MEAN–FIELD DYNAMO THEORY

K.-H. Rädler
Astrophysical Institute Potsdam

- INTRODUCTION
  Electrodynamics of conducting moving matter
  and the kinematic dynamo problem

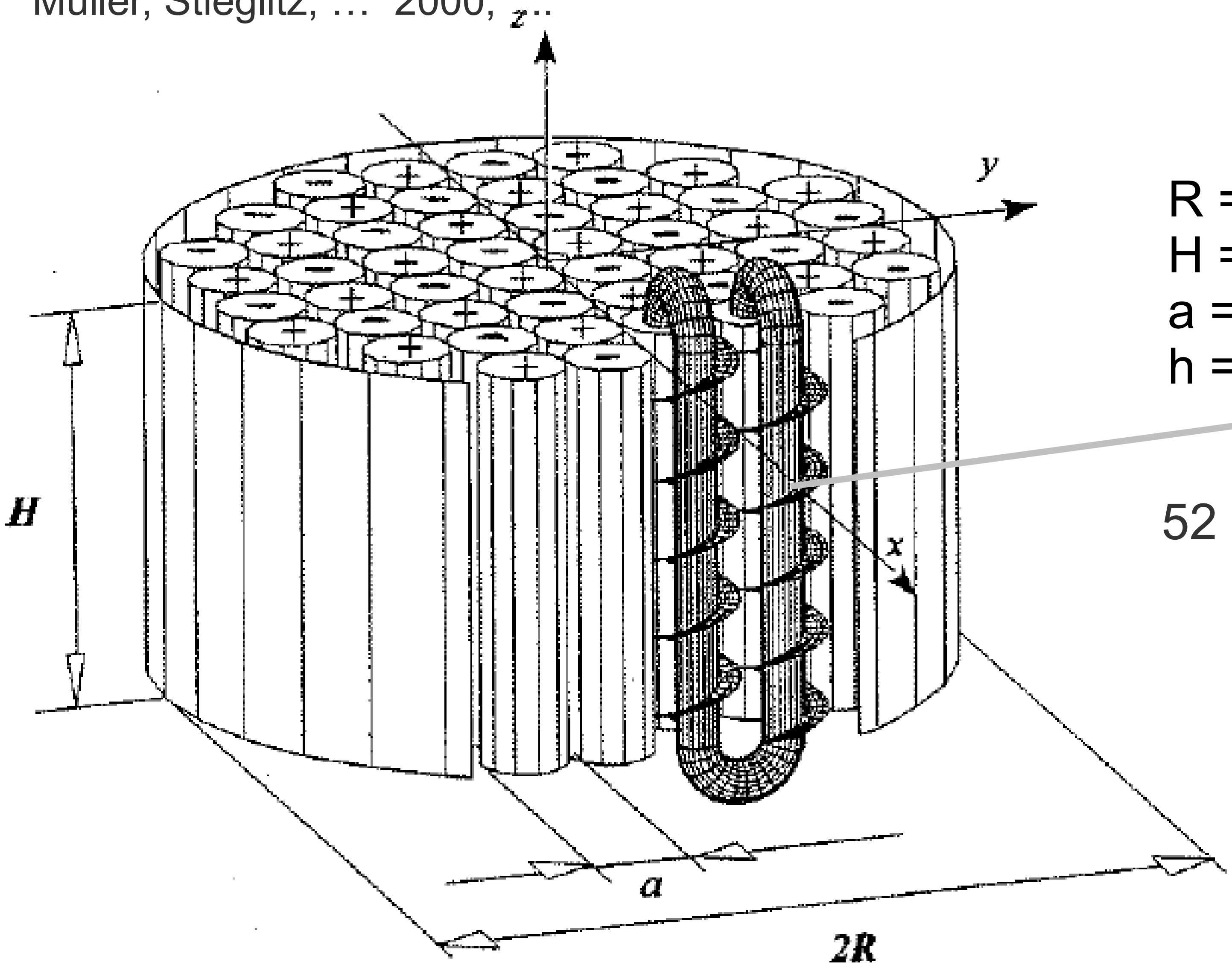
- MEAN–FIELD ELECTRODYNAMICS

- MEAN–FIELD MAGNETOFLUIDDYNAMICS

- MEAN-FIELD ASPECTS OF THE KARLSRUHE DYNAMO

THE KARLSRUHE DYNAMO EXPERIMENT

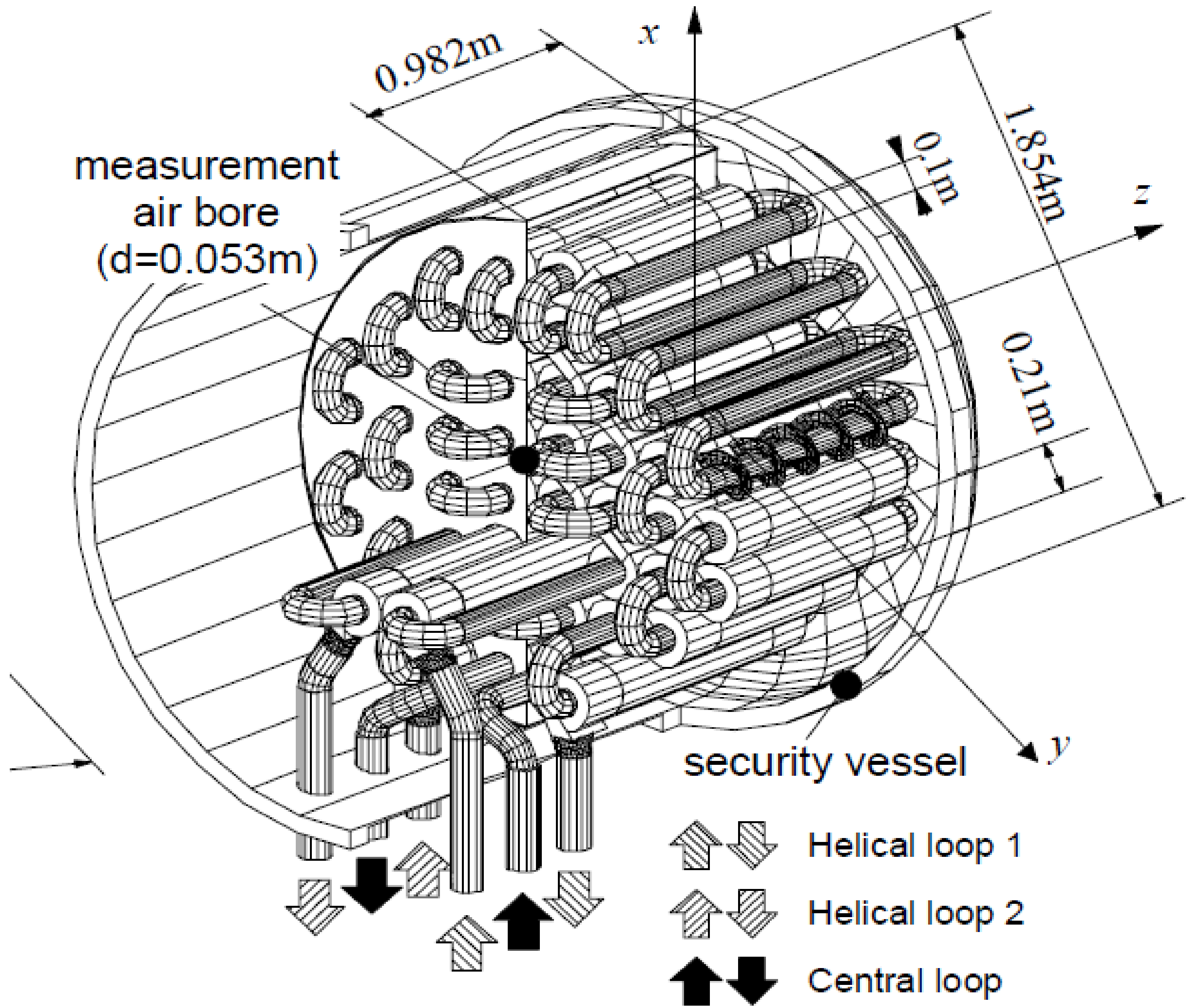
Müller, Stieglitz, … 2000, …



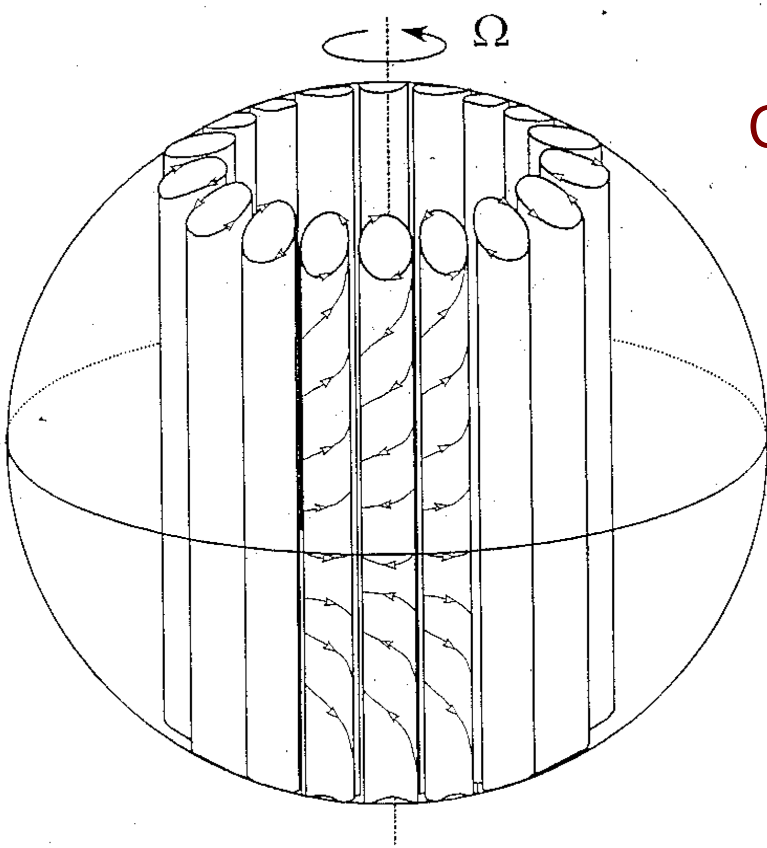
R = 0.85 m
H = 0.71 m
a = 0.21 m
h = 0.19 m - pitch

52 "spin generators"

liquid sodium
η ~ 0.1 m²/s

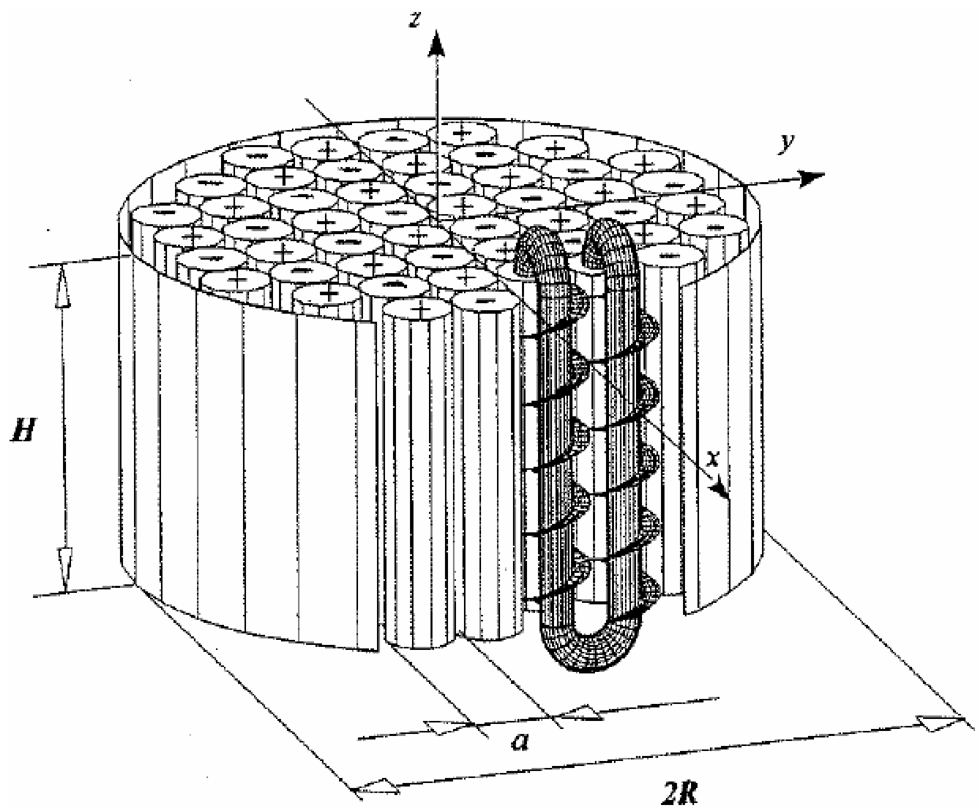


0.982m
x
1.854m
0.1m
z
measurement
air bore
(d=0.053m)
0.21m
y
security vessel
Helical loop 1
Helical loop 2
Central loop

