B}_\phi}{\partial z^2} + \epsilon^2 \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_\phi)\right] - \epsilon R_\alpha \frac{\partial}{\partial r}(\alpha \tilde{B}_z) + \frac{2im\epsilon^2}{r^2}\tilde{B}_r \,,$$

$$\left(\frac{\partial}{\partial t} + imR_\omega\Omega + \frac{\epsilon^2 m^2}{r^2}\right)\tilde{B}_z = \frac{\partial^2 \tilde{B}_z}{\partial z^2} + R_\alpha \frac{\epsilon}{r}\frac{\partial}{\partial r}(r\alpha\tilde{B}_\phi) - im\epsilon R_\alpha \frac{\alpha}{r}\tilde{B}_r + \epsilon^2 \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tilde{B}_z)\right] + \frac{\epsilon^2}{r^2}\tilde{B}_z \,,$$

$$\mathbf{B}(t,\mathbf{r}) = \tilde{\mathbf{B}}(t,r,z)e^{im\phi} \,, \qquad \epsilon = \frac{h_0}{R_0} \ll 1 \,, \qquad G = r\frac{d\Omega}{dr} \,.$$

Thin disc, axisymmetric solutions, $\alpha^2\omega$-dynamo:

$$\begin{aligned}
\frac{\partial B_r}{\partial t} &= -\frac{\partial}{\partial z}(\alpha B_\phi) + \beta \frac{\partial^2 B_r}{\partial z^2} \; , \\
\frac{\partial B_\phi}{\partial t} &= G B_r + \frac{\partial}{\partial z}(\alpha B_r) + \beta \frac{\partial^2 B_\phi}{\partial z^2} \; , \\
\frac{\partial B_z}{\partial t} &= \beta \frac{\partial^2 B_z}{\partial z^2} \; .
\end{aligned}$$

$$G = r \, d\Omega/dr$$

Equation for B_z splits from the system.

B_z is supported through B_r and B_ϕ via $\partial/\partial r$

Dimensionless variables

$$\tilde{z} = \frac{z}{h} \Rightarrow \frac{\partial}{\partial z} = \frac{1}{h}\frac{\partial}{\partial \tilde{z}}\,, \qquad \tilde{t} = \frac{t}{h^2/\beta} \Rightarrow \frac{\partial}{\partial t} = \frac{\beta}{h^2}\frac{\partial}{\partial \tilde{t}}\,,$$

$$\tilde{\alpha} = \frac{\alpha(z)}{\alpha_0}\,.$$

$$\begin{aligned}
\frac{\partial B_r}{\partial \tilde{t}} &= -R_\alpha \frac{\partial}{\partial \tilde{z}}(\tilde{\alpha} B_\phi) + \frac{\partial^2 B_r}{\partial \tilde{z}^2}\,, \qquad R_\alpha = \frac{\alpha_0 h}{\beta} \\
\frac{\partial B_\phi}{\partial \tilde{t}} &= R_\omega B_r + R_\alpha \frac{\partial}{\partial \tilde{z}}(\tilde{\alpha} B_r) + \frac{\partial^2 B_\phi}{\partial \tilde{z}^2}\,, \qquad R_\omega = \frac{G h^2}{\beta}\,.
\end{aligned}$$

Drop ˜ at dimensionless variables:

$$\frac{\partial B_r}{\partial t} = \underbrace{-R_\alpha \frac{\partial}{\partial z}(\alpha B_\phi)}_{B_\phi \to B_r \text{ via } \alpha\text{-effect}} + \frac{\partial^2 B_r}{\partial z^2} ,$$

$$\frac{\partial B_\phi}{\partial t} = \underbrace{R_\omega B_r}_{B_r \to B_\phi \text{ via differential rotation}} + \underbrace{R_\alpha \frac{\partial}{\partial z}(\alpha B_r)}_{B_r \to B_\phi \text{ via } \alpha\text{-effect}} + \frac{\partial^2 B_\phi}{\partial z^2} .$$

$\alpha\omega$-Dynamo: $|R_\omega| >> R_\alpha$

$$\frac{\partial B_r}{\partial t} = -R_\alpha \frac{\partial}{\partial z}(\alpha B_\phi) + \frac{\partial^2 B_r}{\partial z^2} ,$$
$$\frac{\partial B_\phi}{\partial t} = R_\omega B_r + \frac{\partial^2 B_\phi}{\partial z^2} .$$

Introduce new variable $B_r = R_\alpha B'_r$ and drop the dash:

$$\frac{\partial B_r}{\partial t} = -\frac{\partial}{\partial z}(\alpha B_\phi) + \frac{\partial^2 B_r}{\partial z^2} ,$$
$$\frac{\partial B_\phi}{\partial t} = D B_r + \frac{\partial^2 B_\phi}{\partial z^2} ,$$

where $D = R_\alpha R_\omega$ is the dynamo number.

