Star-in-a-box simulations of penetrative convection with the Pencil Code

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What is penetrative convection?

Our previous works in local slab geometries

Our first global star-in-a-box simulations


The Pic-du-Midi Observatory
French Pyrénées (2877 m)

What is penetrative convection?

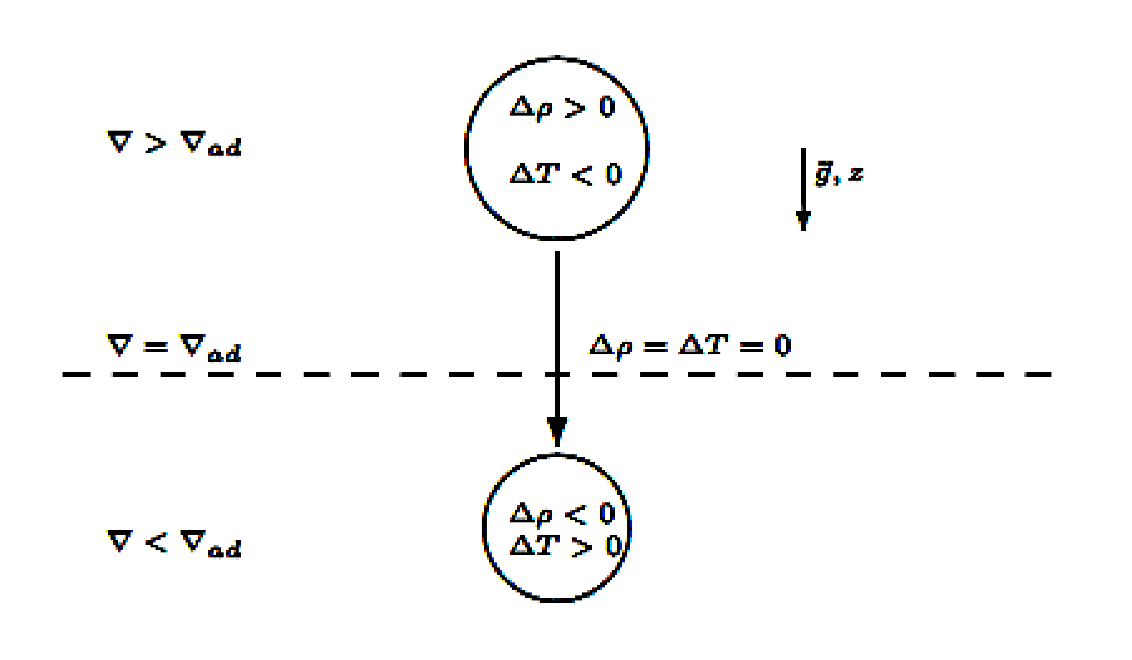
$$v^2 \propto \frac{\delta T}{T} g\ell$$

$$F_{conv} = \rho v c_p \delta T \Rightarrow v = \delta T = F_{conv} = 0 \text{ when } \nabla = \nabla_{ad}$$

Schwarzschild's criterion

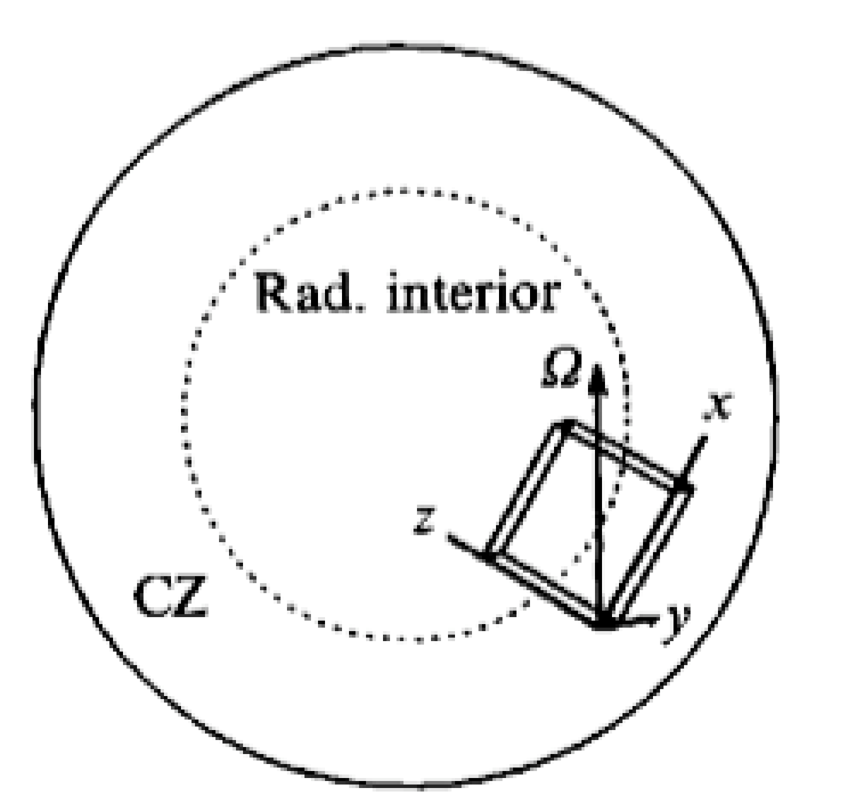
$$\frac{\delta T}{T} \propto \nabla - \nabla_{ad}$$

$$\nabla = \nabla_{ad} \Rightarrow \delta T = \delta\rho = 0 \Rightarrow \frac{d\vec{v}}{dt} \propto \text{Archimedes} + \vec{g} = \vec{0} \text{ and not } \vec{v} = \vec{0}$$



Key parameter governing the penetration: $Pe = \frac{VL}{\chi}$ where $\chi = K/\rho c_p$

Our former local 2-D DNS using the Stein & Nordlund code (2003 & 2005)



$$\frac{D\ln\rho}{Dt} = -\nabla\cdot\vec{v},$$

$$\frac{D\vec{v}}{Dt} = -(\gamma-1)(e\nabla\ln\rho+\nabla e)+\vec{g}+\frac{1}{\rho}\nabla\cdot(2\nu\rho\mathcal{S}),$$