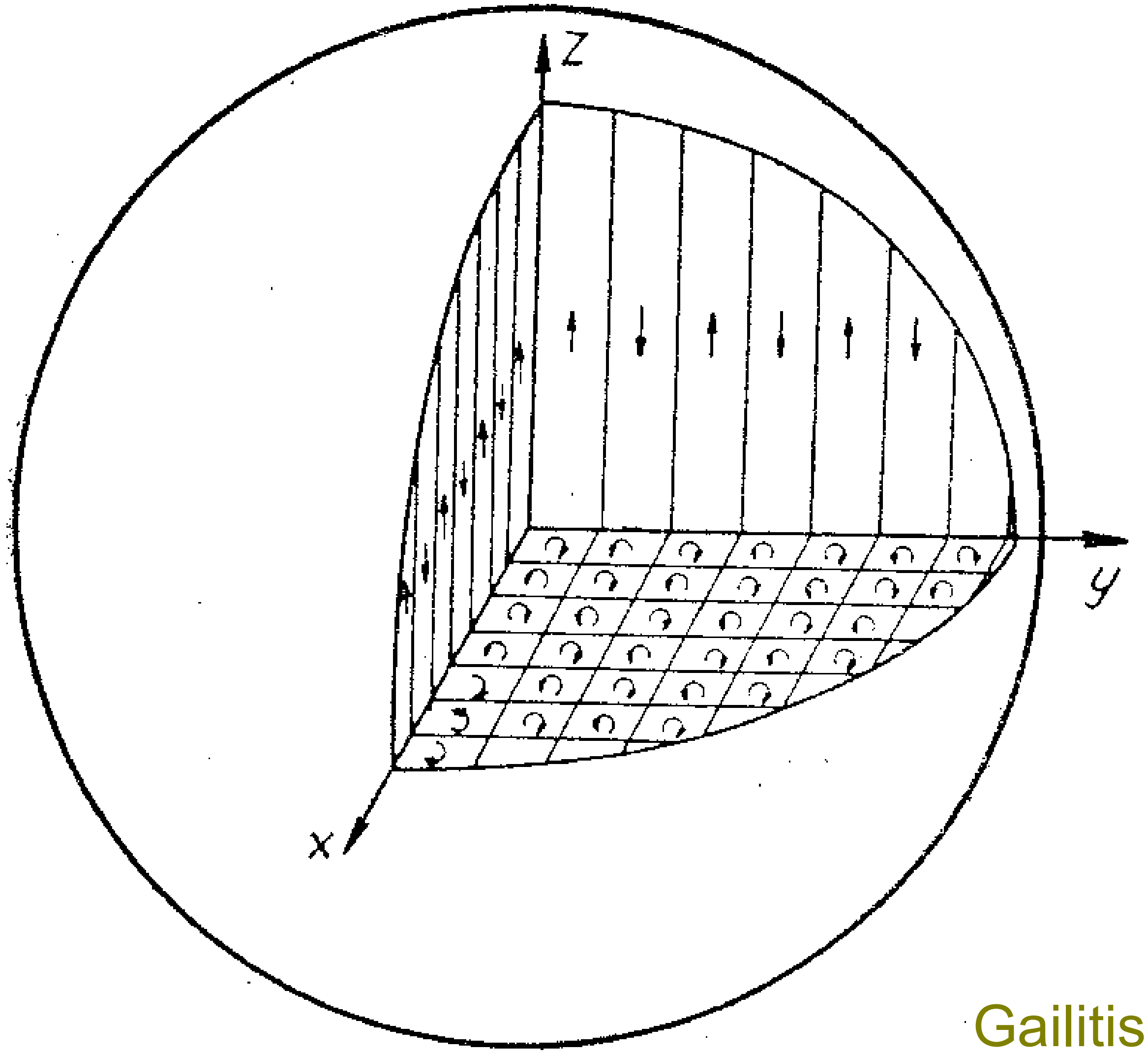


Convection rolls in the Earth's core

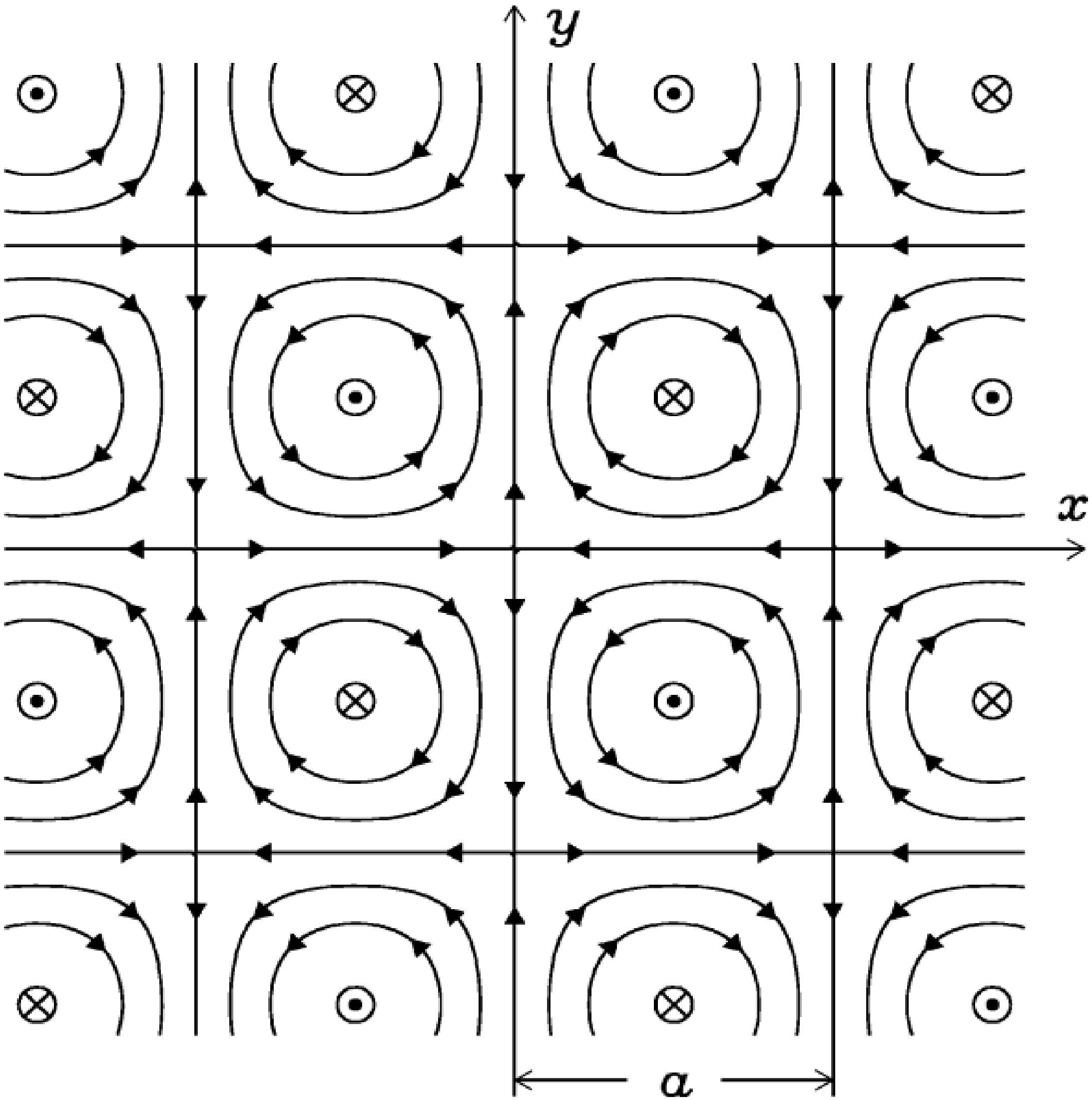
Busse 1970, ...



Experiment proposed by Busse
1975



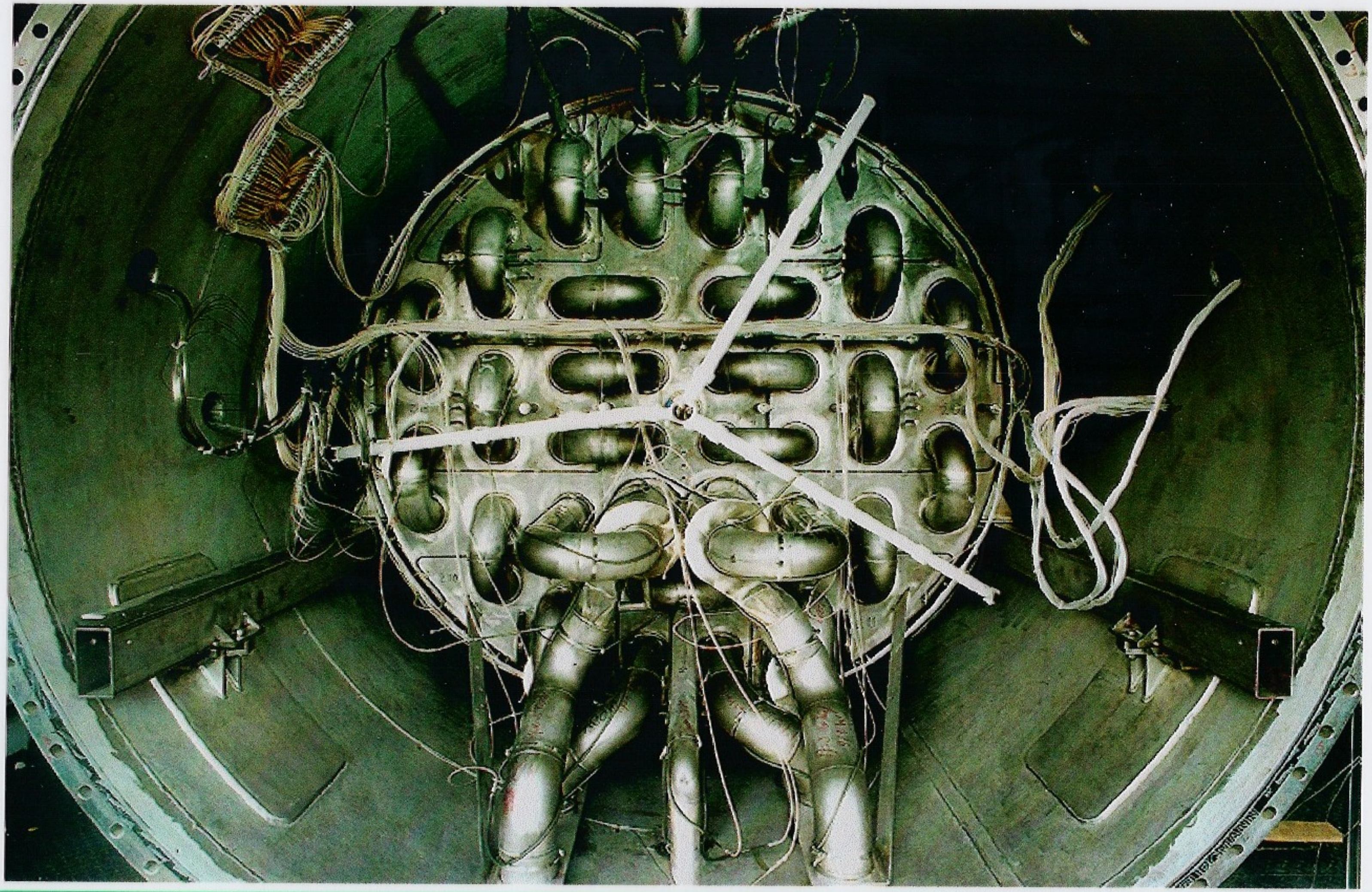
Gailitis 1967



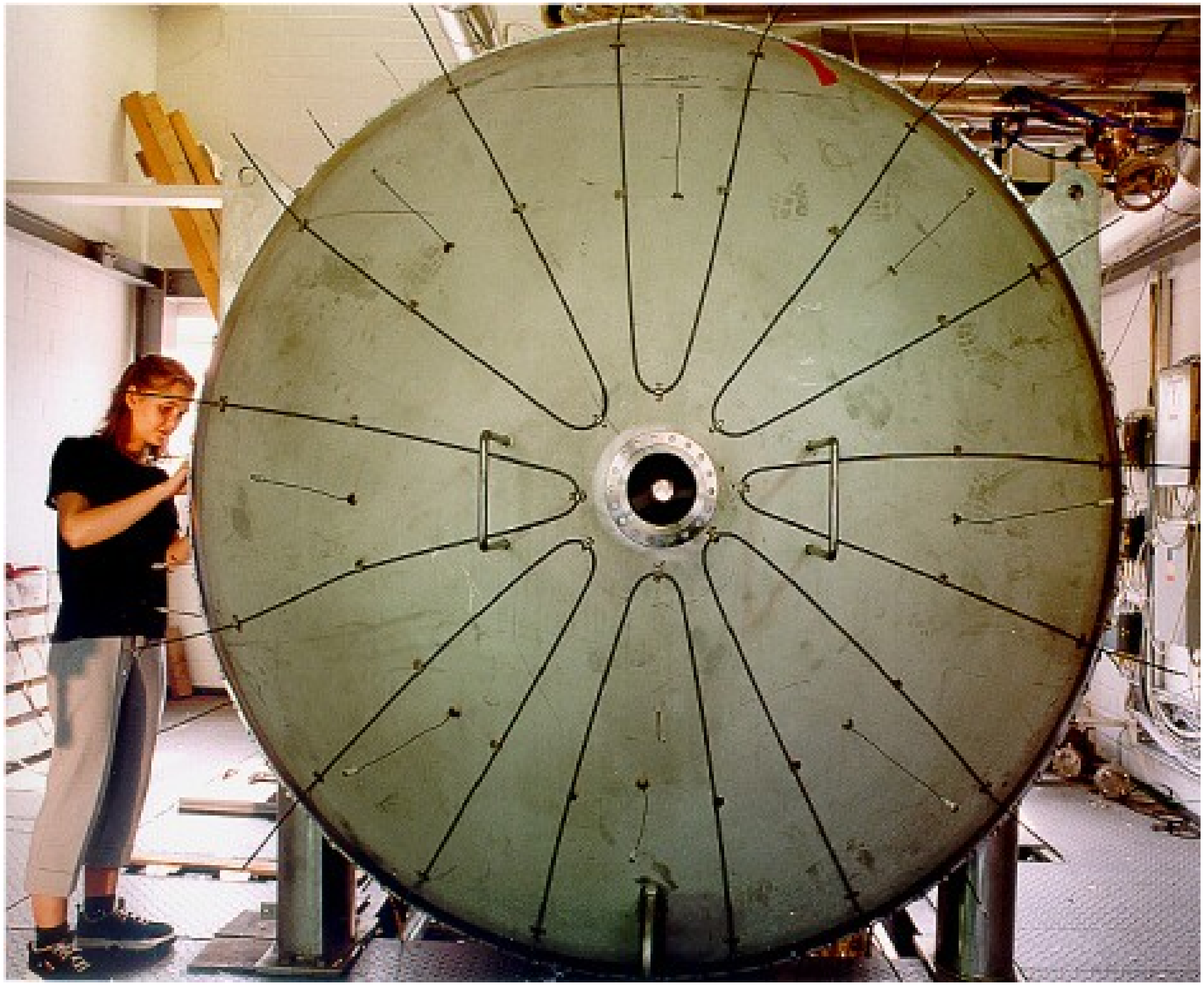
Dynamo action in spatially periodic flows

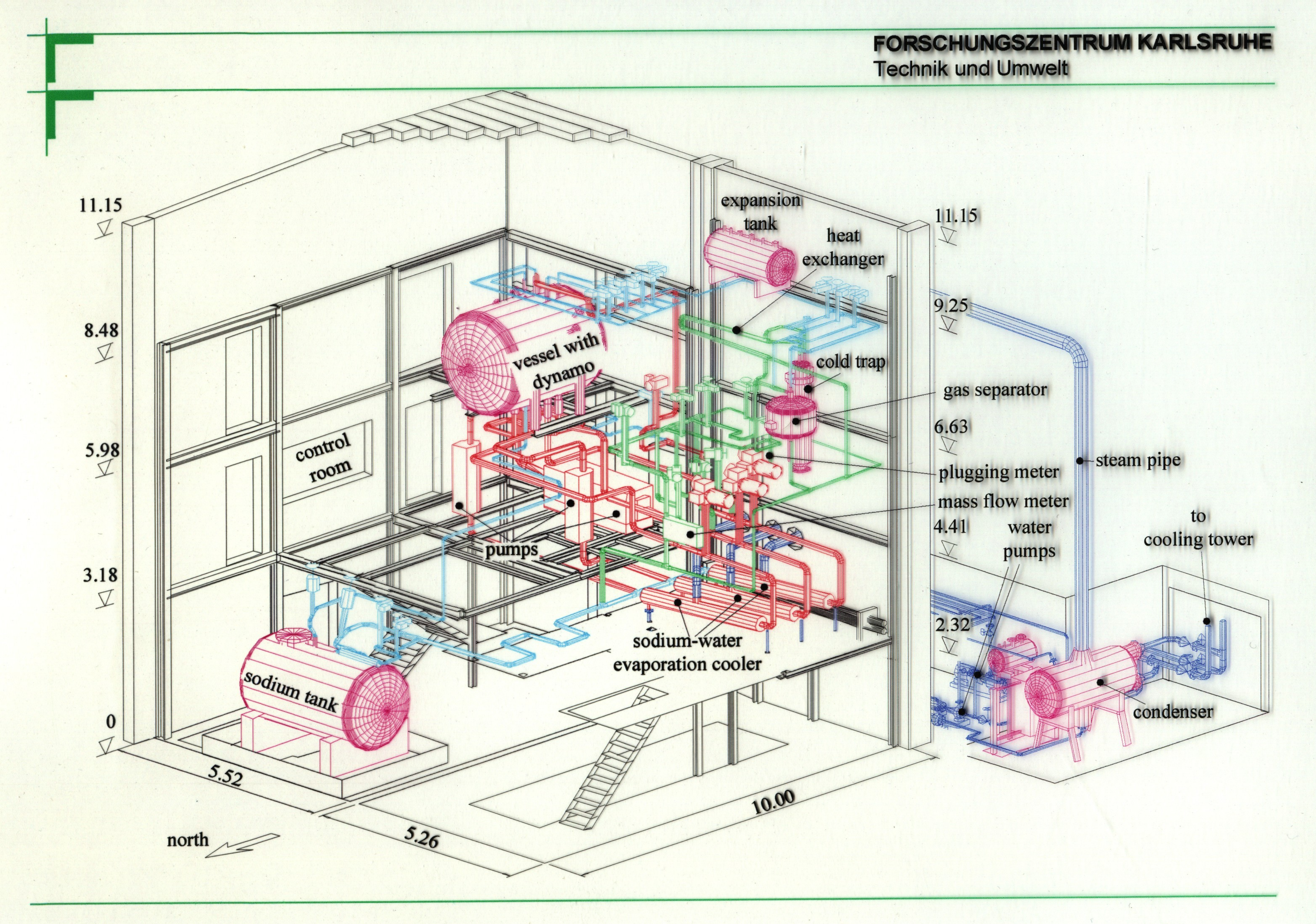
G. O. Roberts 1972

FORSCHUNGSZENTRUM KARLSRUHE
Technik und Umwelt



DYNAMO MODULE





FORSCHUNGSZENTRUM KARLSRUHE
Technik und Umwelt
11.15
8.48
5.98
3.18
0
expansion tank
heat exchanger
11.15
9.25
cold trap
gas separator
6.63
plugging meter
mass flow meter
4.41
water pumps
steam pipe
to cooling tower
2.32
condenser
vessel with dynamo
control room
pumps
sodium-water evaporation cooler
sodium tank
5.52
5.26
10.00
north

Two branches in the theory of the experiment

- Direct nummerical simulations
  Busse, Tilgner 1996 … 2004

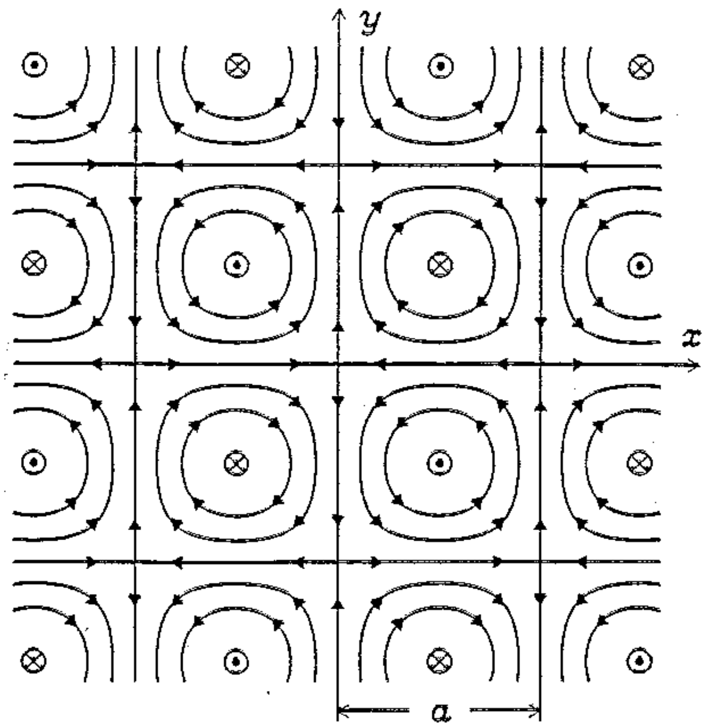
- Investigations based on mean-field electrodynamics
  Apstein, Brandenburg, Fuchs, Rädler,
  Rheinhardt, Schüler 1996 … 2003