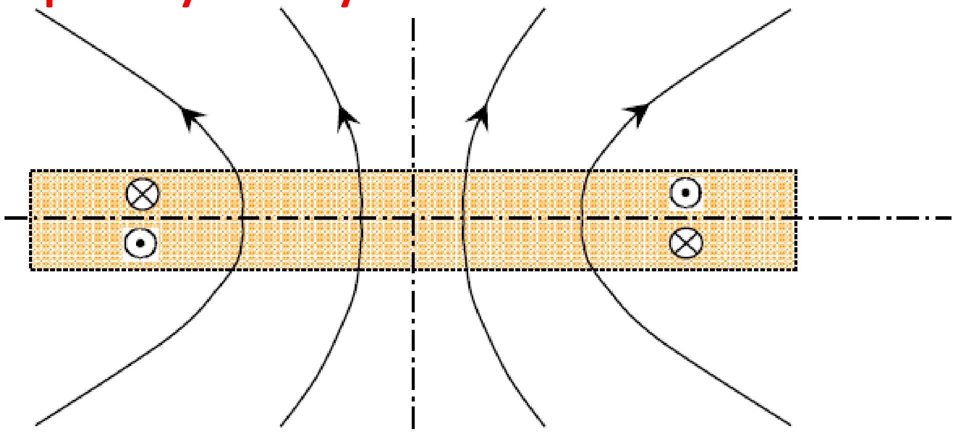
Boundary conditions

$$B_r|_{z=1} = B_\phi|_{z=1} = 0$$ (vacuum boundary conditions)

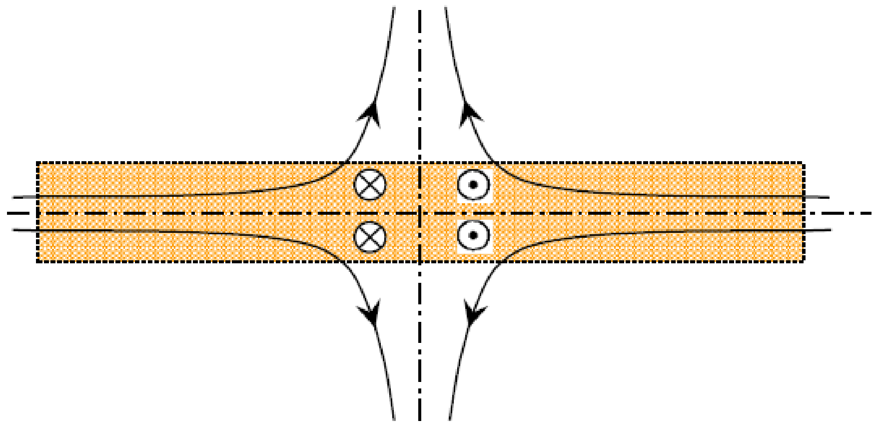
$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_\phi}{\partial z}\right|_{z=0} = 0$$ (quadrupole)

$$B_r|_{z=0} = B_r|_{z=0} = 0$$ (dipole)

Dipolar symmetry



Quadrupolar symmetry



2.2. Dynamo control parameters

NB! The Solar neighbourhood of the Milky Way, where these estimates apply, may not be a typical location.

Rotation $\Omega = \dfrac{V_0}{r}$,

$V_0 \simeq 200\,\mathrm{km/s}$, $r \simeq 10\,\mathrm{kpc}$,

ionised gas scale height
$h \simeq 0.5\,\mathrm{kpc}$,

turbulent velocity $v_0 \simeq 10\,\mathrm{km/s}$,

turbulent scale $l_0 \simeq 0.1\,\mathrm{kpc}$.

$$\alpha_0 \simeq \frac{l_0^2 \Omega}{h} \simeq 0.4\,\mathrm{km/s},$$

$$\beta \simeq \tfrac{1}{3} l_0 v_0 \simeq 10^{26}\,\mathrm{cm^2/s},$$

$$R_\alpha = \frac{\alpha_0 h}{\beta} \simeq 0.6$$

$$R_\omega = \frac{(r\, d\Omega/dr) h^2}{\beta} \simeq -15$$

$$D = R_\alpha R_\omega \simeq -\left(\frac{3\Omega h}{v_0}\right)^2 \simeq -10$$