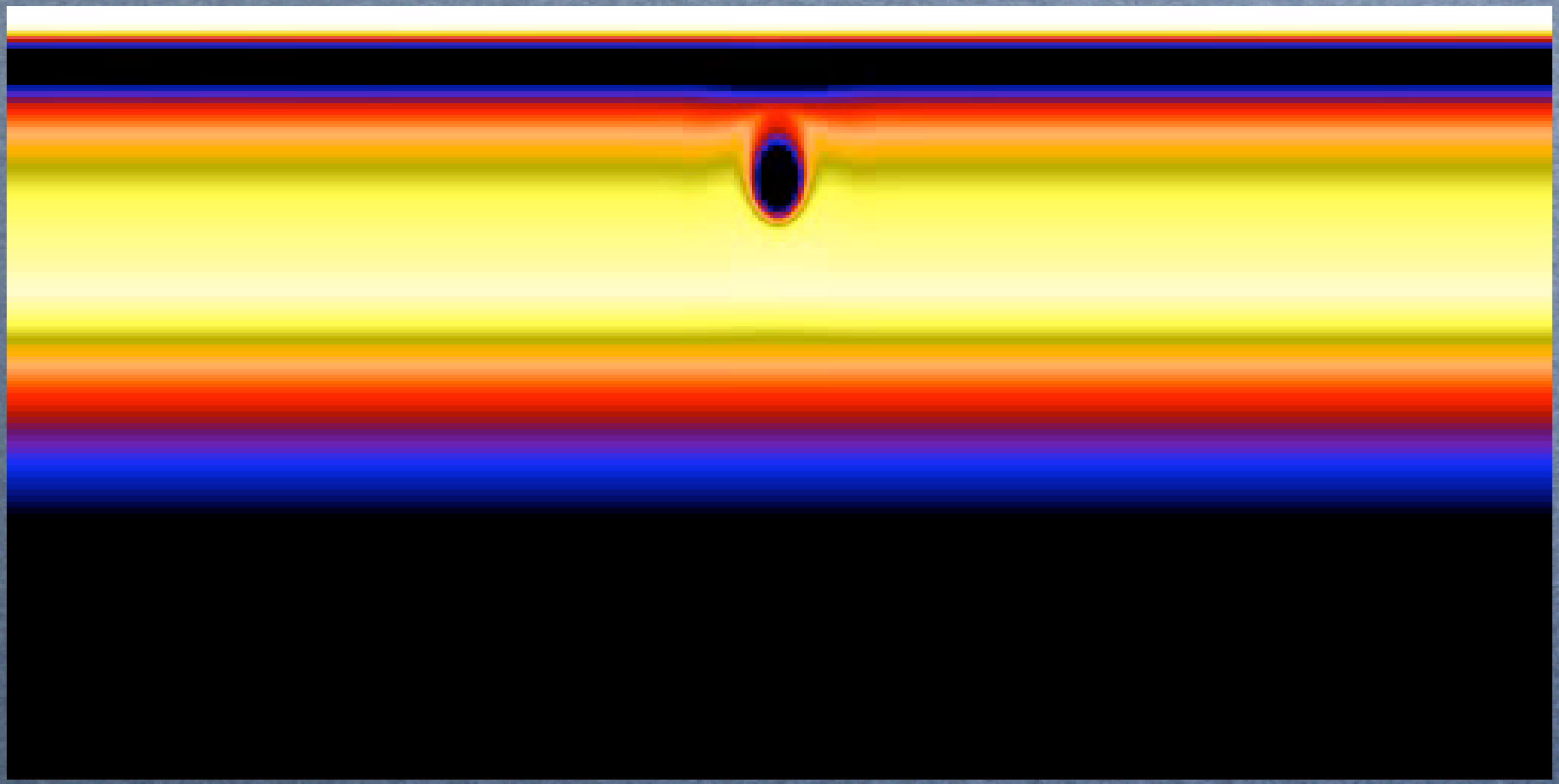
$$\frac{De}{Dt} = -(\gamma-1)e\nabla\cdot\vec{v}+\frac{1}{\rho}\nabla\cdot(\mathcal{K}\nabla e)+2\nu\mathcal{S}^2-\frac{e-e_0}{\tau(z)}$$

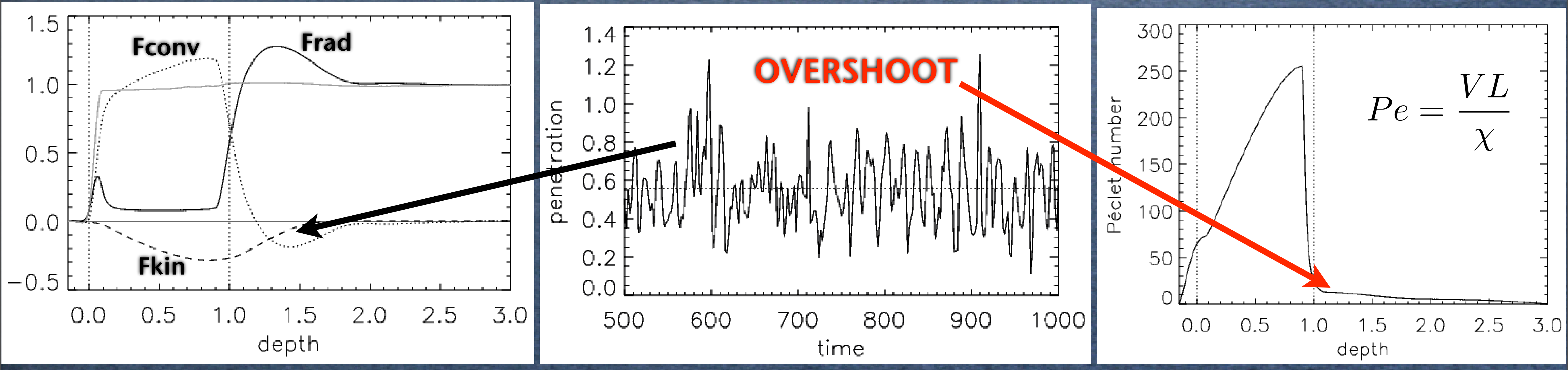
✓ monatomic perfect gas under constant gravity

✓ BCs: horizontal=periodic; vertical=stress-free for velocity/cst temperature at the top/cst flux at the bottom

✓ numerics: 6th-order compact finite differences and 3th-order in time

Example of a 2-D DNS: fluxes, entropy and penetration





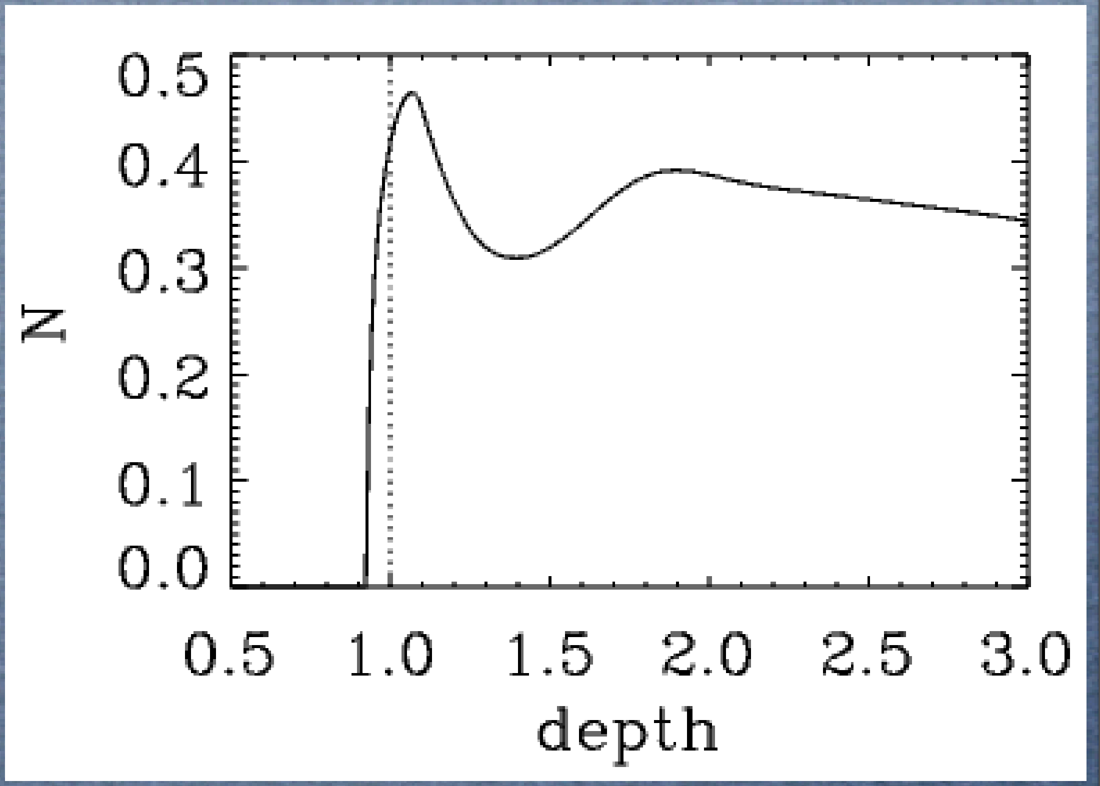
Fluxes

Penetration depth

Peclet's number

Detecting g-modes using the anelastic subspace

Computation of the eigenmodes of the box:

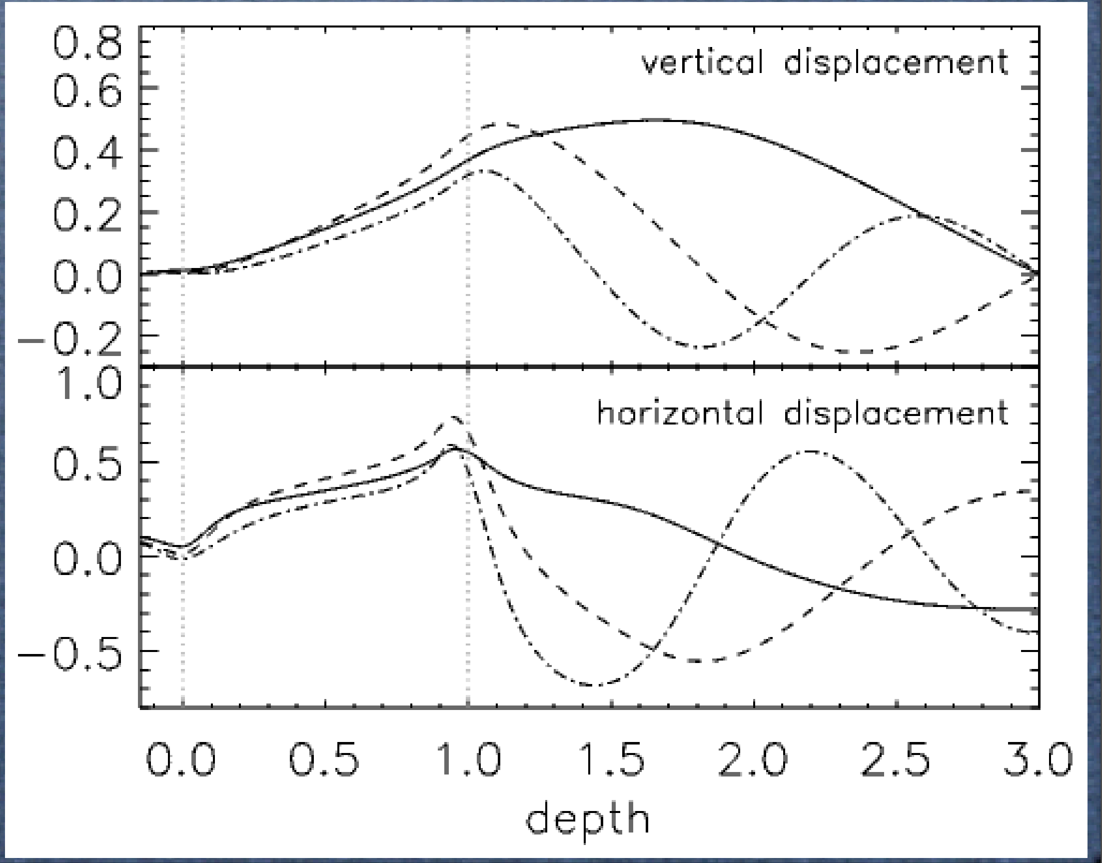


$$\omega^2\left(\xi_z - \frac{1}{k}\frac{d\xi_x}{dz}\right) = N^2\xi_z,$$

$$-k\xi_x + \frac{d\xi_z}{dz} + \frac{d\ln\rho}{dz} = 0,$$

$$\xi_z = 0 \text{ for } z = z_1, z_4$$

$$\Rightarrow \quad [A]\vec{\psi} = \omega^2[B]\vec{\psi}, \qquad \vec{\psi} = \begin{pmatrix} \xi_x \\ \xi_z \end{pmatrix}$$



$$\hat{v}_z(k,z,t) = \sum_{n=0}^{+\infty} < \vec{\psi}_{kn}, \hat{v}_z > \vec{\psi}_{kn}$$