Good agreement of the results of both approaches
concerning onset and behavior of the dynamo

The Roberts dynamo

Fluid velocity, e.g.,

$$u_x = -u_\perp(\pi/2)\,\sin(\pi x/a)\,\cos(\pi y/a)$$
$$u_y = \quad u_\perp(\pi/2)\,\cos(\pi x/a)\,\sin(\pi y/a)$$
$$u_z = -\,u_\parallel(\pi/2)\,\sin(\pi x/a)\,\cos(\pi y/a)$$



Non-decaying magnetic field modes

$$\boldsymbol{B} = \Re(\hat{\boldsymbol{B}}(x,y)\exp(\mathrm{i}kz + pt))$$

if $\quad Rm_\perp\, Rm_\parallel\, \phi(Rm_\perp, k) \geq \dfrac{16}{\pi^2}\, a\, k$

$$Rm_\perp = u_\perp a/2\eta\,, \qquad Rm_\parallel = u_\parallel a/\eta$$

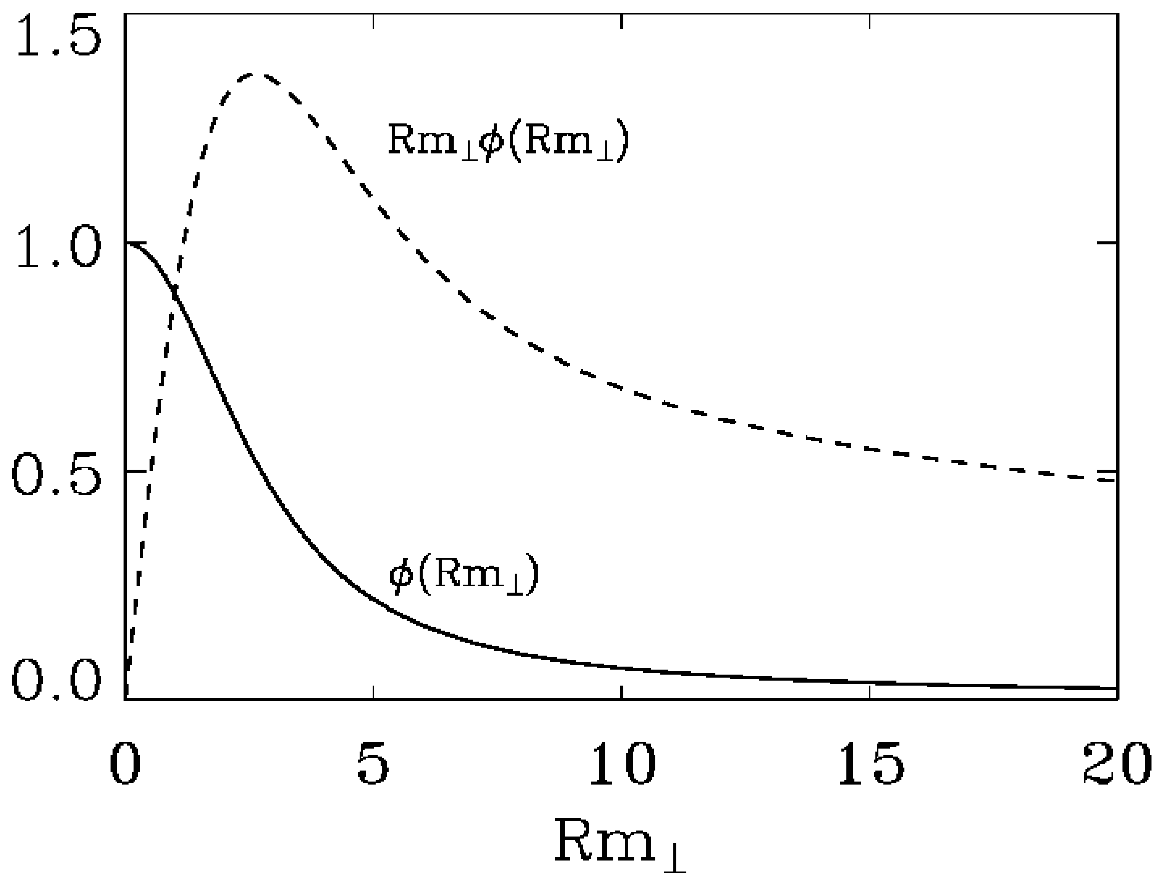
The Roberts dynamo

Non-decaying magnetic modes if

$$Rm_\perp \, Rm_\| \, \phi(Rm_\perp, k) \geq \frac{16}{\pi^2} \, a \, k$$

Most easily excitable modes contain part independent of *x* and *y*

(limit of small *k*)



Mean-field electrodynamics with Roberts-like flows

Start with $\eta\nabla^2\boldsymbol{B}+\nabla\times(\boldsymbol{U}\times\boldsymbol{B})-\partial_t\boldsymbol{B}=\boldsymbol{0}\,,\;\;\nabla\cdot\boldsymbol{B}=0\,,\;\;\nabla\cdot\boldsymbol{U}=0.$

Define mean fields by

$$\overline{F}(x,y,z)=\frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a}F(x-\xi,y-\eta,z)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$

Put $\boldsymbol{B}=\overline{\boldsymbol{B}}+\boldsymbol{b}$ and $\boldsymbol{U}=\overline{\boldsymbol{U}}+\boldsymbol{u}$.

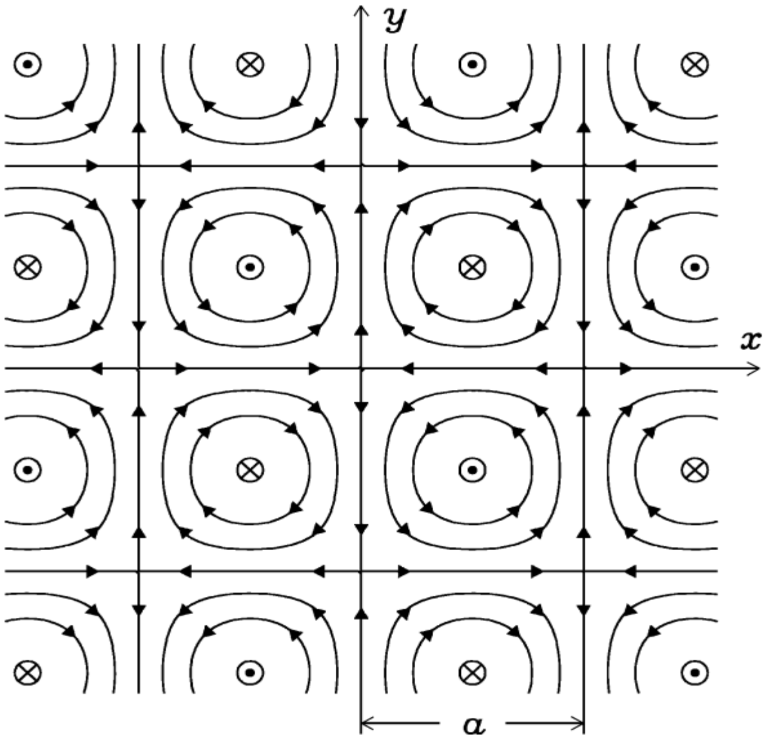
Assume $\overline{\boldsymbol{U}}=\boldsymbol{0}$,

$\boldsymbol{u}$ periodic in x and y,

changes sign if

$x\to x+a$ or $y\to y+a$,

steady.

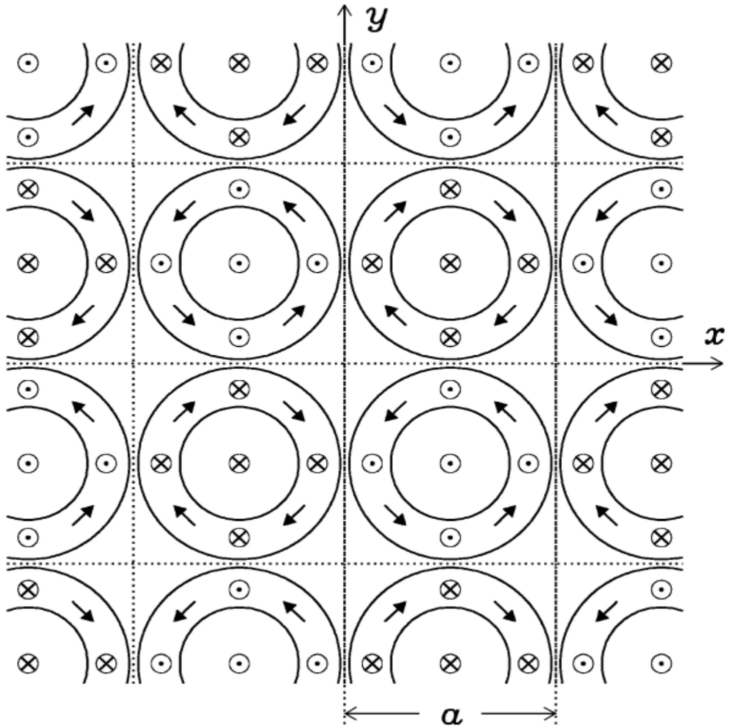


Mean-field electrodynamics with Roberts-like flows

Start with $\eta\nabla^2\boldsymbol{B}+\nabla\times(\boldsymbol{U}\times\boldsymbol{B})-\partial_t\boldsymbol{B}=\boldsymbol{0}\,,\ \ \nabla\cdot\boldsymbol{B}=0\,,\ \ \nabla\cdot\boldsymbol{U}=0.$

Define mean fields by

$$\overline{F}(x,y,z)=\frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a}F(x-\xi,y-\eta,z)\,\mathrm{d}\xi\,\mathrm{d}\eta\,.$$



Put $\boldsymbol{B}=\overline{\boldsymbol{B}}+\boldsymbol{b}$ and $\boldsymbol{U}=\overline{\boldsymbol{U}}+\boldsymbol{u}$.

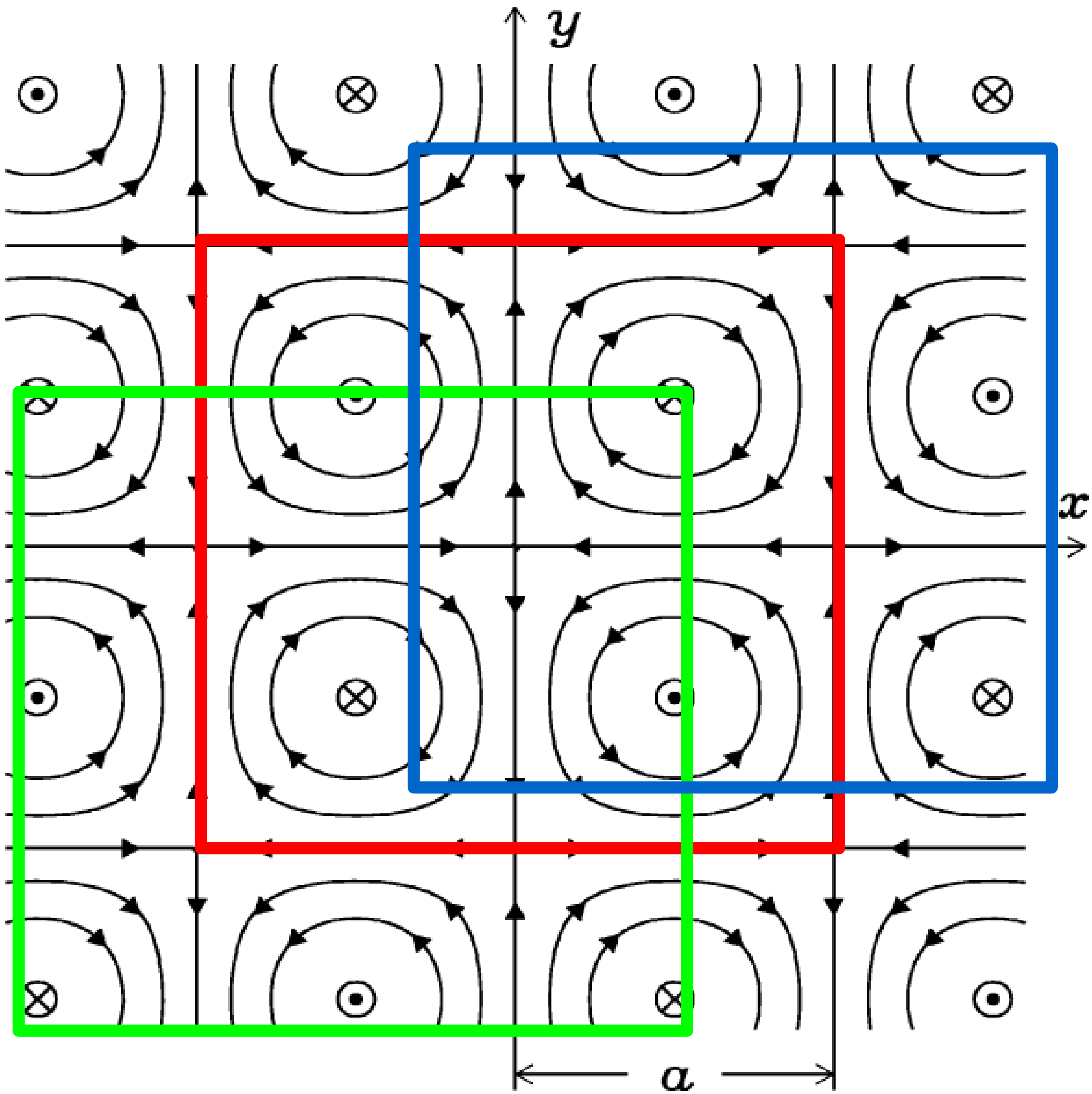
Assume $\overline{\boldsymbol{U}}=\boldsymbol{0}$,

$\boldsymbol{u}$ periodic in x and y,

changes sign if

$x\to x+a$ or $y\to y+a$,

steady.


y
x
a

Mean-field electrodynamics with Roberts-like flows

Mean–field induction equation

$$\eta\nabla^2\overline{\boldsymbol{B}} + \nabla\times\boldsymbol{\mathcal{E}} - \partial_t\overline{\boldsymbol{B}} = \boldsymbol{0}\,,\qquad \nabla\cdot\overline{\boldsymbol{B}} = 0.$$

$$\boldsymbol{\mathcal{E}} = \langle\boldsymbol{u}\times\boldsymbol{b}\rangle$$

Assume $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Definition of the mean fields

$\Rightarrow$ a_{ij} and b_{ijk} independent of x and y

Assumed properties of $\boldsymbol{u}$

$\Rightarrow$ a_{ij} and b_{ijk} axisymmetric tensors w.r.t z axis

Recall:

- Other simple examples:

$\boldsymbol{u}$ axisymmetric turbulence, $\overline{\boldsymbol{U}} = \mathbf{0}$

E.g., inhomogeneous turbulence showing intensity gradient ($\boldsymbol{\nabla}\langle \boldsymbol{u}^2\rangle \neq \mathbf{0}$)

homogeneous turbulence subject to Coriolis force (angular velocity $\boldsymbol{\Omega}$)

homogeneous turbulence influenced by a mean magnetic field ($\overline{\boldsymbol{B}}$)

Assume again $\mathcal{E}_i = a_{ij}\overline{B}_j + b_{ijk}\partial\overline{B}_j/\partial x_k$

Axis of symmetry defined by unit vector $\boldsymbol{e}$

$$
\begin{aligned}
a_{ij} &= a_1\,\delta_{ij} + a_2\,\epsilon_{ijl}\,e_l + a_3\,e_i\,e_j \\
b_{ijk} &= b_1\,\epsilon_{ijk} + b_2\,\delta_{ij}\,e_k + b_3\,\delta_{ik}\,e_j + b_4\,\delta_{jk}\,e_i \\
&\quad + b_5\,\epsilon_{ijl}\,e_k\,e_l + b_6\,\epsilon_{ikl}\,e_j\,e_l + b_7\,\epsilon_{jkl}\,e_i\,e_l + b_8\,e_i\,e_j\,e_k\,.
\end{aligned}
$$

Without loss of generality $b_4 = 0$ and $b_5 = b_6$

$$(\epsilon_{ijl}e_k + \epsilon_{jkl}e_i + \epsilon_{kil}e_j)e_l = \epsilon_{ijk}$$

Recall:

- Other simple examples [2]

...