3. The "no-z" approximation (Subramanian & Mestel, 1993)

Thin disc, dimensional αω-dynamo equations:

$$\frac{\partial B_r}{\partial t} = -\frac{\partial}{\partial z}(\alpha B_\phi) + \beta \frac{\partial^2 B_r}{\partial z^2},$$
$$\frac{\partial B_\phi}{\partial t} = G B_r + \beta \frac{\partial^2 B_\phi}{\partial z^2}.$$

For solutions of a simple form, e.g., $B_{r,\phi} \propto \cos z/h$ (see below),

$$\frac{\partial}{\partial z} \simeq \frac{1}{h}, \qquad \frac{\partial^2}{\partial z^2} \simeq -\frac{1}{h^2}.$$

Kinematic solutions: $\vec{B} = \vec{B}_0 \exp(\gamma t)$.

$$
\begin{aligned}
\left(\gamma + \frac{\beta}{h^2}\right) B_{0r} + \frac{\alpha}{h} B_{0\phi} &= 0, \\
-G B_{0r} + \left(\gamma + \frac{\beta}{h^2}\right) B_{0\phi} &= 0.
\end{aligned}
$$

Nontrivial solutions exist if

$$
\begin{vmatrix} \gamma + \beta/h^2 & \alpha/h \\ -G & \gamma + \beta/h^2 \end{vmatrix} = 0,
$$

i.e., $\quad \gamma \simeq \frac{\beta}{h^2}(-1 + \sqrt{-D}),$

$$
\tan p = \frac{B_r}{B_\phi} \simeq -\sqrt{\frac{\alpha}{-Gh}} = -\sqrt{\frac{R_\alpha}{|R_\omega|}}.
$$

Magnetic field grows if $D \lesssim -1$, with $p \simeq -\arctan\frac{1}{4} \simeq -15^\circ$.

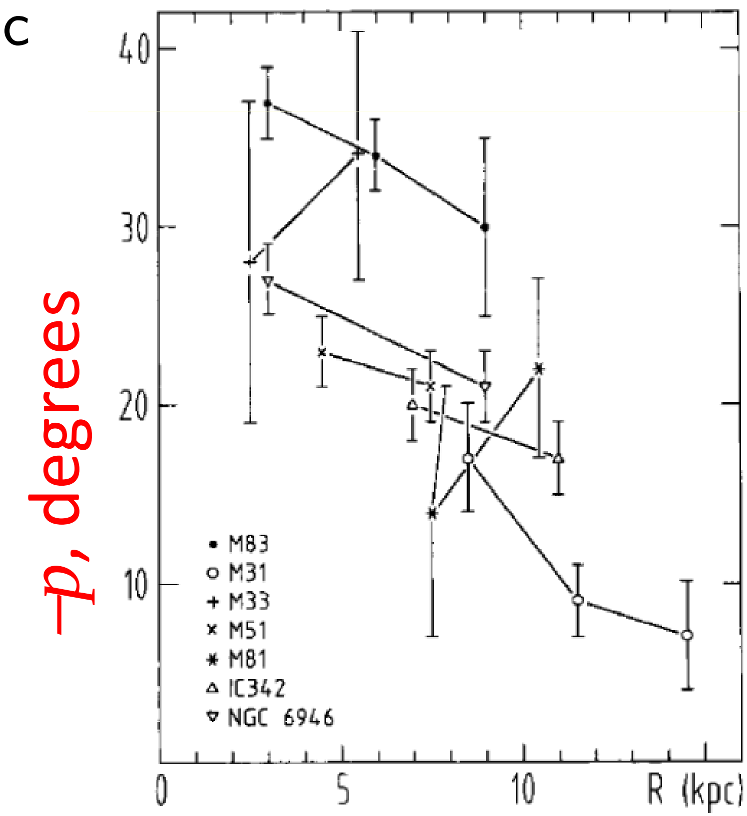
Conclusions

- Mean-field dynamo is a threshold phenomenon:
  $\gamma > 0$ only if $|D| > |D_{\rm cr}|$.

- This inequality is plausible to be satisfied in galactic discs, where $D \simeq -10$, but a more accurate solution is required.

- The pitch angle of the growing magnetic field agrees with observations
  (for parameter values typical of galactic discs).