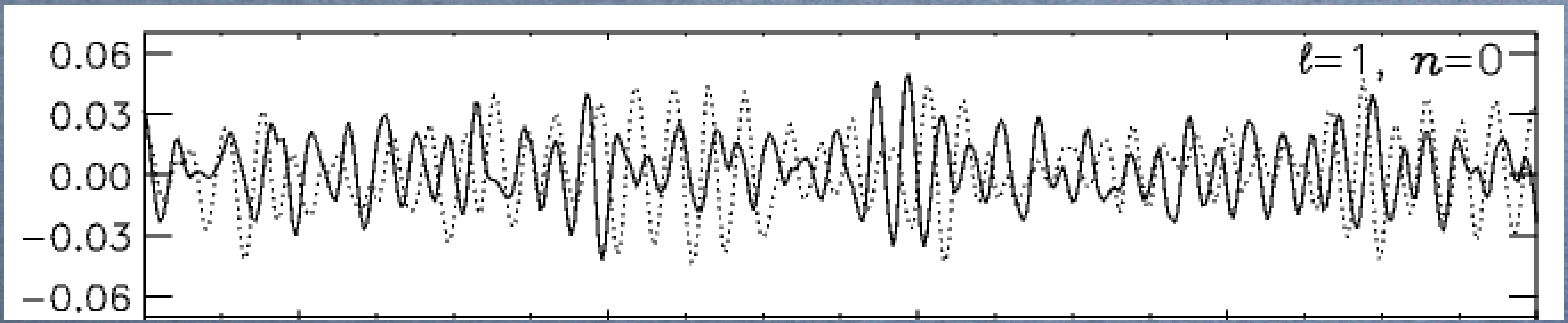
simulation

eigenmode

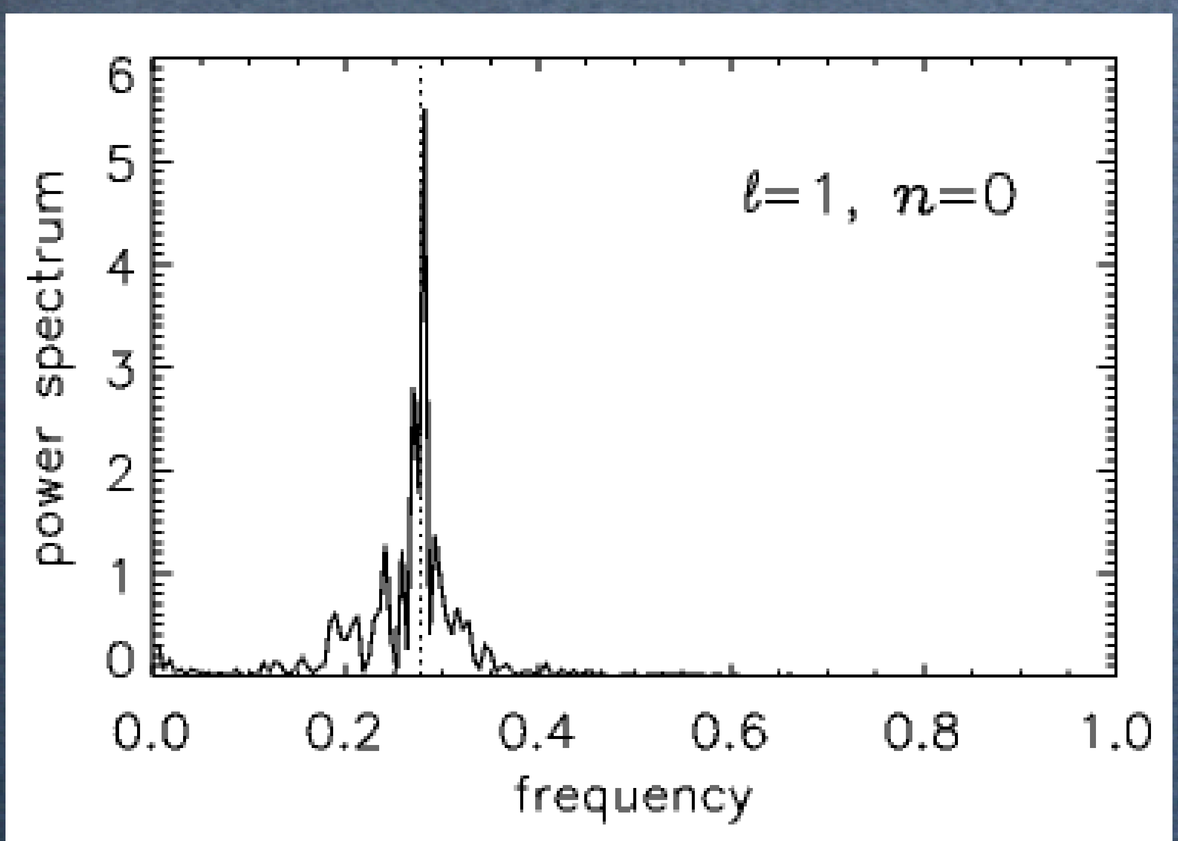
projection coefficient

$$C_{kn}(t) \propto \exp(i\omega_{kn}t)$$

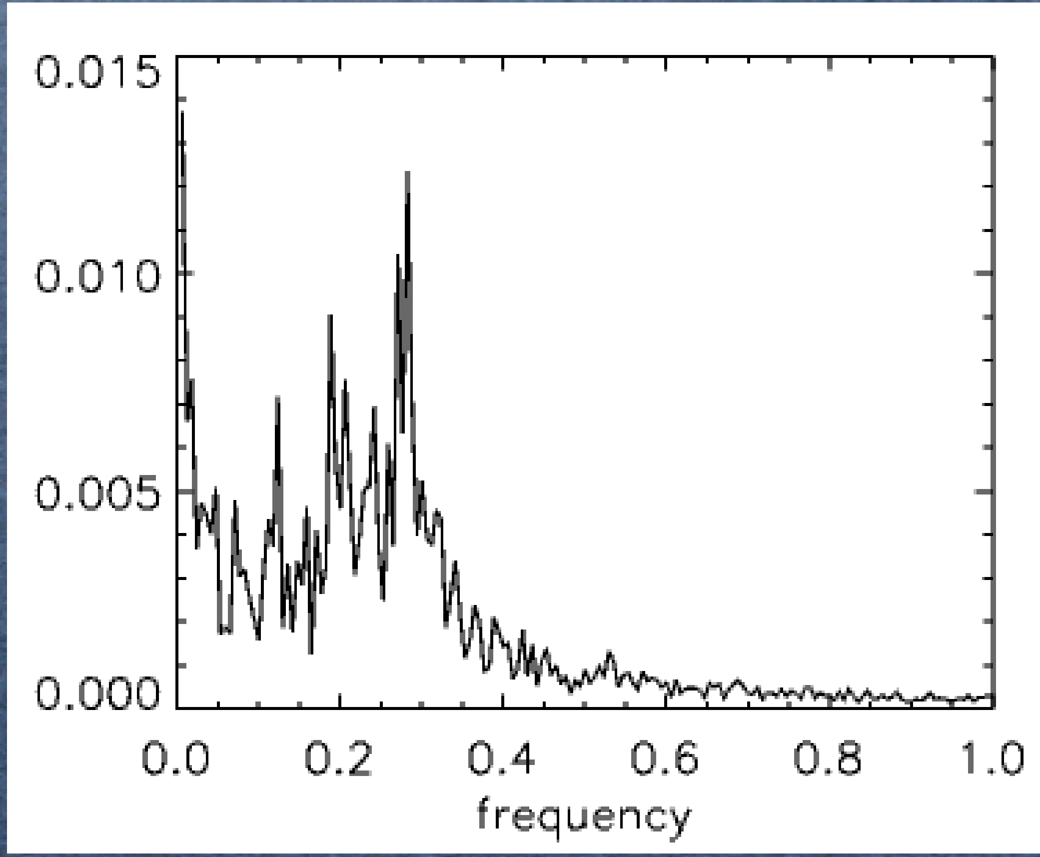
Example: time evolution of the mode l=1, n=0



$C_{kn}(t) \Rightarrow \hat{C}_{kn}(\omega)$



comparison with the 'classical' way



The turning to global simulations

We want to solve the dynamics of the whole star

Spherical harmonics codes (e.g. ASH)	star-in-a-box codes (e.g. Pencil code)
sphere = surface of coordinates spectral precision (Chebyshev+Legendre)	center = no pb highly parallelizable (linear scalability) non-linear termes ($v.\nabla$) = no pb
center = singularity! nonlinear terms ($v.\nabla$) in the physical space (spectral ↔ physical transformations) non-trivial parallelization (transposition...)	~half of the grid points are lost non-spectral precision boundary conditions are quite problematic

Our first global star-in-a-box simulations of penetration convection with the Pencil Code

- ✓ as in a slab: 2 superposed polytropic layers (radiation below convection) + 1 cooling zone at the surface
- ✓ a gaussian volume heating profile inspired from Dobler et al. (2006):

$$- \operatorname{div} \vec{F} + \varepsilon = 0 \quad \text{where} \quad \varepsilon(r) = \frac{L_0}{2\pi\sigma^2} \exp(-r^2/2\sigma^2)$$