Change of notation

$a_1 \longrightarrow -\alpha_1\,,\ a_2 \longrightarrow -\alpha_1\,,\ a_3 \longrightarrow -\alpha_2$

$b_1, b_2, b_3, b_5, b_7, b_8 \ \longrightarrow$ combinations of $\beta_1, \beta_1, \beta_1, \kappa_1, \kappa_2, \kappa_3$

$\Rightarrow$

$$\begin{aligned} \boldsymbol{\mathcal{E}} \;=\; & -\alpha_1\,\overline{\boldsymbol{B}} - \alpha_2\,(\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\,\boldsymbol{e} - \gamma\,\boldsymbol{e}\times\overline{\boldsymbol{B}} \\ & -\beta_1\,\boldsymbol{\nabla}\times\overline{\boldsymbol{B}} - \beta_2\,(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}))\boldsymbol{e} - \delta\,\boldsymbol{e}\times(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) \\ & -\kappa_1\,\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})} - \kappa_2\,\boldsymbol{e}\times(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}) - \kappa_3\,(\boldsymbol{e}\cdot(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}))\,\boldsymbol{e} \end{aligned}$$

$$(\boldsymbol{\nabla}\overline{\boldsymbol{B}})^{(\mathrm{s})}_{ij} = \tfrac{1}{2}(\partial\overline{B}_i/\partial x_j + \partial\overline{B}_j/\partial x_i)$$

Mean-field electrodynamics with Roberts-like flows

Under the assumptions made so far

$$\begin{aligned}\boldsymbol{\mathcal{E}} \;=\; & -\alpha_\perp \overline{\boldsymbol{B}} - (\alpha_{||} - \alpha_\perp)(\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\boldsymbol{e} - \gamma \boldsymbol{e}\times\overline{\boldsymbol{B}} \\ & -\beta_\perp(\nabla\times\overline{\boldsymbol{B}}) - (\beta_{||} - \beta_\perp)(\boldsymbol{e}\cdot(\nabla\times\overline{\boldsymbol{B}}))\boldsymbol{e} \\ & \qquad -\beta_3 \boldsymbol{e}\times(\nabla(\boldsymbol{e}\cdot\overline{\boldsymbol{B}}) + (\boldsymbol{e}\cdot\nabla)\overline{\boldsymbol{B}}) \\ & -\delta_1\nabla(\boldsymbol{e}\cdot\overline{\boldsymbol{B}}) - \delta_2(\boldsymbol{e}\cdot\nabla)\overline{\boldsymbol{B}} - \delta_3(\boldsymbol{e}\cdot\nabla(\boldsymbol{e}\cdot\overline{\boldsymbol{B}}))\boldsymbol{e}\end{aligned}$$

Assume that $\boldsymbol{u}$ independent of z

$$\Rightarrow \alpha_{||} = \gamma = \delta_1 = \delta_2 = \delta_3 = 0$$

$$\begin{aligned}\Rightarrow \boldsymbol{\mathcal{E}} \;=\; & -\alpha_\perp(\overline{\boldsymbol{B}} - (\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\boldsymbol{e}) \\ & -\beta_\perp(\nabla\times\overline{\boldsymbol{B}}) - (\beta_{||} - \beta_\perp)(\boldsymbol{e}\cdot(\nabla\times\overline{\boldsymbol{B}}))\boldsymbol{e} \\ & \qquad -\beta_3 \boldsymbol{e}\times(\nabla(\boldsymbol{e}\cdot\overline{\boldsymbol{B}}) + (\boldsymbol{e}\cdot\nabla)\overline{\boldsymbol{B}})\end{aligned}$$

Mean-field electrodynamics with Roberts flows

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} \;=\; & -\alpha_\perp(\overline{\boldsymbol{B}} - (\boldsymbol{e}\cdot\overline{\boldsymbol{B}})\boldsymbol{e}) \\
& -\beta_\perp(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}) - (\beta_{||}-\beta_\perp)(\boldsymbol{e}\cdot(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}}))\boldsymbol{e} \\
& \qquad -\beta_3\boldsymbol{e}\times(\boldsymbol{\nabla}(\boldsymbol{e}\cdot\overline{\boldsymbol{B}}) + (\boldsymbol{e}\cdot\boldsymbol{\nabla})\overline{\boldsymbol{B}})
\end{aligned}
$$

$$
\boldsymbol{\mathcal{E}} = -\boldsymbol{\alpha}\cdot\overline{\boldsymbol{B}} + \cdots
$$

$$
\boldsymbol{\alpha} = \begin{pmatrix} \alpha_\perp & 0 & 0 \\ 0 & \alpha_\perp & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

If $\overline{\boldsymbol{B}} = \overline{\boldsymbol{B}}(z\, only)$ then

$$
\boldsymbol{\mathcal{E}}_\perp = -\alpha_\perp\overline{\boldsymbol{B}}_\perp - (\beta_\perp + \beta_3)(\boldsymbol{\nabla}\times\overline{\boldsymbol{B}})_\perp
$$

$$
\boldsymbol{\mathcal{E}}_{||} = \boldsymbol{0}
$$

Mean-field dynamo models with Roberts-like flows

$$\eta \nabla^2 \overline{\boldsymbol{B}} - \alpha_\perp \nabla \times (\overline{\boldsymbol{B}} - (\boldsymbol{e} \cdot \overline{\boldsymbol{B}})\boldsymbol{e}) - \partial_t \overline{\boldsymbol{B}} = 0\,, \qquad \nabla \cdot \overline{\boldsymbol{B}} = 0$$

$C = \dfrac{\alpha_\perp R}{\eta}$ dimensionless measure of α effect

C^* marginal value of C

$\xi = \sigma_{\text{ext}}/\sigma$

α-effect region	shape of dynamo module	conductivity in outer space	symmetry of $\overline{\mathbf{B}}$	C^*
all infinite space	cylinder with $R/H = z_1/\pi$ = 1.21		$m = 0$	7.66
infinite slab		$\xi = 0$ $\xi = 1$	$m = 0$	9.21 6.44
infinite cylinder		$\xi = 0$ $\xi = 1$	$m = 0$	7.2 6.1
sphere	sphere	$\xi = 0$	$m = 0$ AS $m = 0$ SA $m = 1$ $m = 2$	8.35 8.34 6.09 8.80

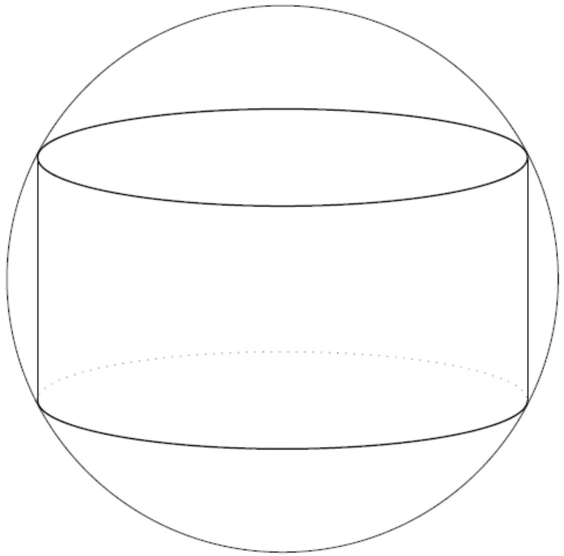
Gailitis 1967

Mean-field dynamo models with Roberts-like flows

$$\eta \nabla^2 \overline{\boldsymbol{B}} - \alpha_\perp \nabla \times (\overline{\boldsymbol{B}} - (\boldsymbol{e} \cdot \overline{\boldsymbol{B}})\boldsymbol{e}) - \partial_t \overline{\boldsymbol{B}} = \boldsymbol{0}\,, \qquad \nabla \cdot \overline{\boldsymbol{B}} = 0$$

$C = \dfrac{\alpha_\perp R}{\eta}$ dimensionless measure of α effect

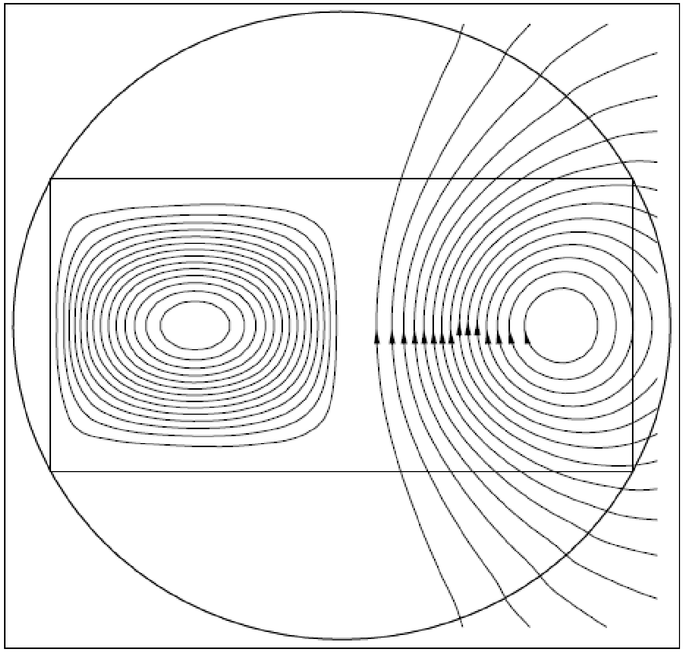
C^* marginal value of C



(i) – $R/H = 1$

(ii) – $R/H = 1.21$

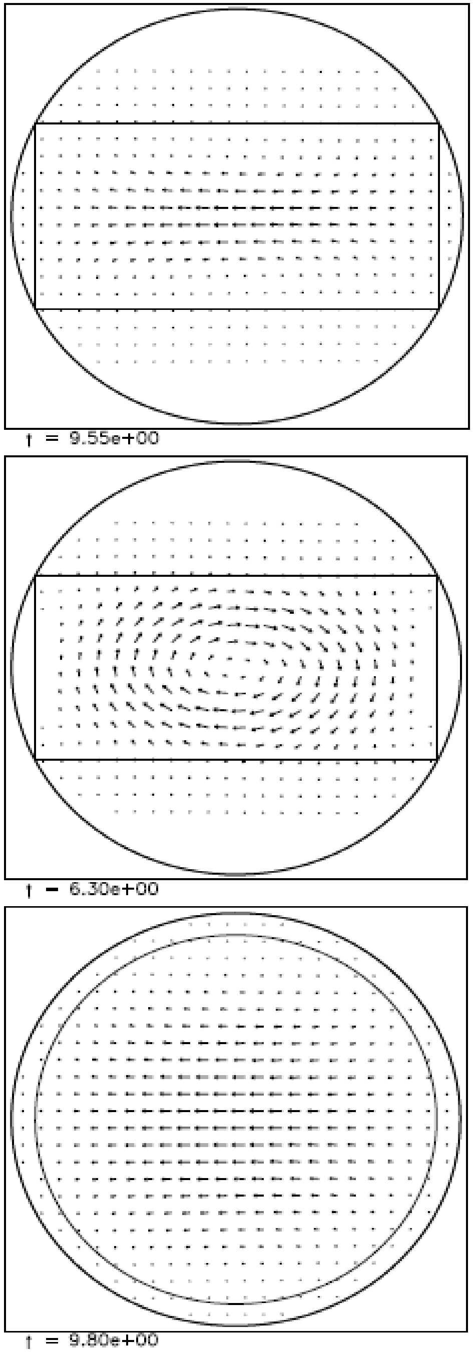
model	$m=0$ AS	$m=0$ SA	$m=1$	$m=2$
(i) $\xi = 1$	8.22	8.46	6.41	8.62
(i) $\xi = 0.01$	8.64	9.18	7.70	9.67
(i) $\xi = 0.001$	9.02	9.60	8.12	10.12
(ii)	8.55	8.55	6.28	8.55