- For $\alpha_0 \simeq l_0^2\Omega/h$, we have $p \simeq -l_0/h$.
  Since h grows with r, $h \propto e^{kr}$,
  $|p|$ decreases with r
  (if l_0 does not grow with r that fast),
  as observed.



4. Perturbation solutions

$$\frac{\partial B_r}{\partial t} = -\frac{\partial}{\partial z}(\alpha B_\phi) + \frac{\partial^2 B_r}{\partial z^2} \,,$$

$$\frac{\partial B_\phi}{\partial t} = D B_r + \frac{\partial^2 B_\phi}{\partial z^2} \,,$$

$$\left.\frac{\partial B_r}{\partial z}\right|_{z=0} = \left.\frac{\partial B_\phi}{\partial z}\right|_{z=0} = 0 \,, \qquad B_r|_{z=1} = B_\phi|_{z=1} = 0 \,.$$

(Quadrupolar solution) (Vacuum around the disc)

α independent of t $\Rightarrow$ $\vec{\mathbf{B}} = \vec{\mathbf{B}}_0(z)e^{\gamma t}$, $\frac{\partial \vec{\mathbf{B}}}{\partial t} = \gamma \vec{\mathbf{B}}$, $\gamma = \text{const}$.

(Kinematic stage, where the Lorentz force is negligible, and so the velocity field is unaffected by magnetic field.)

Introduce new variable $B_{0\phi} = B'_{0\phi}\sqrt{|D|}$:

$$\gamma B_{0r} = -\sqrt{|D|}\,\frac{d}{dz}(\alpha B'_{0\phi}) + \frac{d^2 B_{0r}}{dz^2}\,,$$

$$\gamma B'_{0\phi} = \frac{D}{\sqrt{|D|}} B_{0r} + \frac{d^2 B'_{0\phi}}{dz^2}\,,$$

and drop subscript '0' and dash to simplify the notation:

$$\gamma B_r = -\sqrt{|D|}\,\frac{d}{dz}(\alpha B_\phi) + \frac{d^2 B_r}{dz^2}\,,$$

$$\gamma B_\phi = \sqrt{|D|}\,\mathrm{sign}(D) B_r + \frac{d^2 B_\phi}{dz^2}\,.$$

For $|D|$ << 1, derive an approximate solution for $\vec{B}$

Plan of the solution

1. Rewrite the equations in a matrix operator form and isolate the unperturbed operator $\widehat{W}$ and the perturbation operator $\widehat{V}$:

$$\gamma\vec{\mathbf{B}} = \left(\widehat{W} + |D|^{1/2}\widehat{V}\right)\vec{\mathbf{B}}\,, \qquad \vec{\mathbf{B}} = \begin{pmatrix} B_r \\ B_\phi \end{pmatrix}.$$

2. Derive the EIGENVALUES λ_n and EIGENFUNCTIONS $\vec{\mathbf{b}}_n$ of the unperturbed equation

$$\lambda_n \vec{\mathbf{b}}_n = \widehat{W}\vec{\mathbf{b}}_n\,, \qquad n = 0, 1, 2 \ldots\,, \qquad \lambda_n < 0\,.$$

The eigenfunctions should satisfy the same boundary conditions as $\vec{\mathbf{B}}$.

The eigenfunctions form an ORTHOGONAL set on $0 \le z \le 1$:

$$\int_0^1 \vec{\mathbf{b}}_n \cdot \vec{\mathbf{b}}_m \, dz = 0 \quad \text{if} \quad n \ne m\,.$$

3. Expand $\gamma = \gamma_0 + \epsilon\gamma_1 + \ldots$, $\vec{B} \sim \sum_{n=0}^{\infty} \epsilon^n (C_n \vec{b}_n + C'_n \vec{b}'_n)$ substitute into the equations, take a dot product with $\vec{b}_m$, integrate over z from 0 to 1 to obtain algebraic equation for C_m.