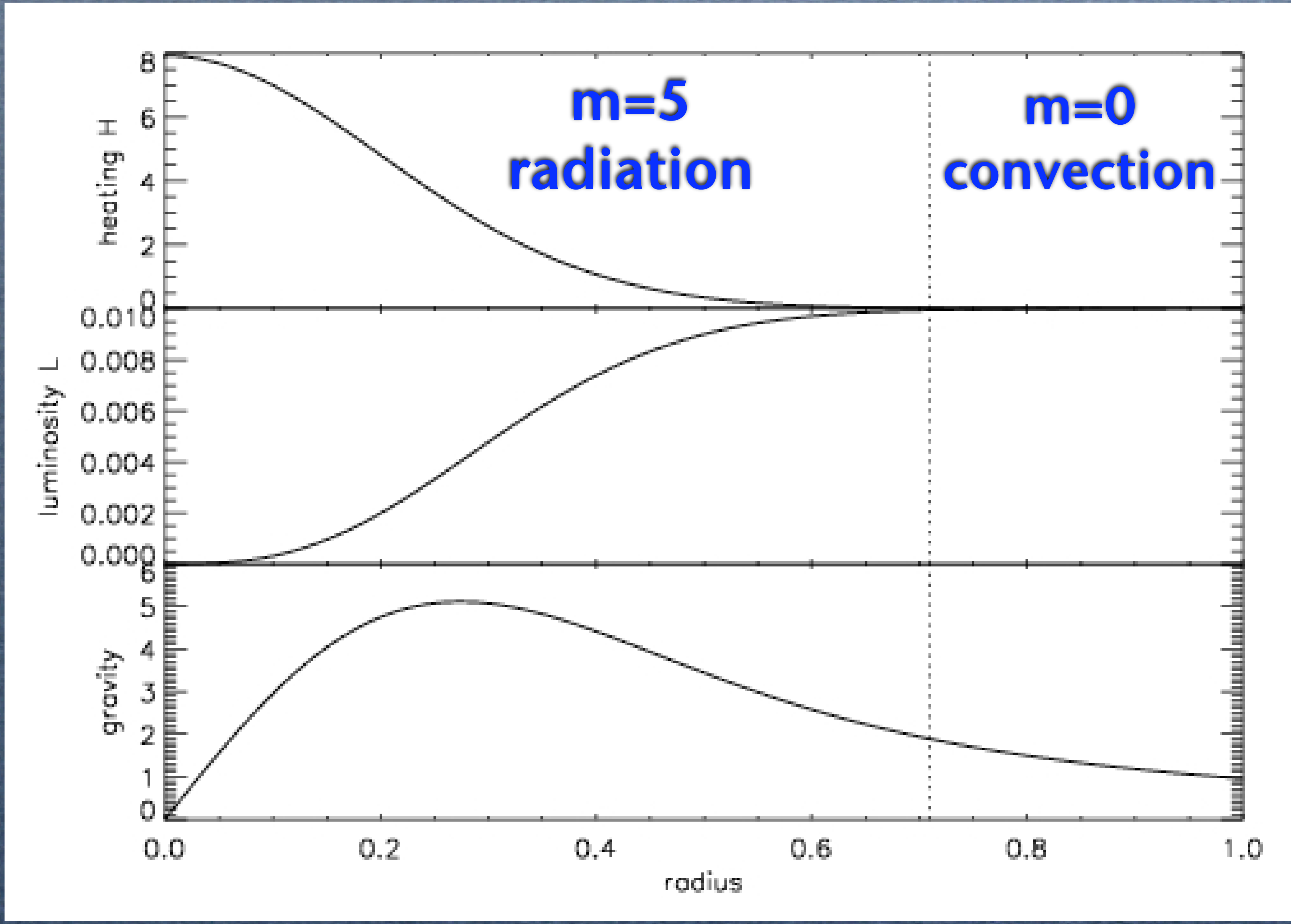
$$\Rightarrow \frac{1}{r}\frac{d(rF)}{dr} = \varepsilon \quad \text{and} \quad 2\pi r F = \int_0^r 2\pi t \; \varepsilon \; dt = L(r)$$

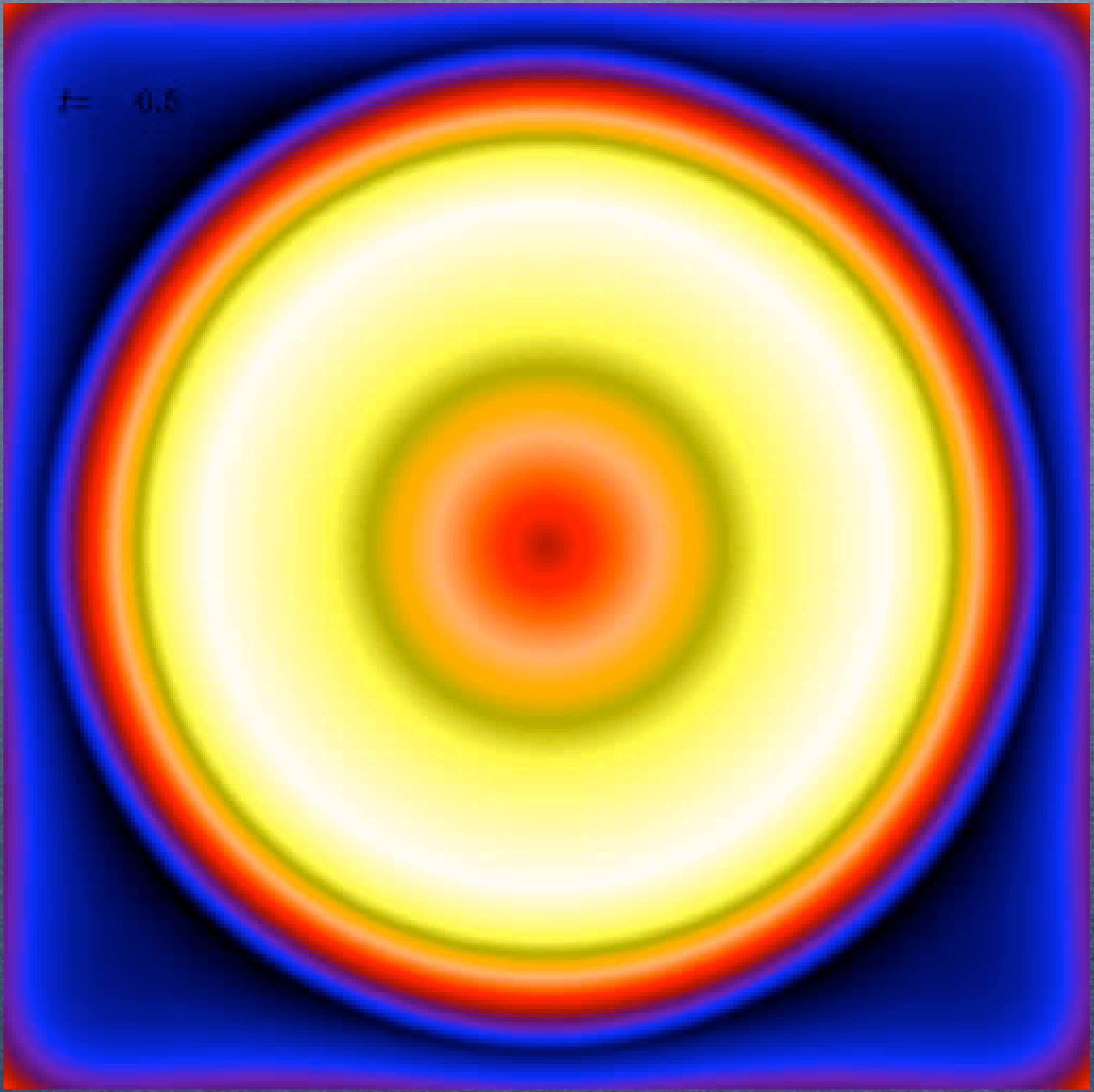
$$\Rightarrow L(r) = L_0 \left[1 - \exp(-r^2/2\sigma^2)\right]$$

$$\Rightarrow F = \frac{L}{2\pi r} = \frac{L_0}{2\pi r} \left[1 - \exp(-r^2/2\sigma^2)\right] = -K \frac{dT}{dr}$$

The hydrostatic equilibrium gives the analytic gravity profile:

$$\frac{dT}{dr} = -\frac{L(r)}{2\pi r K} = -\frac{g(r)}{(m+1)R_*} \Rightarrow g(r) = \frac{L(r)}{2\pi r}\frac{m+1}{K}R_*$$




t= 0.5

Excitation of radial acoustic modes by kappa-mecanism: the semi-implicit approach