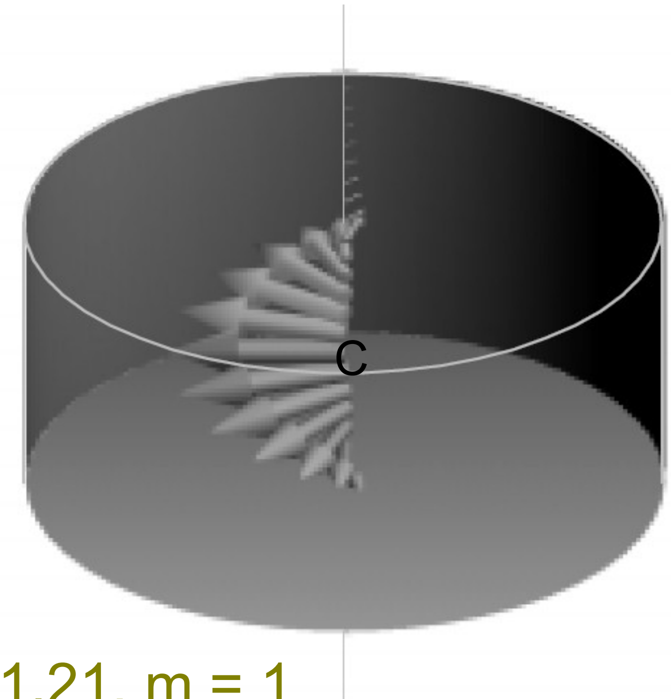


AS m = 0

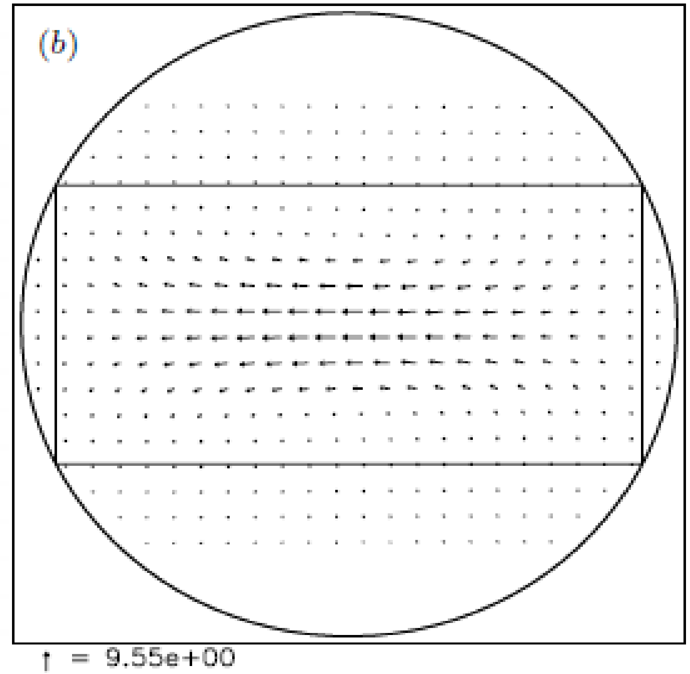
R/H = 1.21

m = 1



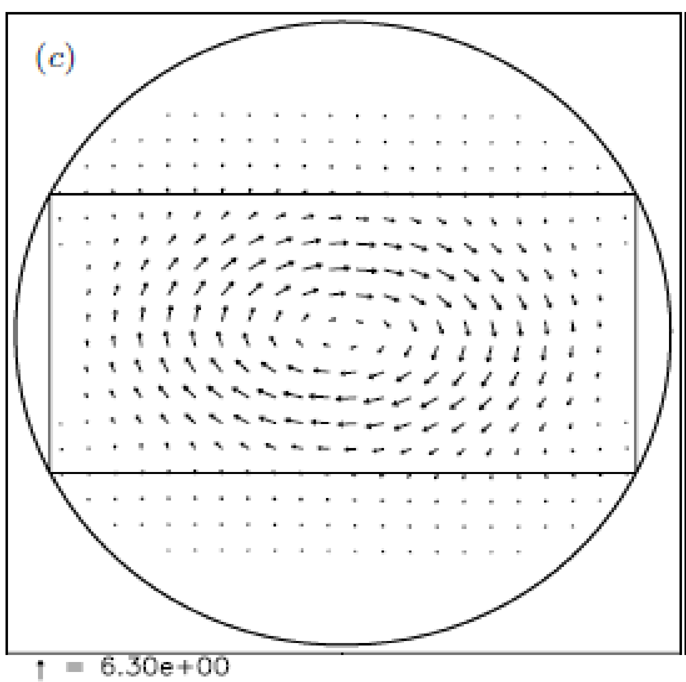


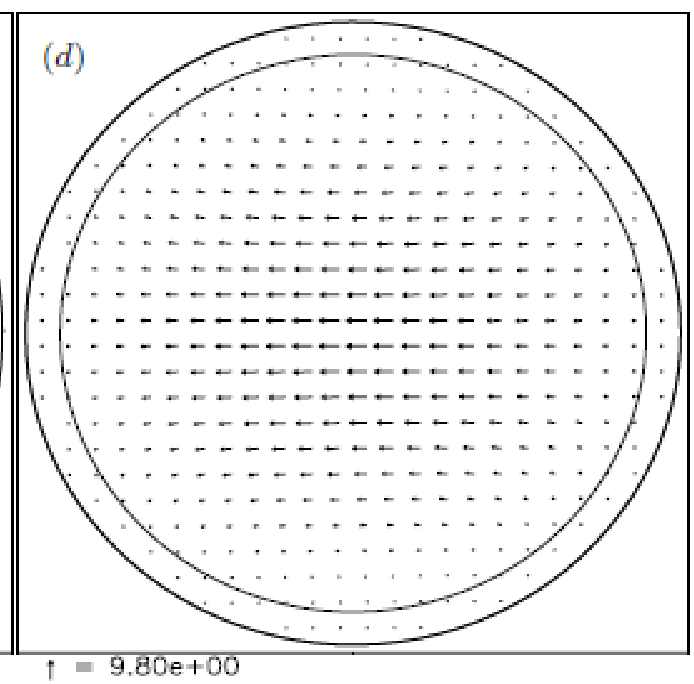
R/H = 1.21, m = 1



xz plane

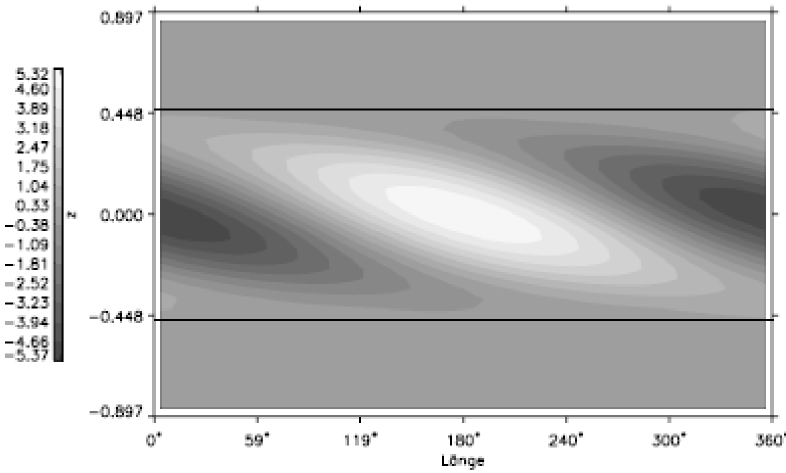
yz plane

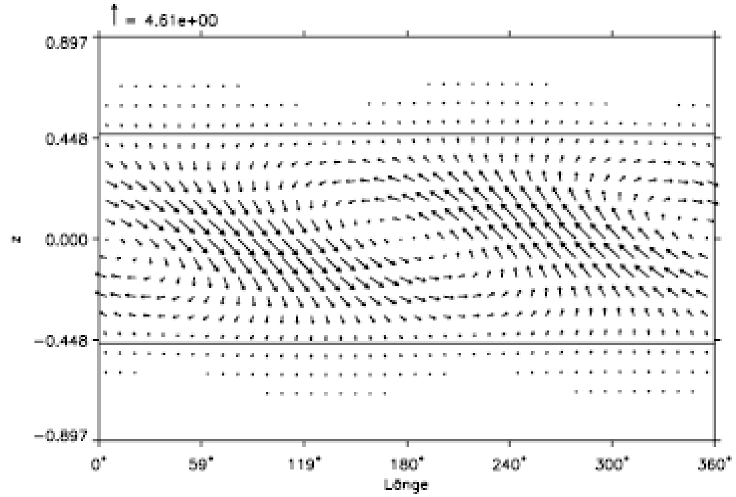




xy plane

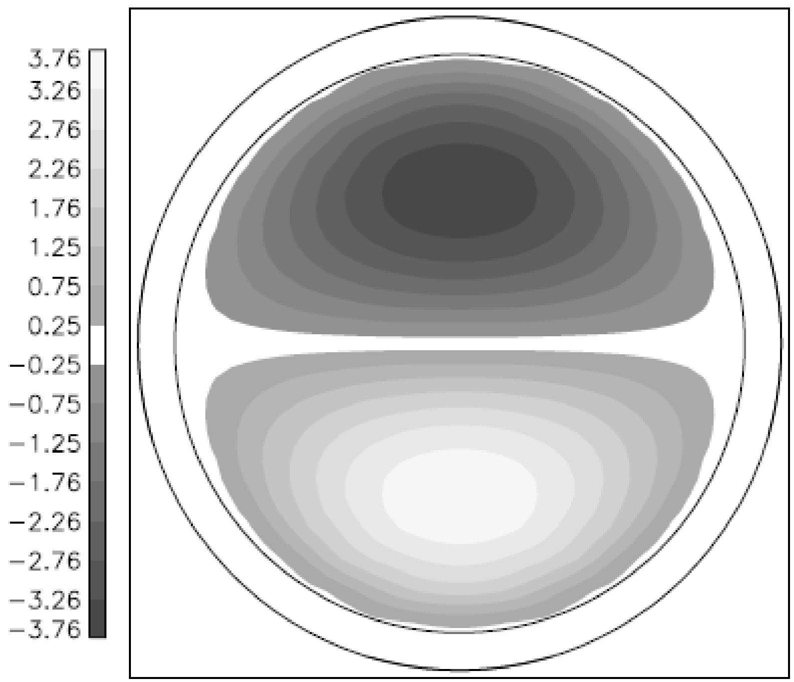
Br
r/R=0.54

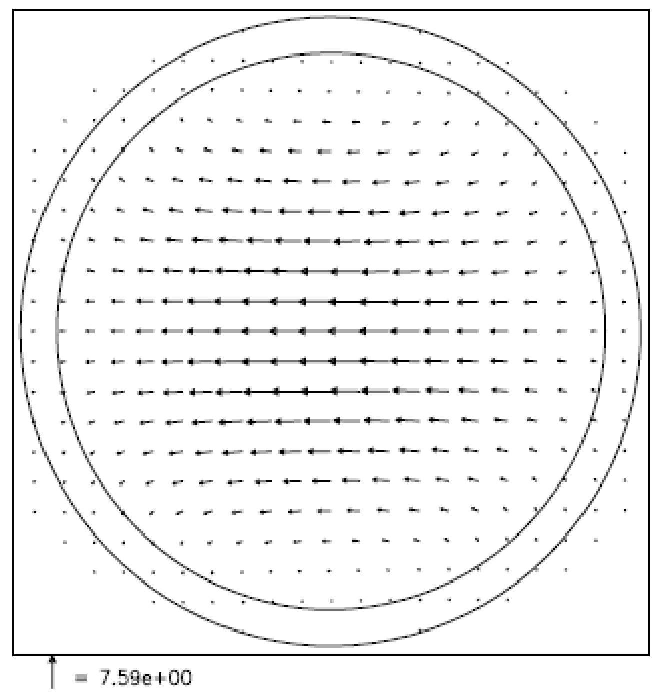




Bz,Bϕ
r/R=0.54

Bz
xy plane





Bx,By
xy plane

R/H = 1.21

Dependence of the *α*-effect on the flow rates

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle = -\alpha_\perp (\overline{\boldsymbol{B}} - (\boldsymbol{e} \cdot \overline{\boldsymbol{B}})\boldsymbol{e}) + \cdots$$

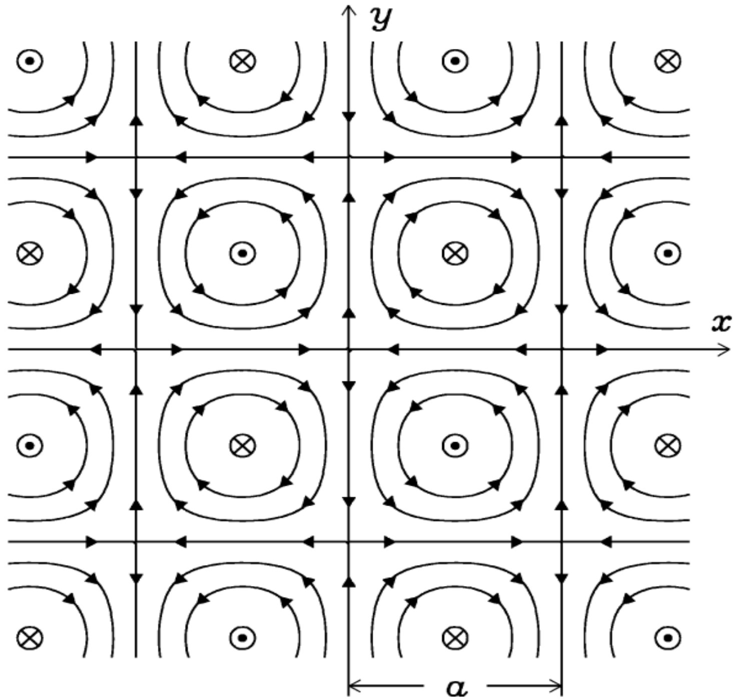
$$\eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b})' - \partial_t \boldsymbol{b} = -\nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}), \quad \nabla \cdot \boldsymbol{b} = 0$$

Roberts flow

$$u_x = -u_\perp (\pi/2) \sin(\pi x/a) \cos(\pi y/a)$$

$$u_y = u_\perp (\pi/2) \cos(\pi x/a) \sin(\pi y/a)$$

$$u_z = -u_\parallel (\pi/2) \sin(\pi x/a) \cos(\pi y/a)$$



Dependence of the *α*-effect on the flow rates

$$\boldsymbol{\mathcal{E}} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle = -\alpha_\perp (\overline{\boldsymbol{B}} - (\boldsymbol{e} \cdot \overline{\boldsymbol{B}})\boldsymbol{e}) + \cdots$$

$$\eta \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b})' - \partial_t \boldsymbol{b} = -\nabla \times (\boldsymbol{u} \times \overline{\boldsymbol{B}}), \quad \nabla \cdot \boldsymbol{b} = 0$$

Roberts flow

$$u_x = -u_\perp (\pi/2) \sin(\pi x/a) \cos(\pi y/a)$$
$$u_y = u_\perp (\pi/2) \cos(\pi x/a) \sin(\pi y/a)$$
$$u_z = -u_\parallel (\pi/2) \sin(\pi x/a) \cos(\pi y/a)$$

Second-order correlation approximation

$$\alpha_\perp = \frac{\pi^2 a}{32\eta} u_\perp u_\parallel = \frac{\pi^2 \eta}{16 a} Rm_\perp Rm_\parallel = \frac{\pi^2}{16} \frac{V_\perp V_\parallel}{a^2 h \eta}$$

$$Rm_\perp = \frac{u_\perp a}{2\eta} \quad Rm_\parallel = \frac{u_\parallel a}{\eta}$$

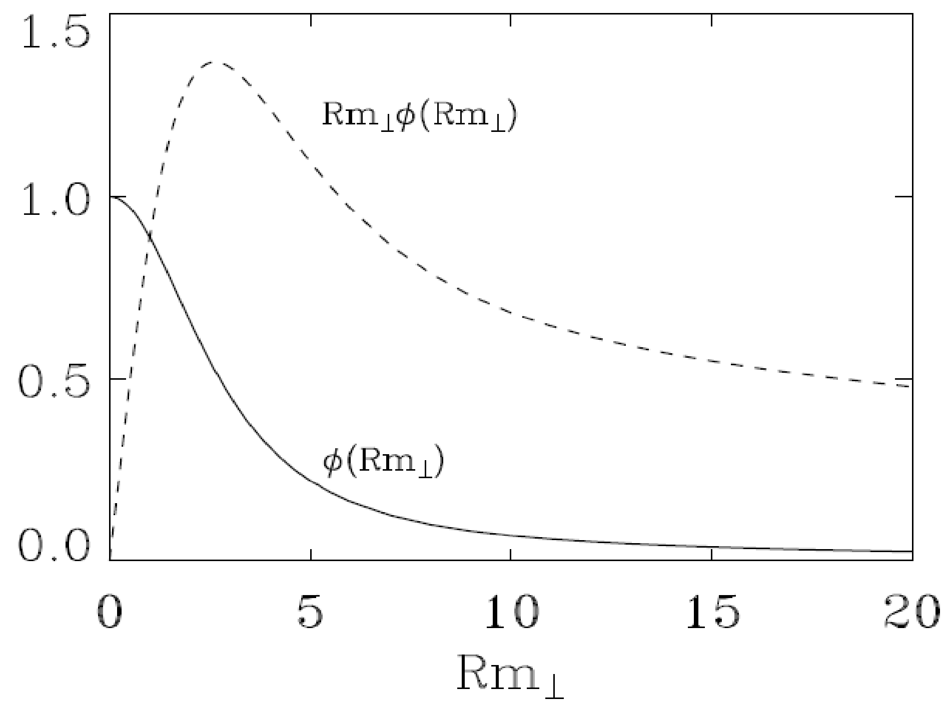
$$V_\perp = \frac{ah}{2} u_\perp \quad V_\parallel = a^2 u_\parallel$$

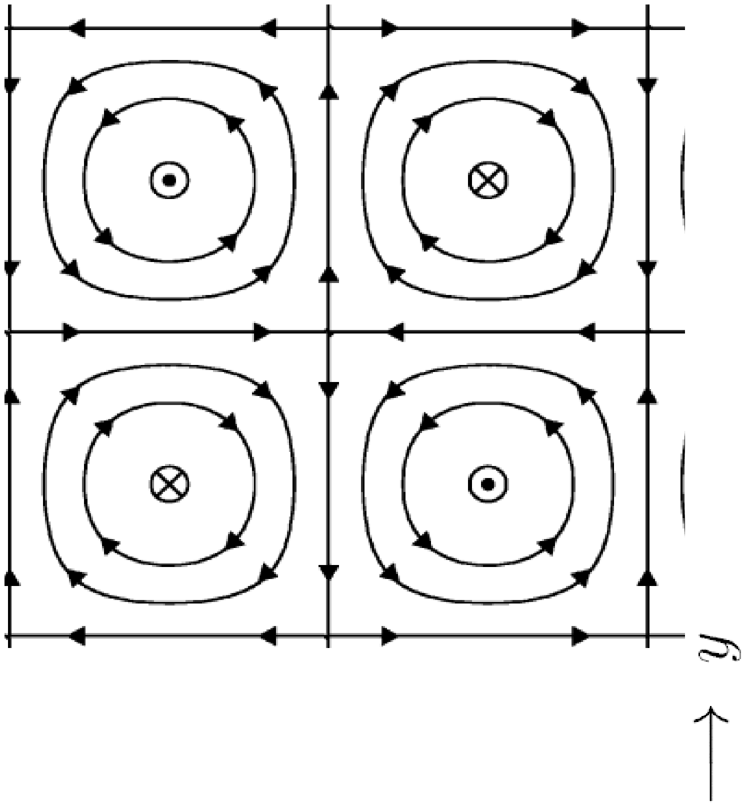
Dependence of the *α*-effect on the flow rates

Roberts flow

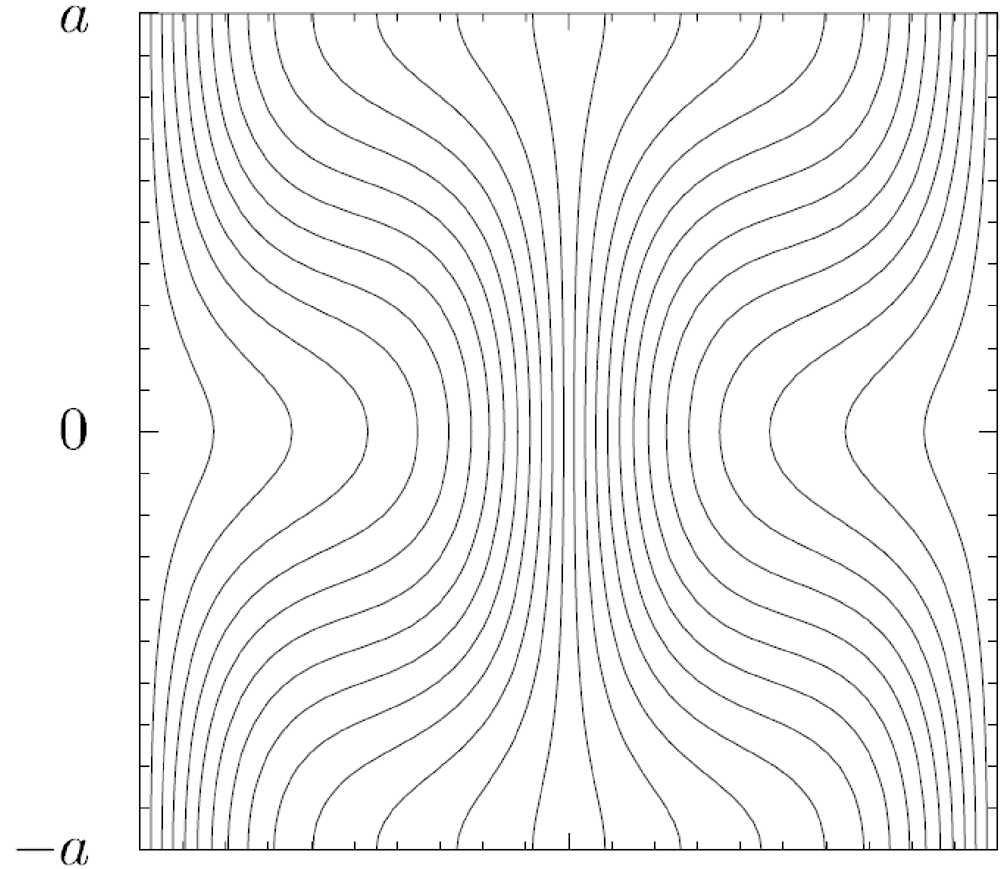
General result (for arbitrary *Rm*)

$$\alpha_\perp = \frac{\pi^2 a}{32\eta} u_\perp u_\| \phi(\frac{u_\perp a}{2\eta})$$
$$= \frac{\pi^2 \eta}{16 a} Rm_\perp Rm_\| \phi(Rm_\perp)$$
$$= \frac{\pi^2}{16} \frac{V_\perp V_\|}{a^2 h\eta} \phi(\frac{V_\perp}{h\eta})$$