4.1. An operator form of the equations

$$\gamma B_r = -|D|^{1/2} \frac{d}{dz}(\alpha B_\phi) + \frac{d^2 B_r}{dz^2} ,$$

$$\gamma B_\phi = |D|^{1/2} \operatorname{sign}(D)\, B_r + \frac{d^2 B_\phi}{dz^2} ,$$

$$\Rightarrow \quad \gamma \vec{\mathbf{B}} = \left(\widehat{W} + |D|^{1/2} \widehat{V}\right) \vec{\mathbf{B}} ,$$

$$\widehat{W} = \begin{pmatrix} \frac{d^2}{dz^2} & 0 \\ 0 & \frac{d^2}{dz^2} \end{pmatrix}, \qquad \widehat{V} = \begin{pmatrix} 0 & -\frac{d}{dz}(\alpha \cdot \ldots) \\ \operatorname{sign} D & 0 \end{pmatrix}$$

$D < 0$, sign $D = -1$; $\quad \epsilon = |D|^{1/2}, \quad \alpha = \sin \pi z$

4.2. Free decay modes of quadrupolar symmetry

$$\lambda\vec{b} = \widehat{W}\vec{b}, \qquad \frac{d}{dz}\vec{b}(0) = \vec{b}(1) = 0$$

$$\lambda_n = -\pi^2\left(n + \tfrac{1}{2}\right)^2 , \quad n = 0, 1, 2 \ldots ,$$

with two eigenfunctions corresponding to each eigenvalue,

$$\vec{\mathbf{b}}_n = \begin{pmatrix} b_r \\ b_\phi \end{pmatrix}_n = \begin{pmatrix} \sqrt{2}\cos z\sqrt{-\lambda_n} \\ 0 \end{pmatrix},$$

$$\vec{\mathbf{b}}'_n = \begin{pmatrix} b'_r \\ b'_\phi \end{pmatrix}'_n = \begin{pmatrix} 0 \\ \sqrt{2}\cos z\sqrt{-\lambda_n} \end{pmatrix}.$$

Normalization and orthogonality:

$$\int_0^1 \vec{b}_n \cdot \vec{b}_m \, dz = \int_0^1 \vec{b}'_n \cdot \vec{b}'_m \, dz = \delta_{nm},$$

$\vec{b}_n \cdot \vec{b}'_m = 0,$ for any n, m

4.3. The form of the asymptotic solution

$$\gamma \vec{B} = \widehat{W}\vec{B} + \epsilon \widehat{V}\vec{B}, \qquad \widehat{W}\vec{b}_n = \lambda_n \vec{b}_n, \quad \widehat{W}\vec{b}'_n = \lambda_n \vec{b}'_n$$

3a. First order: the perturbation removes the degeneracy giving an O(1) correction to the eigenfunction and an O(ε) correction to the eigenvalue (e.g., Landau & Lifshitz, *Quantum Mechanics*):

$$\vec{B} = C_0 \vec{b}_0 + C'_0 \vec{b}'_0 + \dots, \qquad \gamma = \lambda_0 + \epsilon\gamma_1 + \dots$$

Substitute these expansions into the equation,

take dot product with $\vec{b}_0$ and integrate over z from 0 to 1.

Repeat with $\vec{b}'_0$.

Remember that the free decay modes are orthonormal.

$$\gamma\vec{B} = (\widehat{W} + \epsilon\widehat{V})\vec{B}, \quad \vec{B} = C_0\vec{b}_0 + C_0'\vec{b}_0' + \dots, \quad \gamma = \gamma_0 + \epsilon\gamma_1 + \dots,$$

$$\widehat{W}\vec{b}_0 = \lambda_0\vec{b}_0, \quad \widehat{W}\vec{b}_0' = \lambda_0\vec{b}_0', \qquad \int_0^1 \vec{b}_0^2\, dz = \int_0^1 \vec{b}_0'^2\, dz = 1.$$

Terms of order ϵ^0 : $\quad \gamma_0 = \lambda_0.$

Terms of order ϵ : $\quad C_0\gamma_1\vec{b}_0 + C_0'\gamma_1\vec{b}_0' = C_0\widehat{V}\vec{b}_0 + C_0'\widehat{V}\vec{b}_0'.$

$$\int_0^1 (\cdots)\cdot\vec{b}_0\, dz : \qquad C_0(\gamma_1 - V_{00}) - C_0'V_{00'} = 0.$$

$$\int_0^1 (\cdots)\cdot\vec{b}_0'\, dz : \qquad -C_0V_{0'0} + C_0'(\gamma_1 - V_{0'0'}) = 0.$$

Matrix elements: $V_{mn} = \int_0^1 \vec{b}_m \cdot \widehat{V}\vec{b}_n\, dz, \quad V_{m'n'} = \int_0^1 \vec{b}_m' \cdot \widehat{V}\vec{b}_n'\, dz.$