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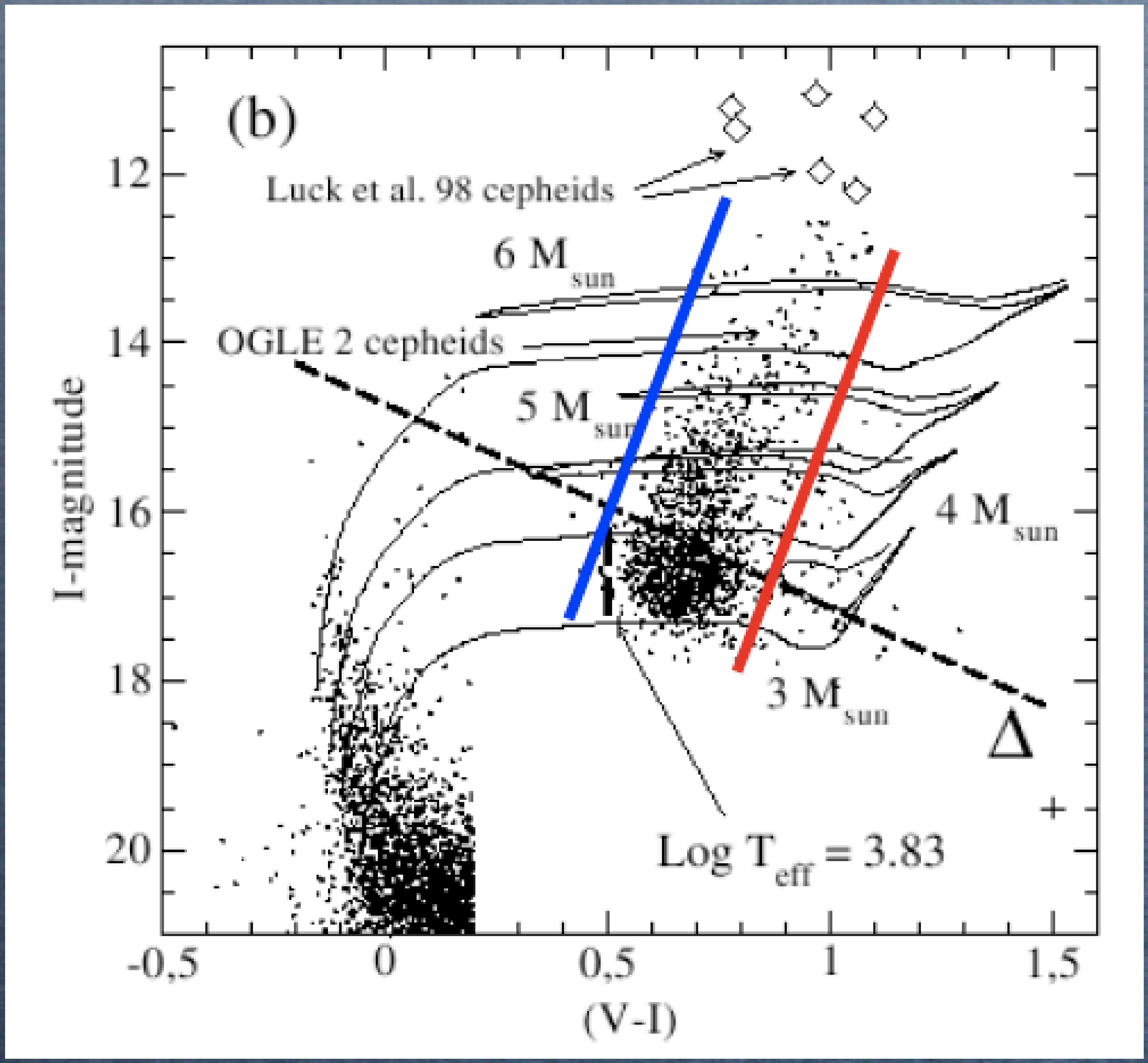
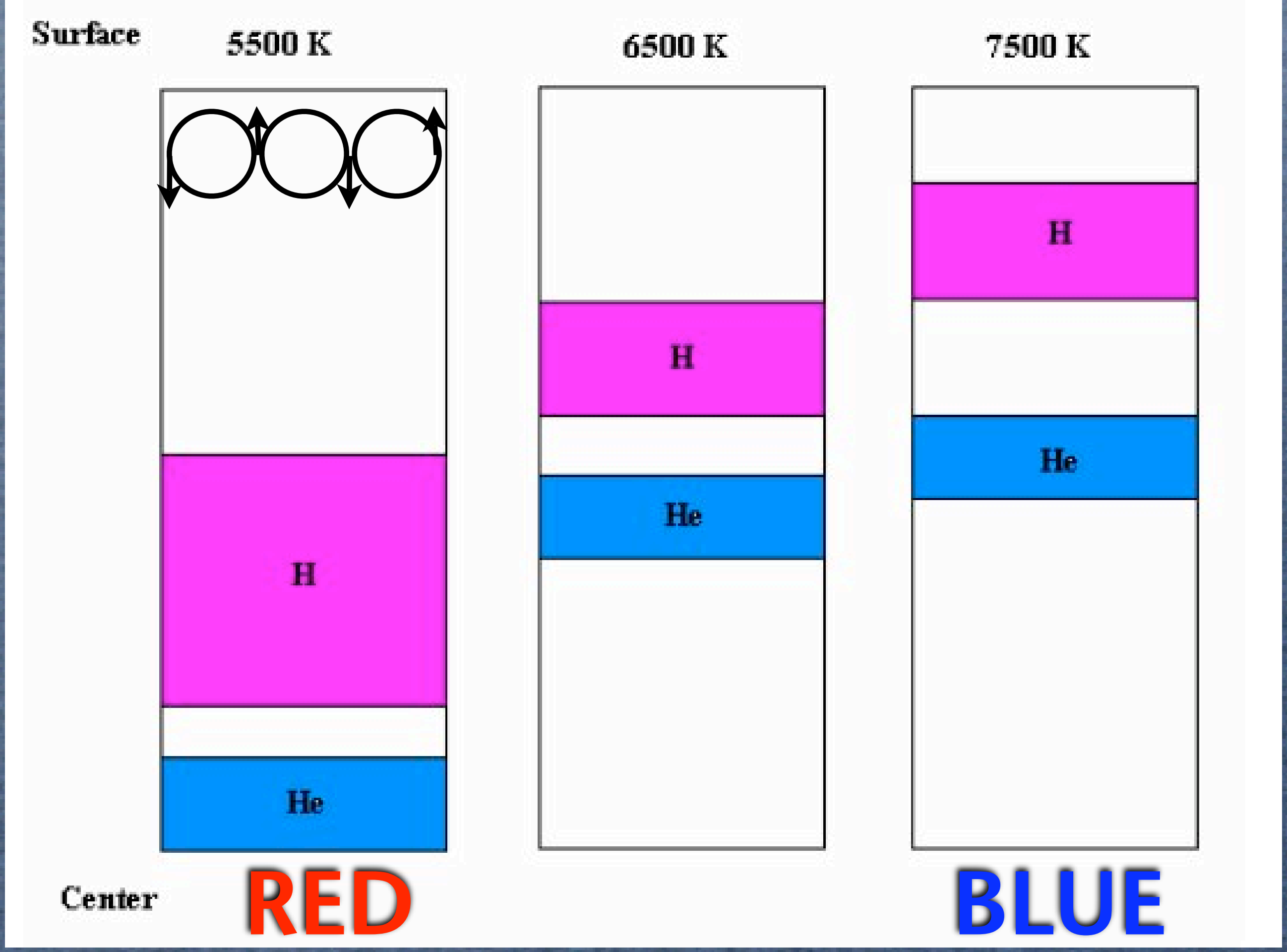
- The astrophysical context
- Our simplified model
- The linear stability analysis
- The first DNS using a semi-implicit approach

The astrophysical context: the Cepheids instability strip





What is the influence of the convection onto the growth rate of a mode when the convective turnover timescale ≈ mode period?