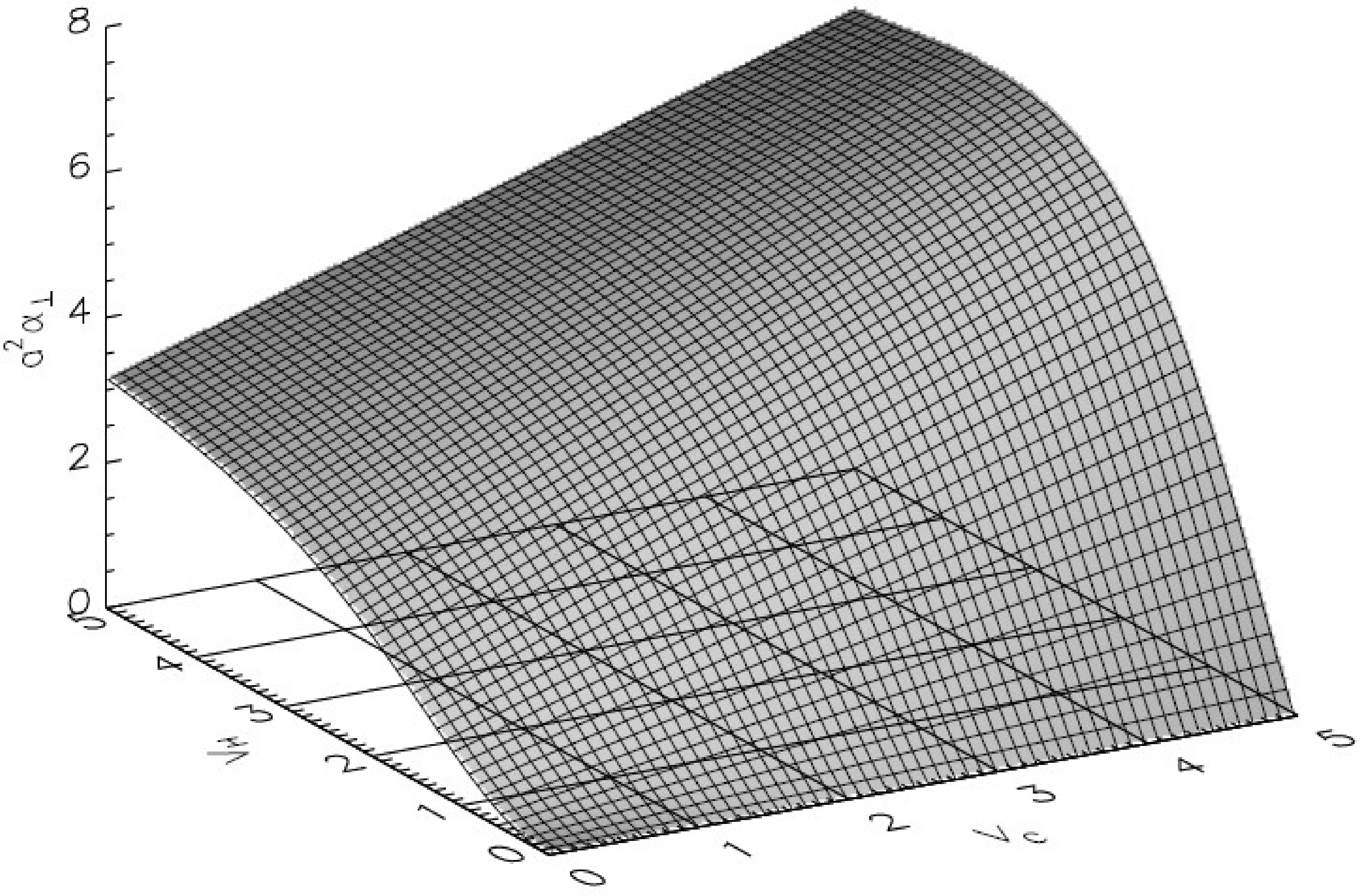


$$\mathrm{Rm}_\perp = 3$$



Dependence of the *α*-effect on the flow rates

Roberts flow

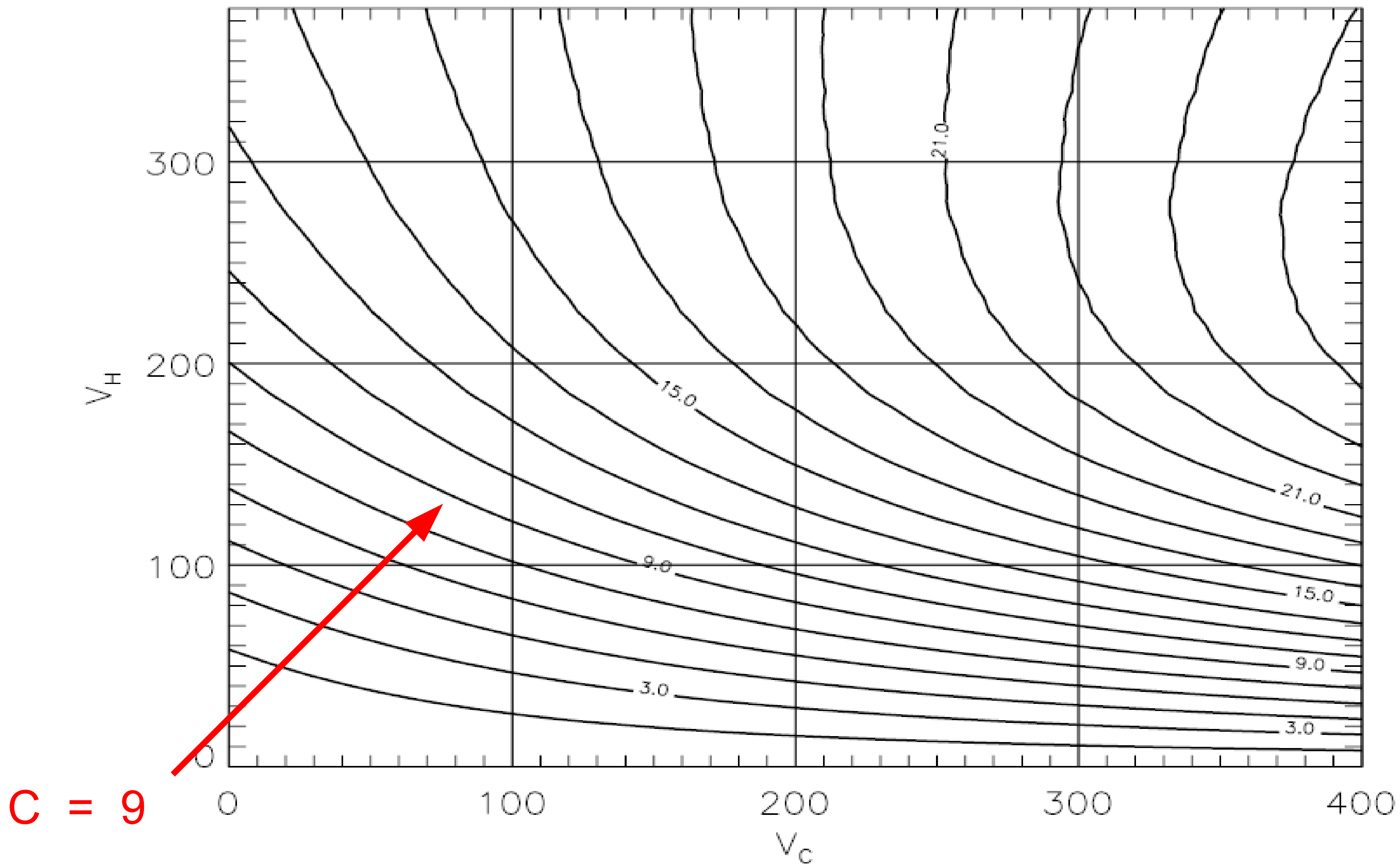


All quantities in units of *aη*, *h/a = 0.905*

Dependence of the *α*-effect on the flow rates

Roberts flow

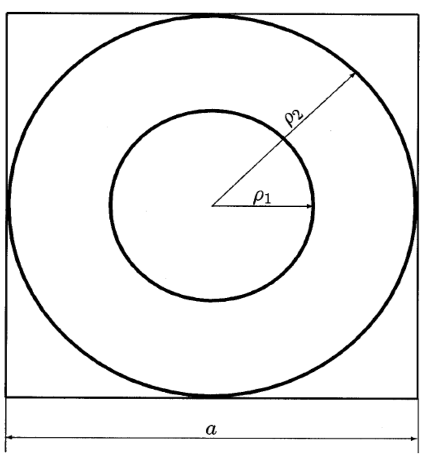
Isolines of *C*

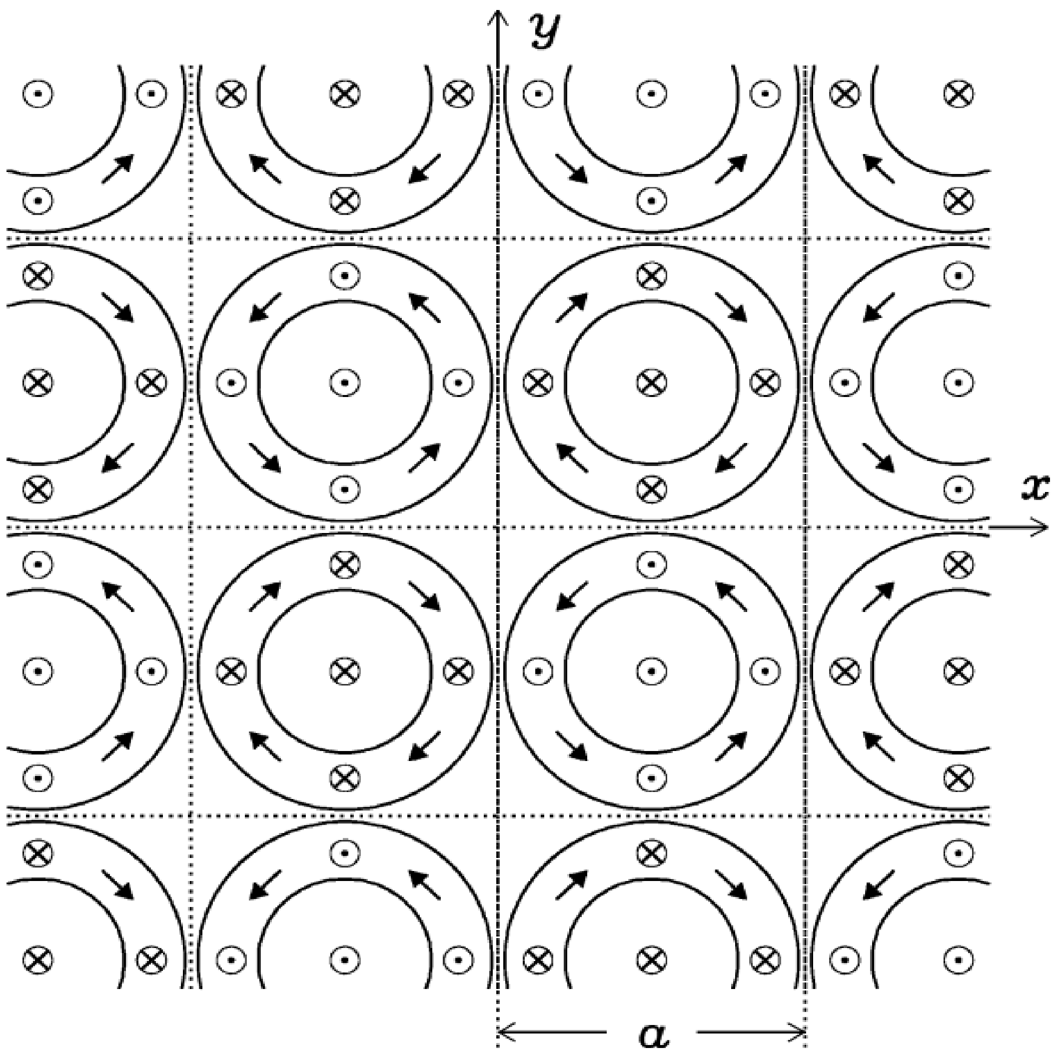


V_H and V_C in m^3 /h

Dependence of the α-effect on the flow rates

Spin generator flow





Dependence of the *α*-effect on the flow rates

Spin generator flow

Second-order correlation approximation

$$\begin{aligned}
\alpha_\perp &= \frac{a}{2\eta} u_\perp \left(u_{\|\mathrm{C}} + \frac{1}{2} u_{\|\mathrm{H}}\right) \\
&= \frac{\eta}{a} Rm_\perp \left(Rm_{\|\mathrm{C}} + \frac{1}{2} Rm_{\|\mathrm{H}}\right) \\
&= \frac{1}{a^2 h \eta} V_\mathrm{H} \left(V_\mathrm{C} + \frac{1}{2} V_\mathrm{H}\right)
\end{aligned}$$

$$Rm_\perp = \frac{u_\perp a}{2\eta} \quad Rm_{\|\mathrm{C}} = \frac{u_{\|\mathrm{C}}\, a}{\eta} \quad Rm_{\|\mathrm{H}} = \frac{u_{\|\mathrm{H}}\, a}{\eta}$$

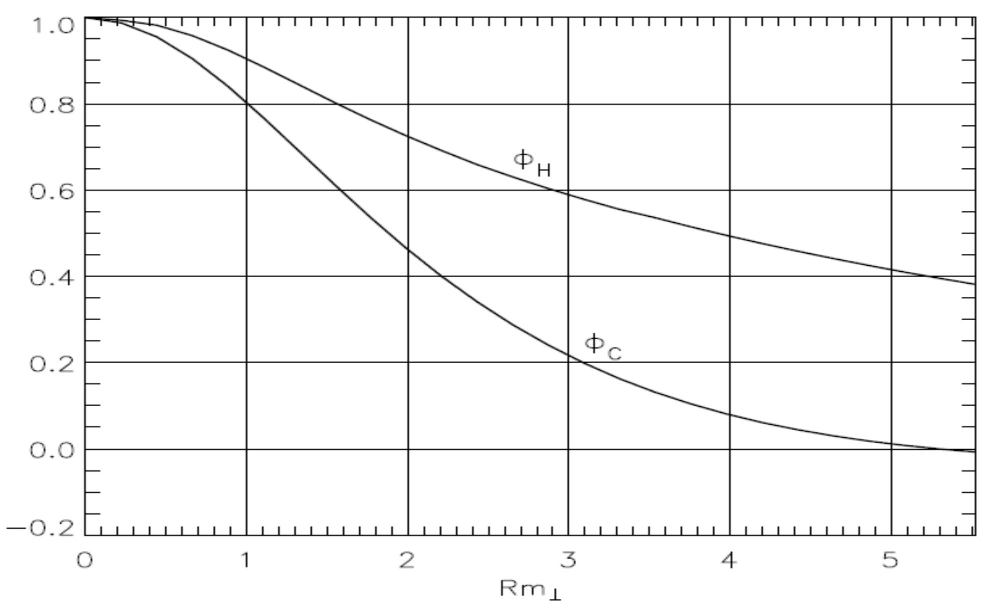
$$V_\mathrm{H} = \frac{ah}{2} u_\perp = a^2 u_{\|\mathrm{H}} \quad V_\mathrm{C} = a^2 u_{\|\mathrm{C}}$$