4.4. First order results, $V_{nm} = \int_0^1 \vec{b}_n \cdot \widehat{V} \vec{b}_m \, dz$

$$V_{00} = V_{0'0'} = V_{10} = V_{1'0} = V_{1'0'} = 0,$$

$$V_{0'0} = -1, \; V_{10'} = -\frac{3\pi}{4}, \; V_{00'} = -\frac{\pi}{4}.$$

$$C_0' = C_0 \sqrt{\frac{V_{0'0}}{V_{00'}}} = -\frac{2}{\sqrt{\pi}} C_0.$$

C_0 : normalisation $\int_0^1 B^2 \, dz = 1$

$$\vec{B} \equiv \vec{\tilde{b}}_0 \approx \sqrt{\frac{2}{1 + 4/\pi}} \begin{pmatrix} 1 \\ -2/\sqrt{\pi} \end{pmatrix} \cos\frac{\pi z}{2}, \quad \gamma_1 = \frac{\sqrt{\pi}}{2}$$

4.5. Second order results

$$\gamma\vec{B} = \widehat{W}\vec{B} + \epsilon\widehat{V}\vec{B}, \qquad \widehat{W}\vec{b}_n = \lambda_n\vec{b}_n, \quad \widehat{W}\vec{b}'_n = \lambda_n\vec{b}'_n$$

$$\vec{B} = \vec{\tilde{b}}_0 + \epsilon\sum_{n=1}^{\infty}(C_n\vec{b}_n + C'_n\vec{b}'_n) + \ldots,$$

$$\gamma = \lambda_0 + \epsilon\gamma_1 + \epsilon^2\gamma_2\ldots$$

Proceed as before.

$$C_n = \frac{V_{n\tilde{0}}}{\lambda_0 - \lambda_n}, \qquad C'_n = \frac{V_{n'\tilde{0}}}{\lambda_0 - \lambda_n},$$

$$V_{n\tilde{0}} = \frac{1}{2}\sqrt{\frac{\pi}{1+4/\pi}} \times \begin{cases} 1, & n = 0; \\ 3, & n = 1; \\ 0, & n \neq 0, 1. \end{cases}$$

$$V_{n'0} = -\frac{1}{\sqrt{1+4/\pi}} \times \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

$$V_{\tilde{0}n} = \frac{2}{\sqrt{\pi+4}} \times \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

$$C_n = 0 \text{ for } n \neq 0, 1; \qquad C_1 = \frac{3}{4\pi^{3/2}\sqrt{1 + 4/\pi}};$$

$$C'_n = 0 \text{ for } n \neq 0.$$

$$\gamma_2 = \sum_{n=1}^{\infty} \frac{V_{n\tilde{0}} V_{\tilde{0}n} + V_{n'\tilde{0}} V_{\tilde{0}n'}}{\lambda_0 - \lambda_n} = 0 \quad \text{for any } \alpha(z).$$

Second order results

$$\begin{aligned}
\vec{B} &= \vec{\tilde{b}}_0 + \epsilon \frac{3}{4\pi^{3/2}} \frac{1}{\sqrt{1+4/\pi}} \vec{b}_1 + \dots \\
&\approx \sqrt{\frac{2}{1+4/\pi}} \begin{pmatrix} \cos\frac{\pi z}{2} + \epsilon \frac{3}{4\pi^{3/2}} \cos\frac{3\pi z}{2} \\ -\frac{2}{\sqrt{\pi}} \cos\frac{\pi z}{2} \end{pmatrix}, \\
\gamma &= -\frac{\pi^2}{4} + \epsilon \frac{\sqrt{\pi}}{2} + O(\epsilon^3).
\end{aligned}$$

In terms of original (physical) variables:

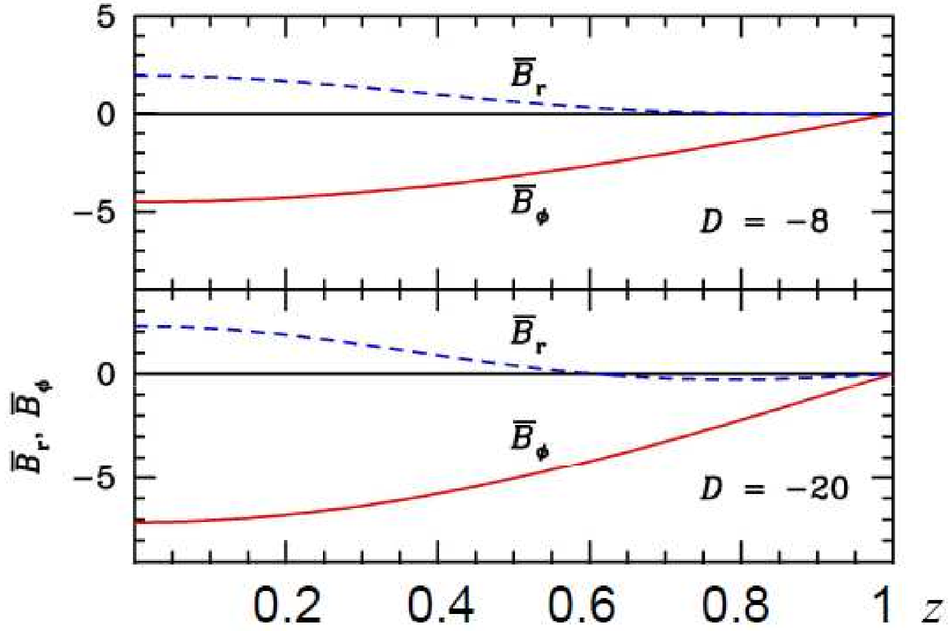
$$B_r \approx R_\alpha C_0 \left[\cos\frac{\pi z}{2} + \frac{3}{4\pi^{3/2}}\sqrt{-D}\cos\frac{3\pi z}{2}\right],$$

$$B_\phi \approx -2C_0\sqrt{-\frac{D}{\pi}}\cos\frac{\pi z}{2},$$

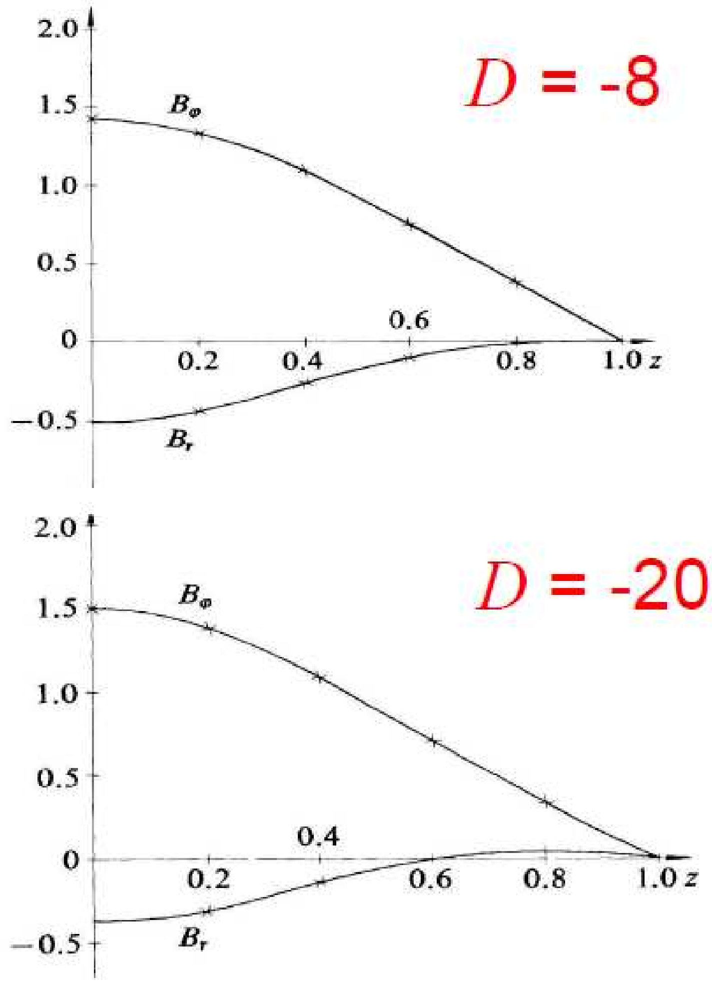
$$\gamma \approx -\frac{\pi^2}{4} + \frac{1}{2}\sqrt{-\pi D} + O(|D|^{3/2}), \quad D < 0.$$