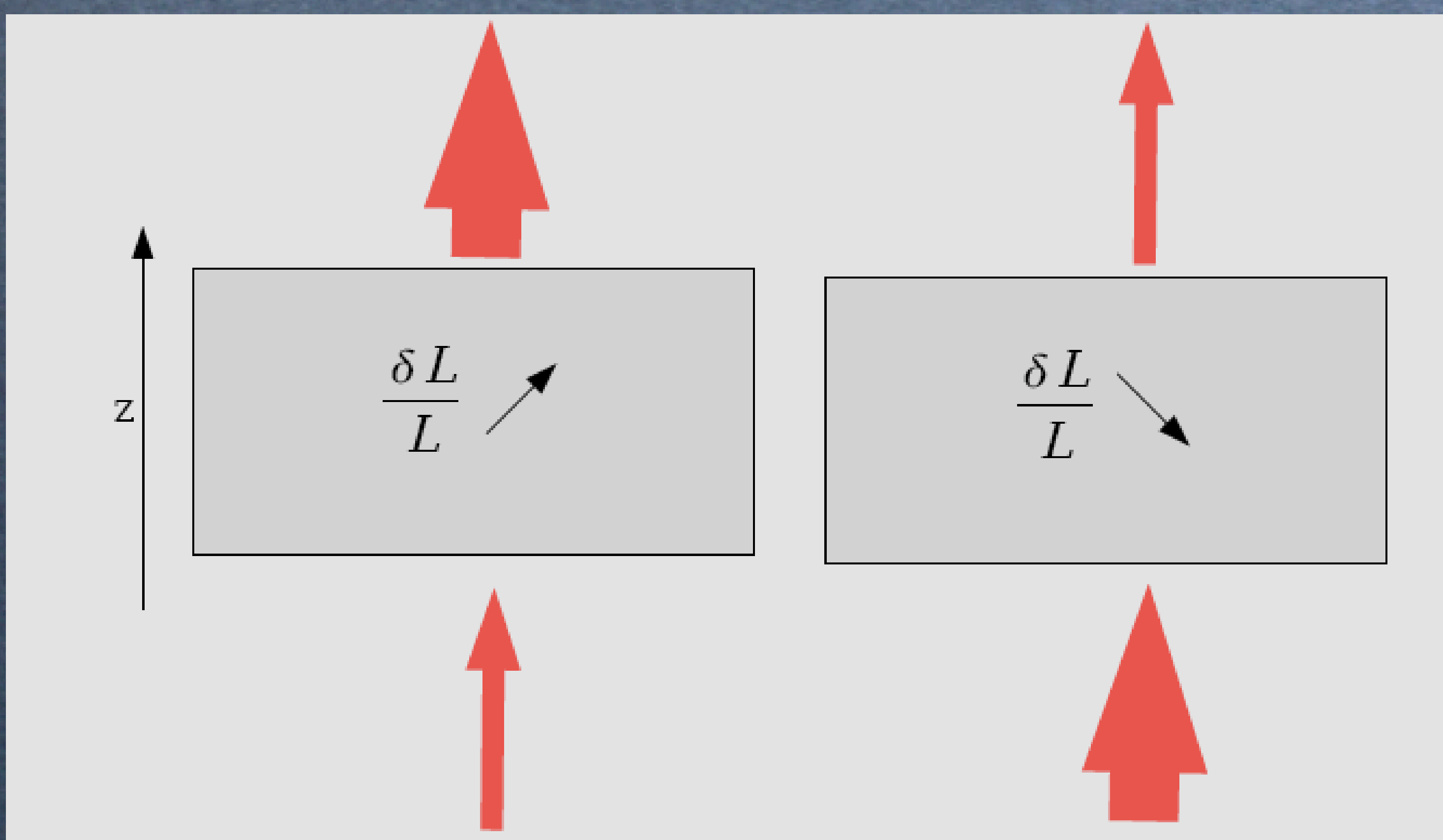
Mode excitation by the kappa-mecanism

pulsation. But, in order to convert heat of any kind into work, the star, or some part of it, must behave as an engine in the thermodynamical sense: that is to say, it must take in heat when it is at a higher temperature than the average and give out heat at a lower temperature—just the opposite to what usually happens in natural conditions. An apparent exception to this rule must

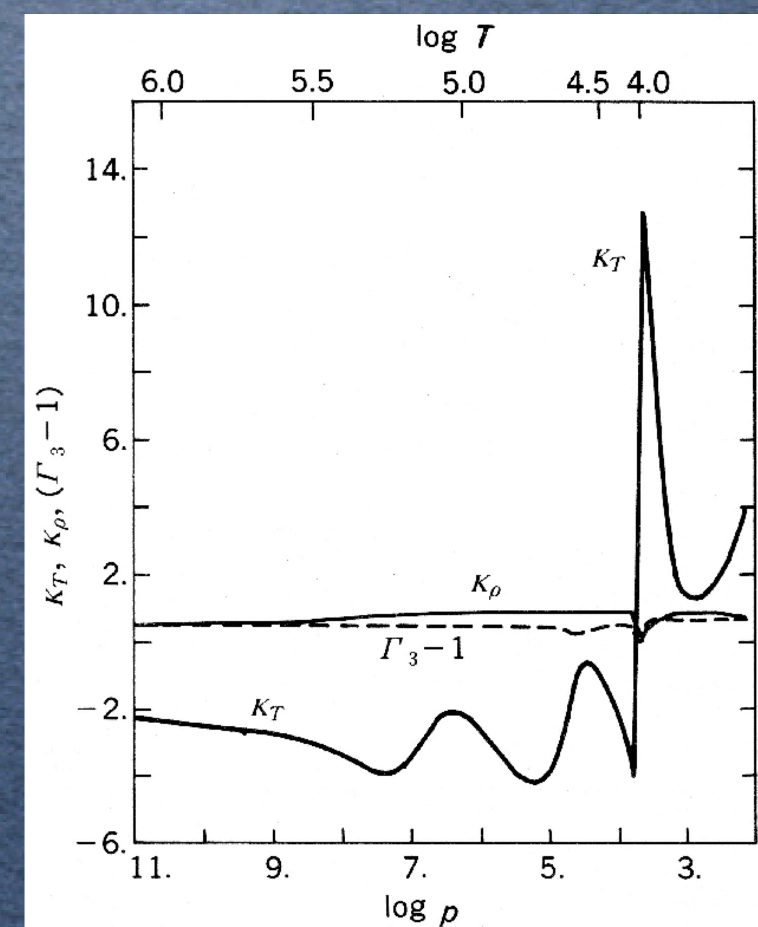
could behave in the manner of an engine. Possibly during the pulsation, variations of the transparency, which governs the flow of heat, might cause the engine to be fed in the required manner.