Dependence of the *α*-effect on the flow rates

Spin generator flow

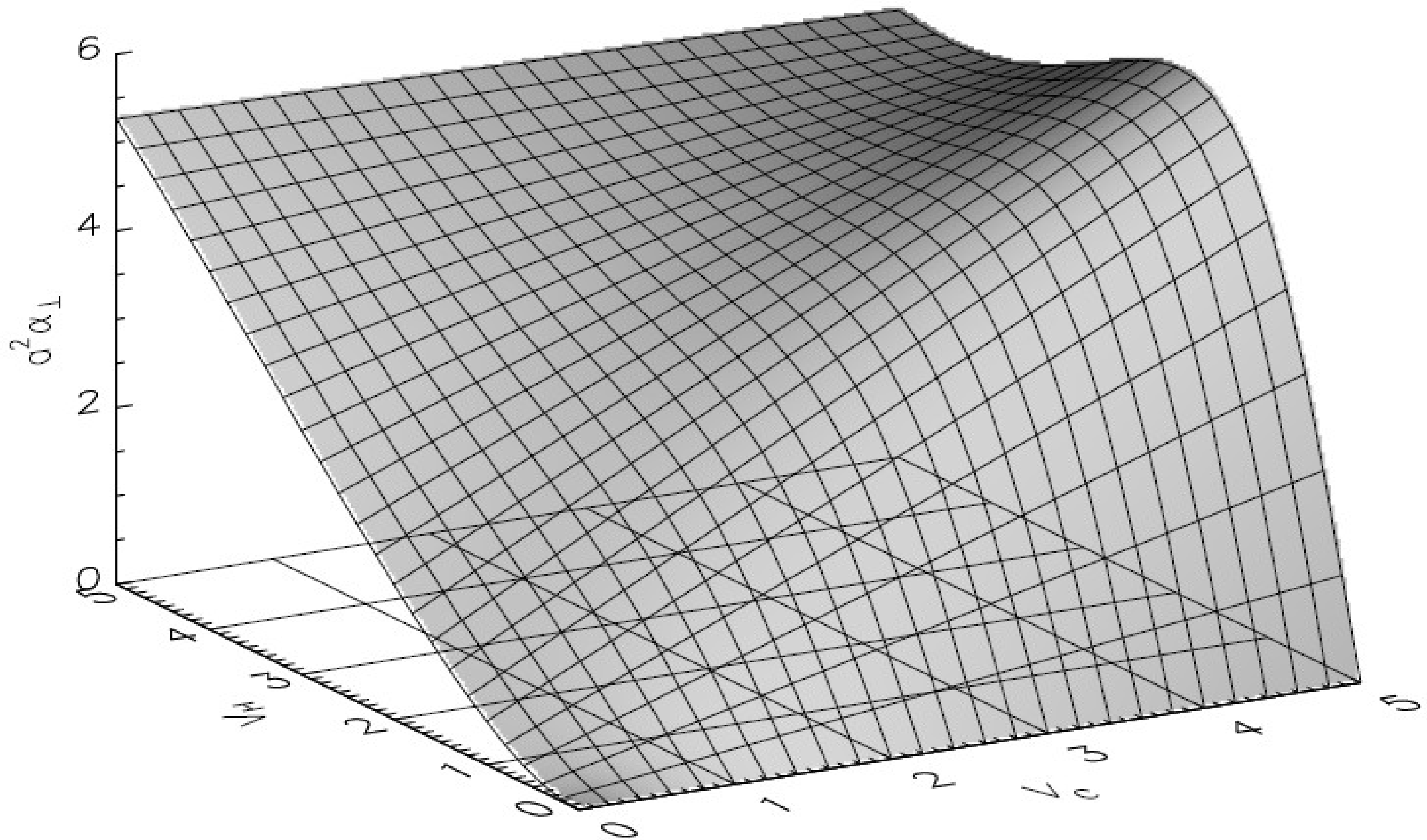
General result (for arbitrary *Rm*)

$$
\begin{aligned}
\alpha_\perp &= \frac{a}{2\eta} u_\perp \left(u_{\|\mathrm{C}}\, \phi_\mathrm{C}\left(\frac{u_\perp a}{2\eta}\right) + \frac{1}{2} u_{\|\mathrm{H}}\, \phi_\mathrm{H}\left(\frac{u_\perp a}{2\eta}\right)\right) \\
&= \frac{\eta}{a} Rm_\perp \left(Rm_{\|\mathrm{C}}\, \phi_\mathrm{C}(Rm_\perp) + \frac{1}{2} Rm_{\|\mathrm{H}}\, \phi_\mathrm{H}(Rm_\perp)\right) \\
&= \frac{1}{a^2 h\eta} V_\mathrm{H} \left(V_\mathrm{C}\, \phi_\mathrm{C}\left(\frac{V_\perp}{h\eta}\right) + \frac{1}{2} V_\mathrm{H}\, \phi_\mathrm{H}\left(\frac{V_\perp}{h\eta}\right)\right)
\end{aligned}
$$