$$\tan p = \frac{B_r}{B_\phi} \approx -\sqrt{\frac{\pi R_\alpha}{4|R_\omega|}}$$

Perturbation solution



Numerical solution

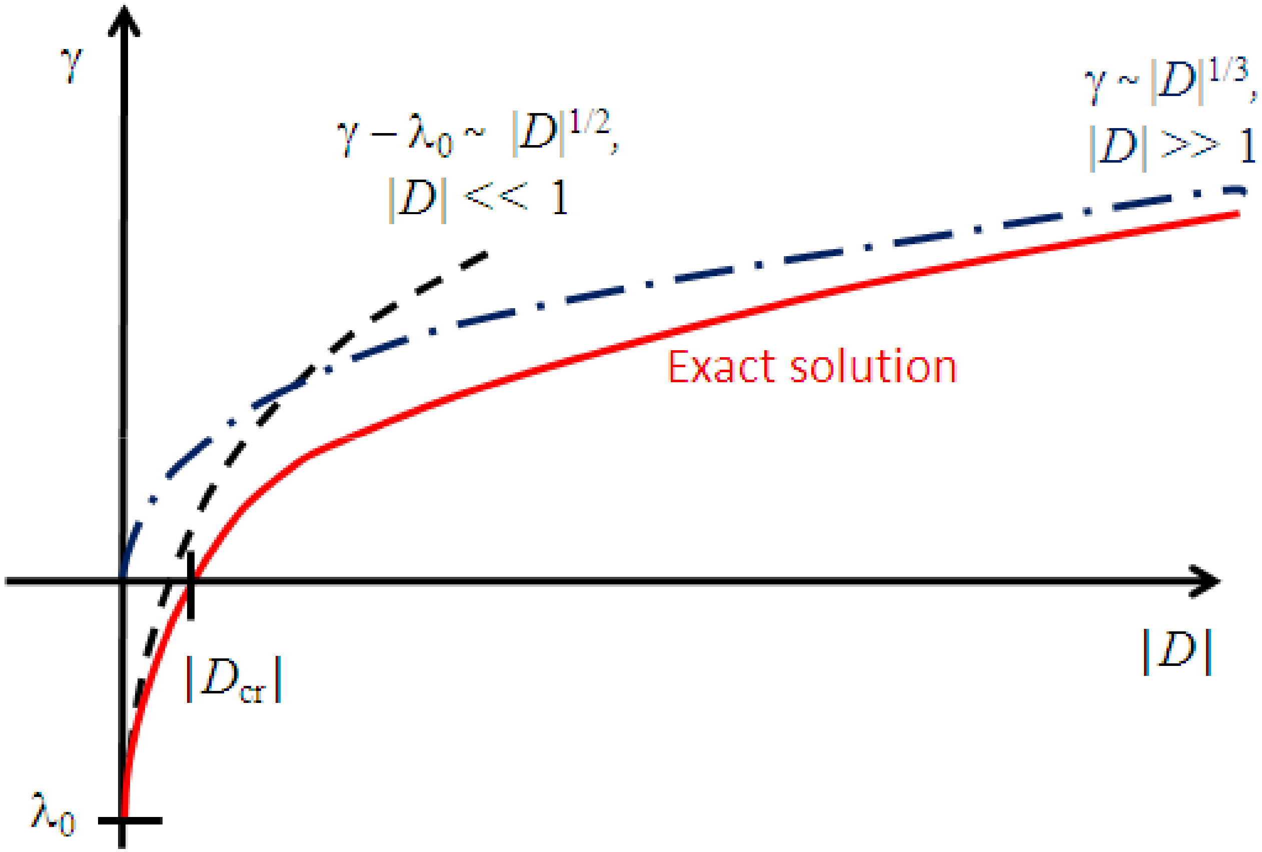


$$\gamma = 0 \quad \Rightarrow \quad D_{\rm cr} \approx -7.8$$

$$D_{\rm cr} \approx -8$$

The asymptotic solution developed for $|D| << 1$ remains applicable at $|D| \cong |D_{cr}| > 1$.

Why?



Asymptotics formally applicable for $|D| << 1$ and $|D| >>1$ happen to be consistent with each other, so both are reasonably accurate for $|D|$ = O(1).

5. Random magnetic fields in the ISM

The fluctuation dynamo in the ISM:

- ★ Dynamo time scale $l_0/v_0 \simeq 10^7$ years.
- ★ Field within the ropes $b_{\max} \simeq \sqrt{4\pi\rho v_0^2} \simeq 5\,\mu$G.
- ★ Ropes of length $l_0 \simeq 100$ pc, thickness $l_B \simeq l_0 R_{\mathrm{m,cr}}^{-1/2} \simeq 15$ pc.
- ★ Volume filling factor $f \simeq \dfrac{l_0 l_B^2}{l_0^3} = R_{\mathrm{m,cr}}^{-1} \simeq 3\%$ assuming that there is one rope per turbulent cell, and $3n\%$ if there are n ropes.
- ★ Other sources of turbulent magnetic fields: tangling of the large-scale magnetic field, compression by shock waves.