Dependence of the α-effect on the flow rates

Spin generator flow

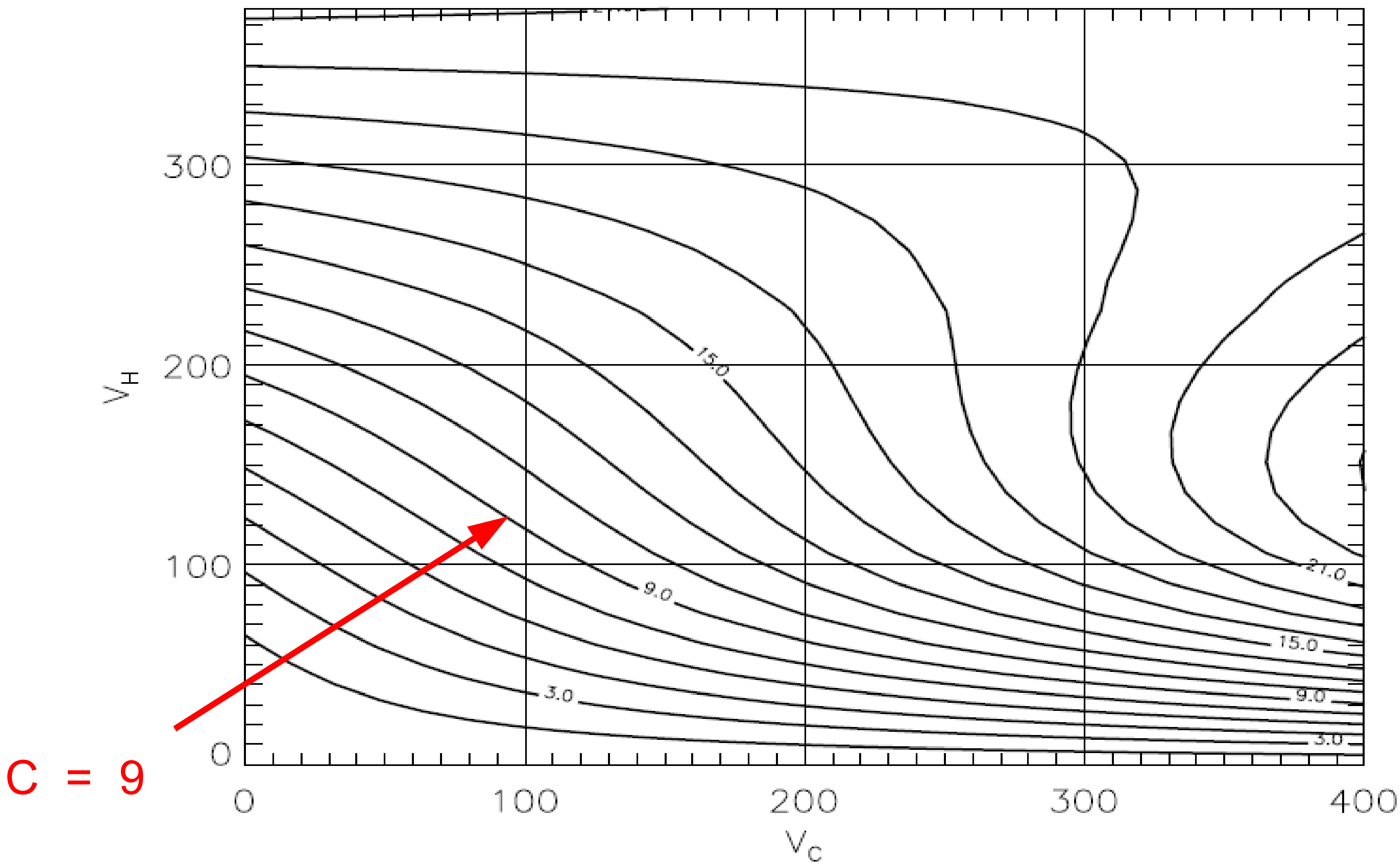


All quantities in units of *aη, h/a = 0.905*

Dependence of the *α*-effect on the flow rates

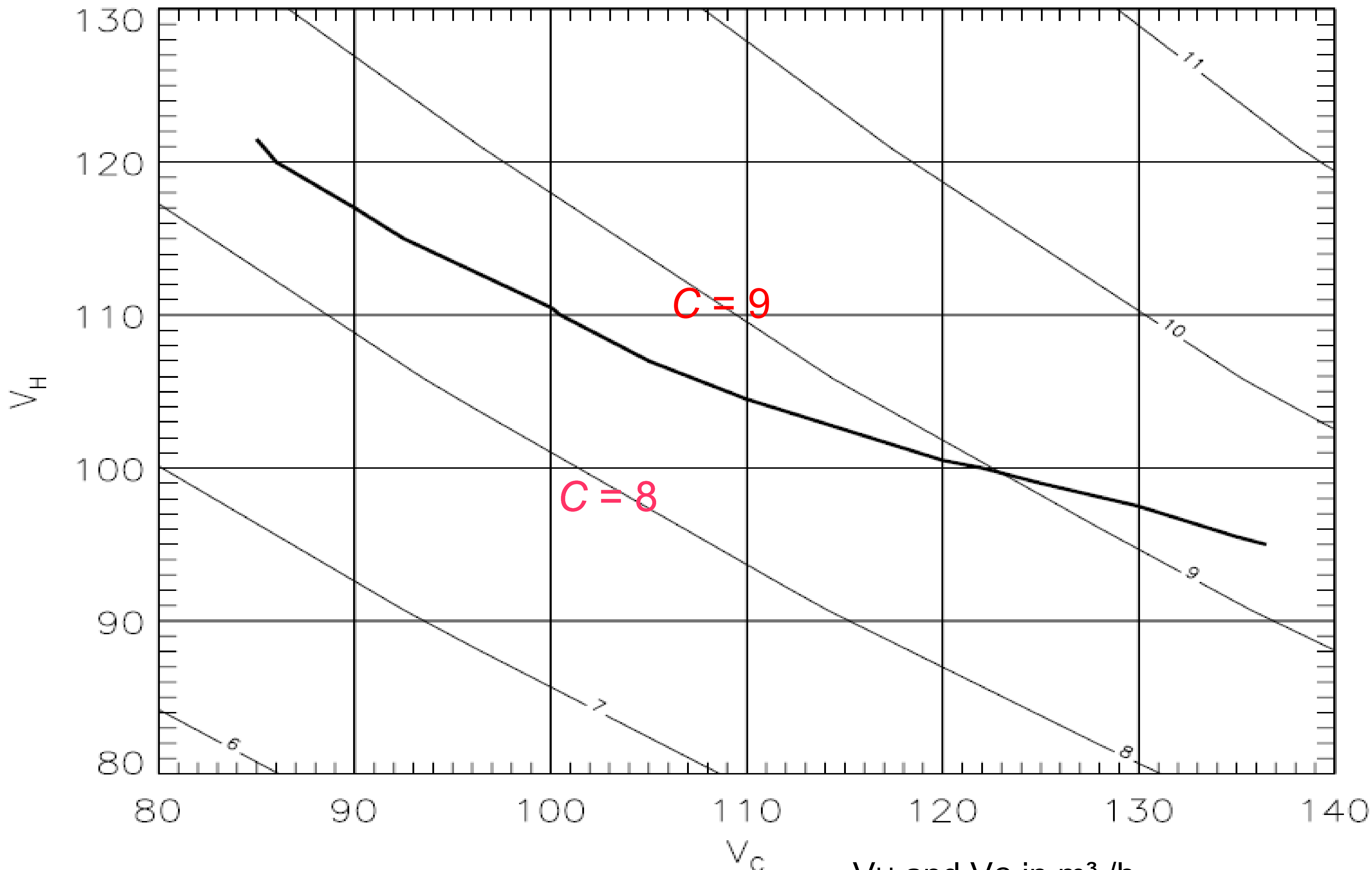
Spin generator flow

Isolines of *C*



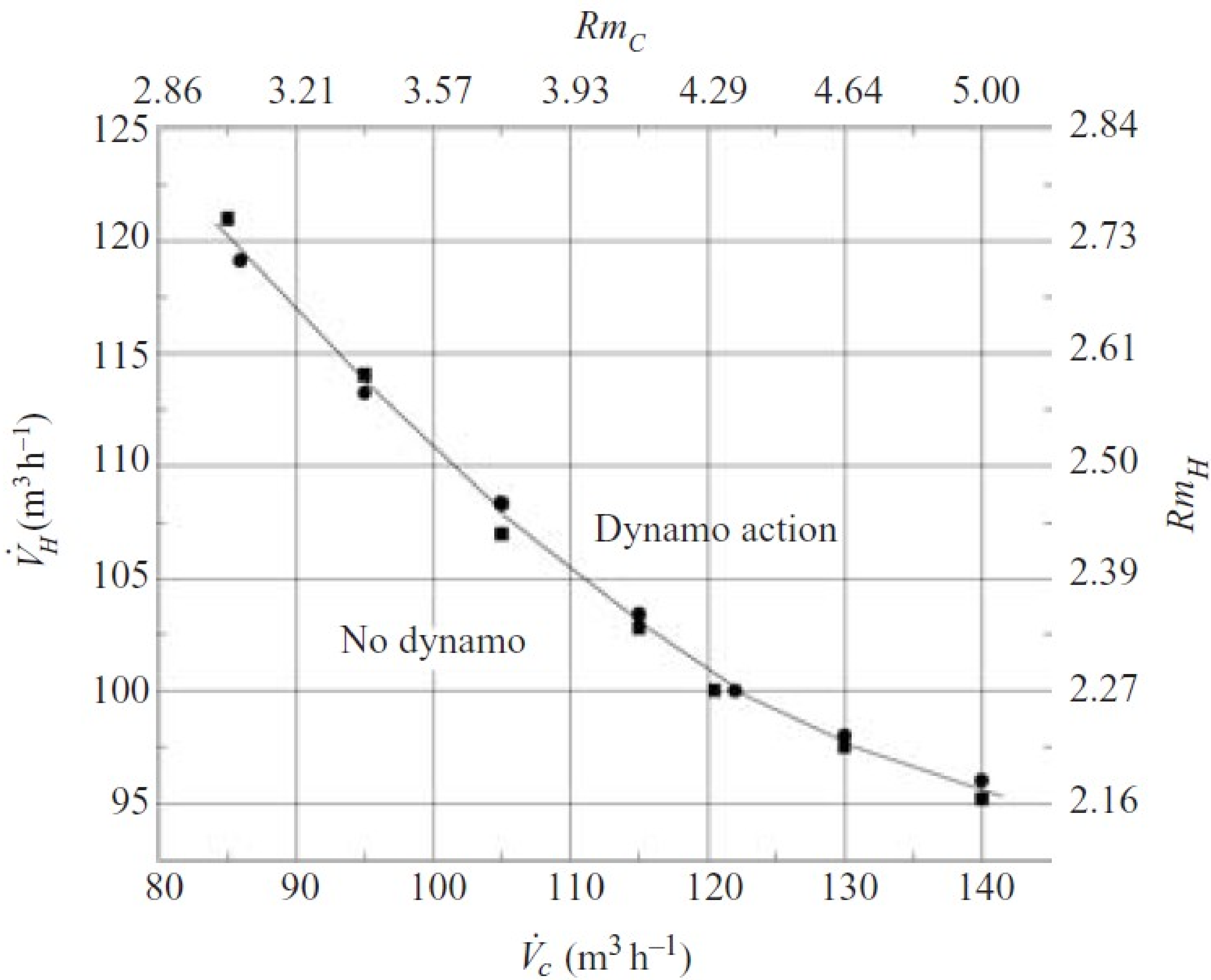
V_H and V_C in m³ /h

Experimental results

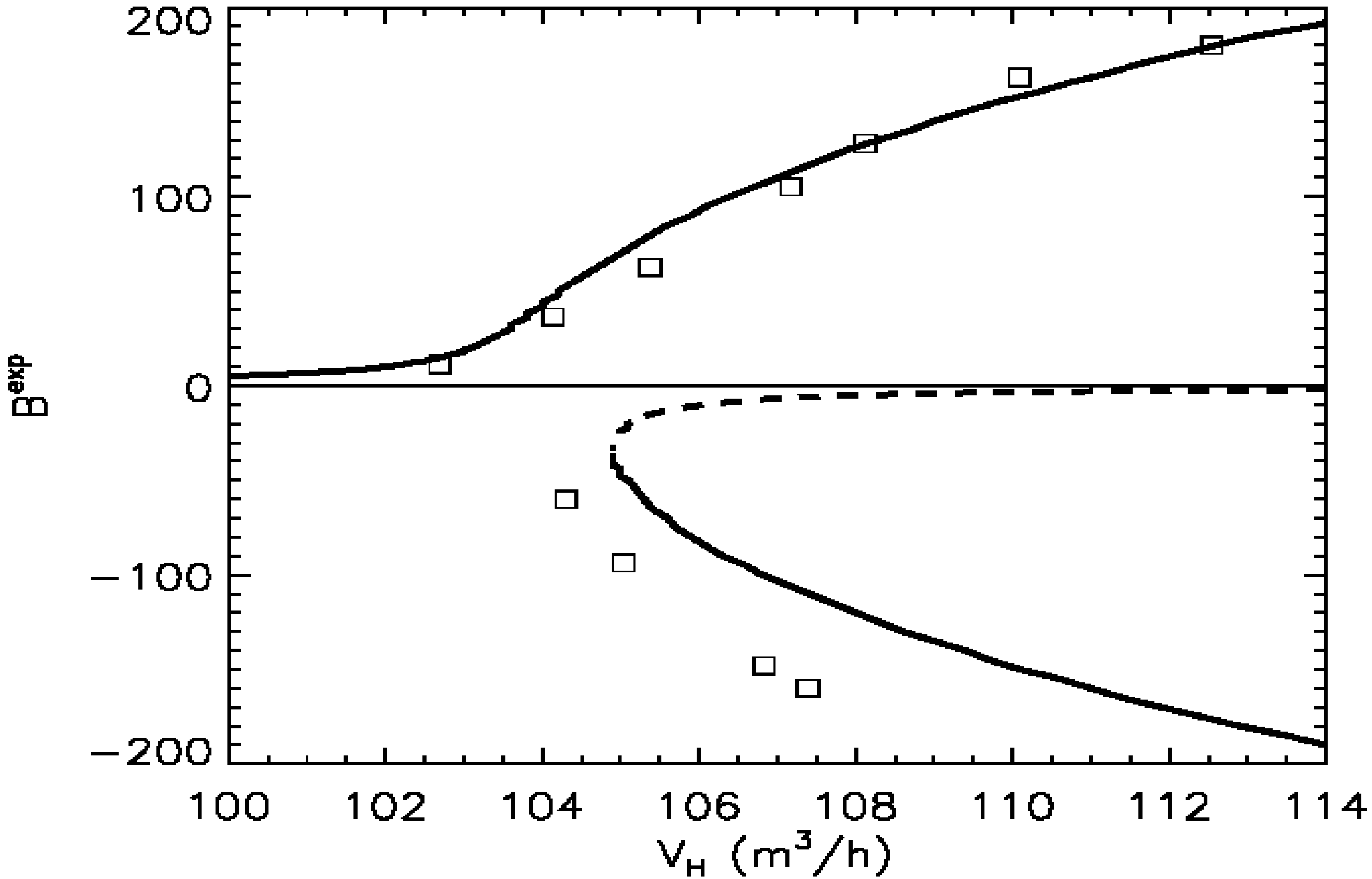


V_H and V_C in m³ /h

Experimental results



Experimental results



Vc = 112.5 m^3/h

Steps towards a refined mean-field theory of the experiment

– Boundary effects

  – at plane bottom and top boundaries of the dynamo module
  – at the curved boundary of the dynamo module

-- Mean-field conductivity etc due to the Roberts-like flow

-- Turbulence effects

– Limits of the mean-field concept

-- Saturation of the dynamo

The Karlsruhe dynamo with $\langle \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \rangle = 0$

Recall:

Homogeneous isotropic turbulence, SOCA, low-conductivity limit

$$\alpha = -\frac{1}{12\pi\eta} \int_\infty \langle \boldsymbol{u}(\boldsymbol{x}, t) \cdot (\nabla \times \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{\xi}, t)) \rangle \frac{\mathrm{d}^3\xi}{\xi} = -\frac{1}{3\eta} \langle \boldsymbol{u} \cdot \boldsymbol{\psi} \rangle$$

where $\boldsymbol{u} = \nabla \times \boldsymbol{\psi}$

For Roberts-like flows applies analogously

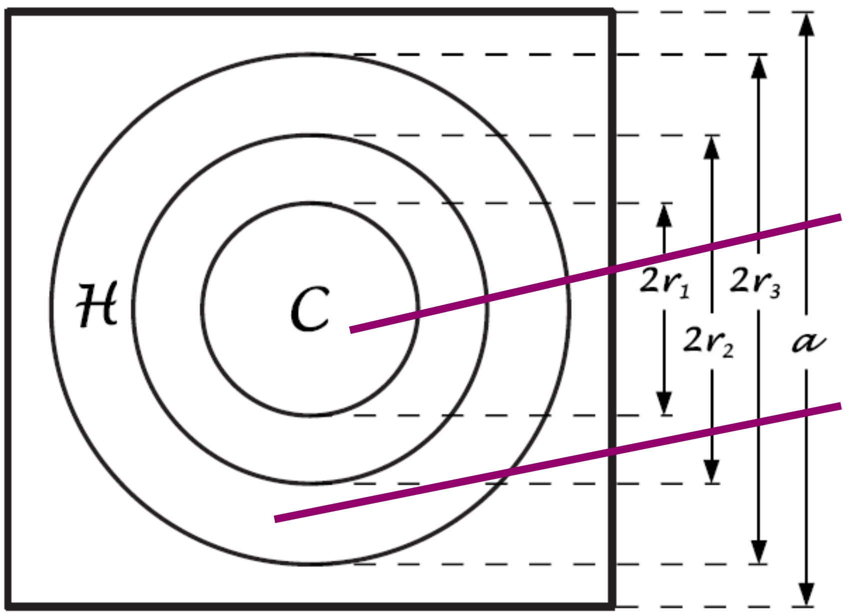
$$\alpha_\perp = -\frac{1}{2\eta} \langle \boldsymbol{u} \cdot \boldsymbol{\psi} \rangle$$

$\langle \boldsymbol{u} \cdot \boldsymbol{\psi} \rangle$ can well be unequal zero

even if $\langle \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \rangle = 0$!!!

The Karlsruhe dynamo with $\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle = 0$

An example



$$u_r = u_\varphi = 0\,,\ u_z = u$$

$$u_r = 0\,, u_\varphi = \omega r\,, u_z = \varepsilon \omega a/2$$

otherwise $\boldsymbol{u} = \boldsymbol{0}$

$$\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle = c\,\varepsilon V_{\mathrm{H}}^2$$

$$\langle \boldsymbol{u} \cdot \boldsymbol{\psi} \rangle = (c_1 V_{\mathrm{C}} + \varepsilon c_2 V_{\mathrm{H}}) V_{\mathrm{H}}$$

The Karlsruhe dynamo with $\langle \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \rangle = 0$

An example

Marginal dynamo states with *ak* = 0.9

$$\tilde{V} = V/a\eta$$

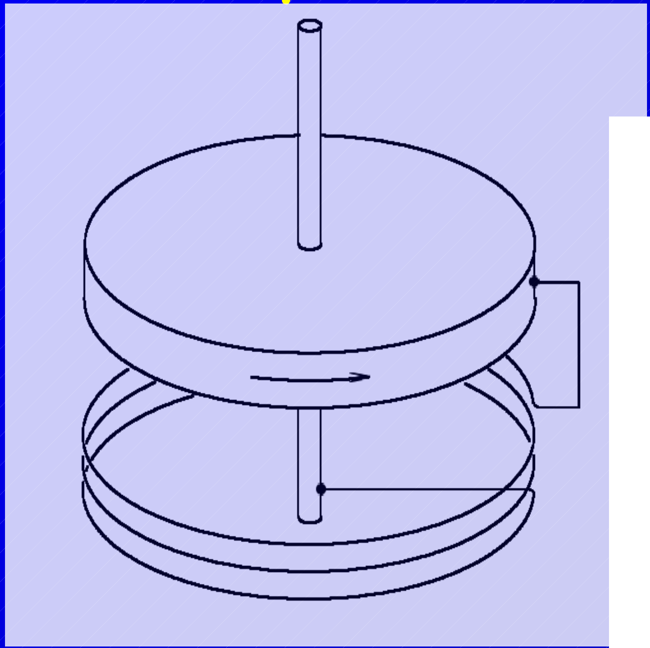
Case	$2r_1/a$	$2r_2/a$	$2r_3/a$	ε	$\tilde{V}_C$	$\tilde{V}_H$
(i)	0.5	0.5	1.0	0.228	2	0.736
(ii)	0.5	0.5	1.0	0	2	0.805
(iii)	0.5	0.7	1.0	0	2	0.965

Rädler and Brandenburg 2008

Helicity is unnecessary for Roberts dynamos, but it helps!!!

Cf. Gilbert et al. 1988
Helicity is unnecessary for α-effect dynamos, but it helps

Disc dynamo



$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + R\,I = \frac{\omega}{2\pi}\,\phi\,, \qquad \phi = L'\,I$$

$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + (R - \Omega^*)\,I = 0\,, \qquad \Omega^* = \frac{\omega}{2\pi}\,L'$$

$$\Rightarrow \qquad I(t) = I(0)\,\exp\left(-\,\frac{R - \Omega^*}{L}\,t\right)$$

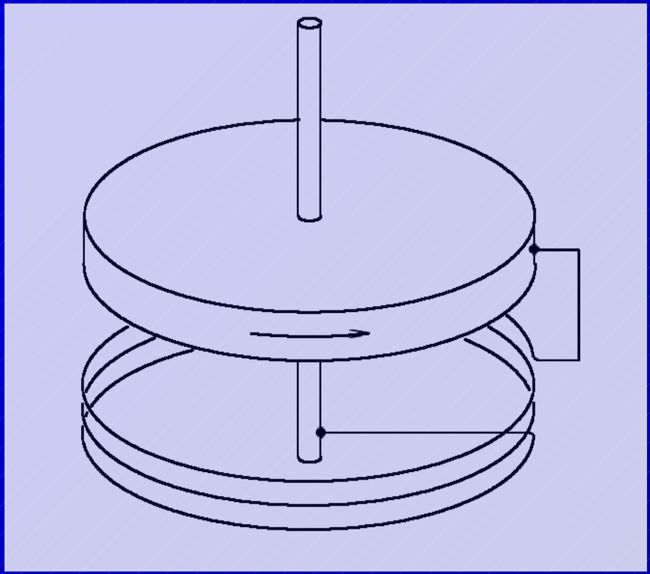
$R > \Omega^*$ decay

$R \leq \Omega^*$ dynamo

$$\omega_{\mathrm{crit}} = 2\pi\frac{R}{L'}$$

Disc dynamo

considering the back-reaction
of the magnetic field
on the rotation rate
(„eddy current braking")



$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + R\,I = \frac{\omega}{2\pi}\,\phi\,, \qquad \phi = L'\,I$$

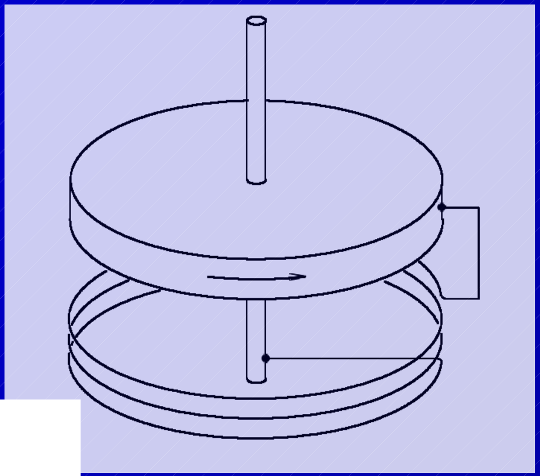
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Theta\,\omega^2 + L\,I^2) = D\,\omega - \frac{\Theta\,\omega^2}{\tau^*} - I^2\,R$$

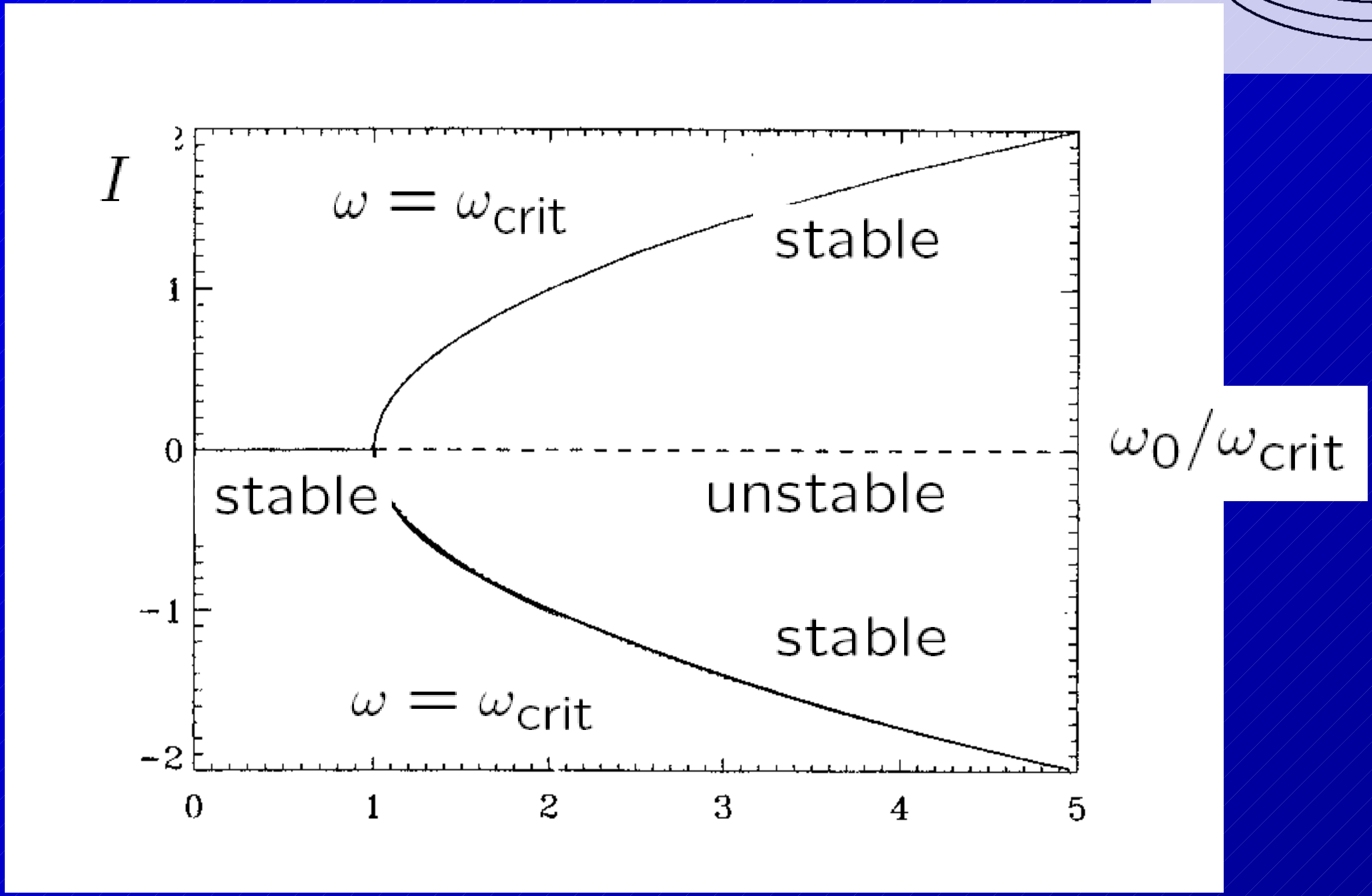
$$I = 0,\ \omega\ \text{steady:}\ \omega = \omega_0 = \frac{D\,\tau^*}{\Theta}$$

Disc dynamo

Back-reaction of the magnetic field on the rotation rate

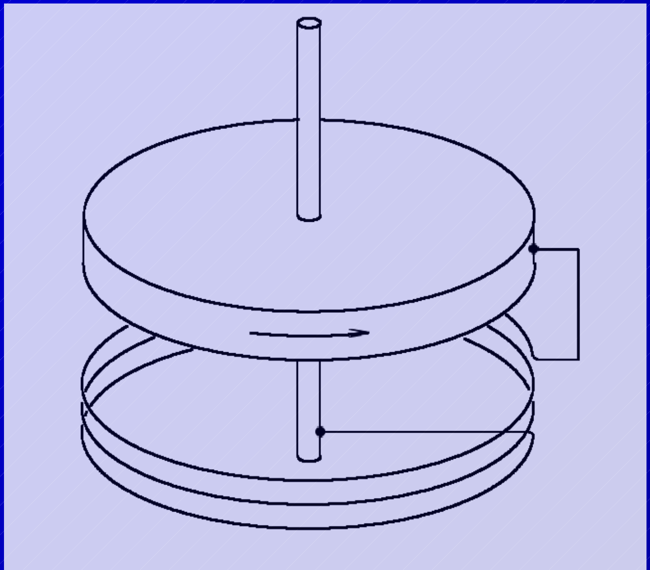
Steady case





Disc dynamo

considering the back-reaction
of the magnetic field
on the rotation rate
and an imposed magnetic field



$$L\,\frac{\mathrm{d}I}{\mathrm{d}t} + R\,I = \frac{\omega}{2\pi}\,\phi\,, \qquad \phi = \phi_0 + L'I$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Theta\,\omega^2 + L\,I^2) = D\,\omega - \frac{\Theta\,\omega^2}{\tau^*} - I^2\,R$$

$$\omega_0 = \frac{D\,\tau^*}{\Theta}$$

Disc dynamo

Back-reaction
of the magnetic field
plus imposed magnetic field

Steady case