Eddington's valve mecanism (1917)

$$L \propto \frac{T^4}{\kappa}\frac{d\ln T}{dr} \quad \text{with Kramer's law} \quad \kappa \propto \rho T^{-3.5} \Rightarrow \frac{\delta L}{L} \sim 4\frac{\delta T}{T}$$



'Normal way' Ionization zone



Our simplified model

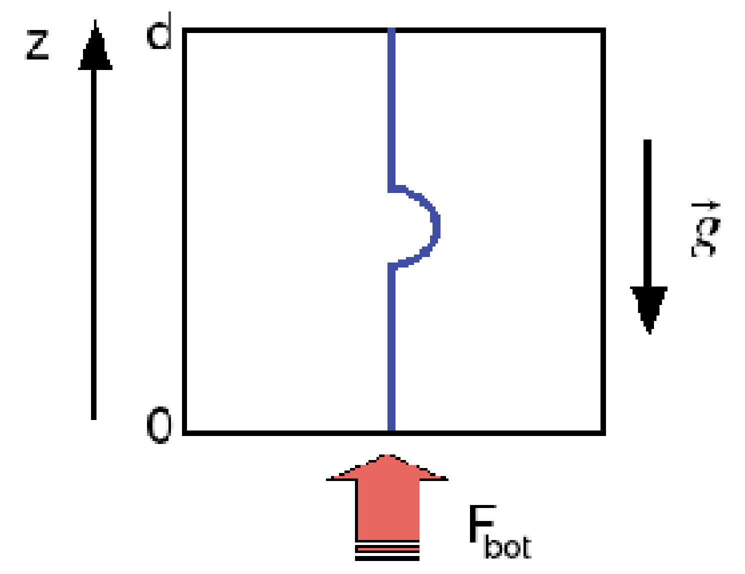


Fig. 1. Scheme corresponding to our model. Gravity is pointing downward, contrary to the vertical coordinate. The (blue) curve represents the radiative conductivity profile that we are going to discuss further whereas large (red) arrow expresses the radiative flux entering in the bottom of the layer.

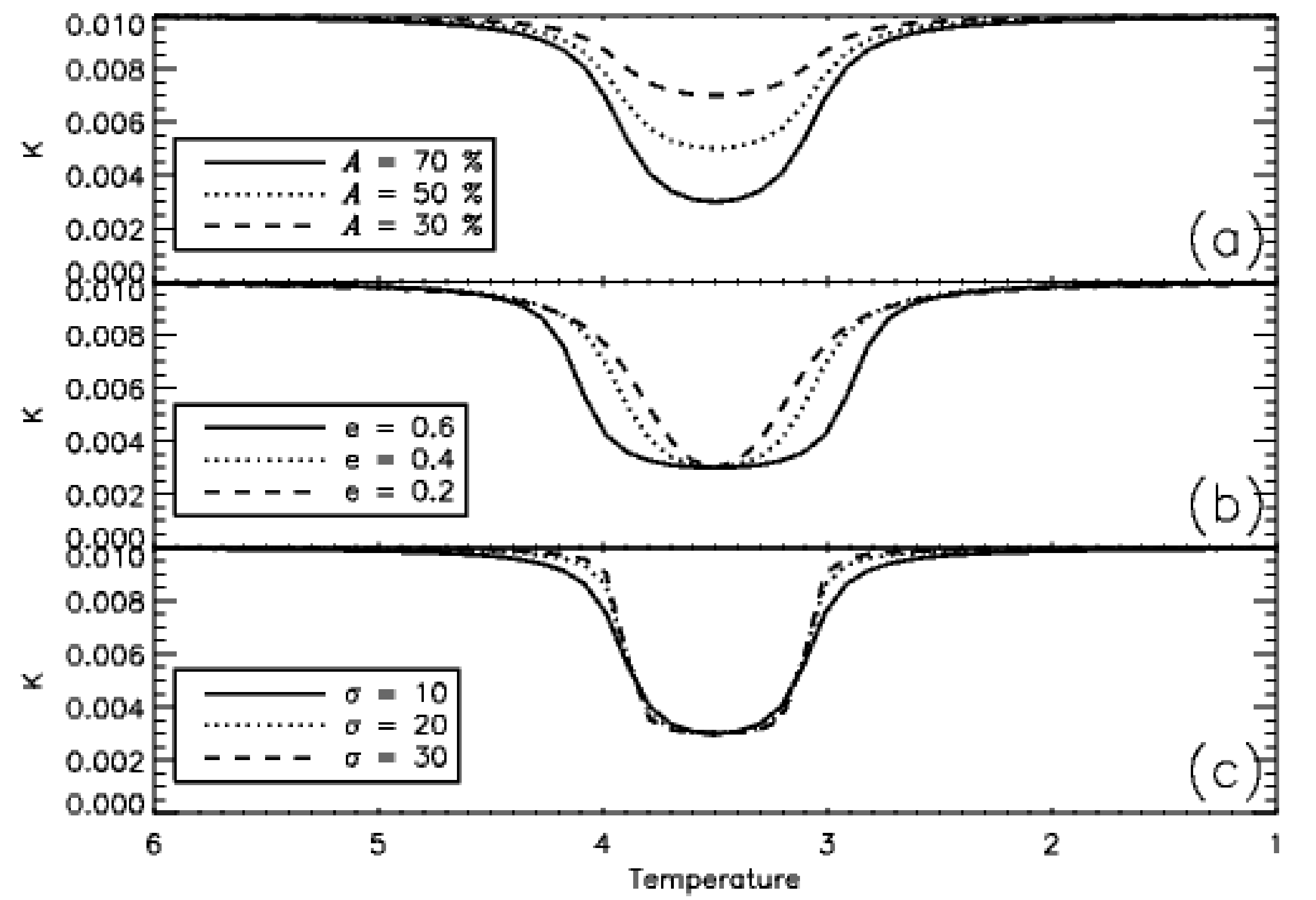


Fig. 2. Influence of hollow parameters on the conductivity profile: amplitude $\mathcal{A}$ (**a**), width e (**b**) and slope σ (**c**) for a constant $T_{\rm bump} = 3.5$.

$$K_0(T) = K_{\max}\left[1 + \mathcal{A}\frac{-\pi/2 + \arctan(\sigma T^+ T^-)}{\pi/2 + \arctan(\sigma e^2)}\right], \qquad (24)$$

with

$$\mathcal{A} = \frac{K_{\max} - K_{\min}}{K_{\max}}, \quad T^{\pm} = T - T_{\rm bump} \pm e, \qquad (25)$$

The linear stability analysis (Gastine & Dintrans 2007)

General oscillations equations

$$
\begin{cases}
\lambda \rho' = -\rho_0 \,\mathrm{div}\, \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \ln \rho_0 \\
\lambda \boldsymbol{u} = -\dfrac{1}{\rho_0} \boldsymbol{\nabla} p' + \dfrac{\rho'}{\rho_0} \boldsymbol{g} \\
\lambda T' = -\dfrac{1}{\rho_0 c_v} \,\mathrm{div}\, \boldsymbol{F}' - (\gamma - 1) T_0 \,\mathrm{div}\, \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} T_0,
\end{cases}
\tag{3}
$$

$$\boldsymbol{F}' = -K_0 \boldsymbol{\nabla} T' - K' \boldsymbol{\nabla} T_0,$$

Pb: one first needs to solve a non-linear equilibrium (hydrostatic + radiative)

$$
\begin{cases}
\boldsymbol{\nabla} p_0 = \rho_0 \boldsymbol{g} \\
\mathrm{div}\,[K_0(T_0) \boldsymbol{\nabla} T_0] = 0.
\end{cases}
\tag{26}
$$

fixed-point algorithm

$$A \psi_{n+1} = B(\psi_n).$$

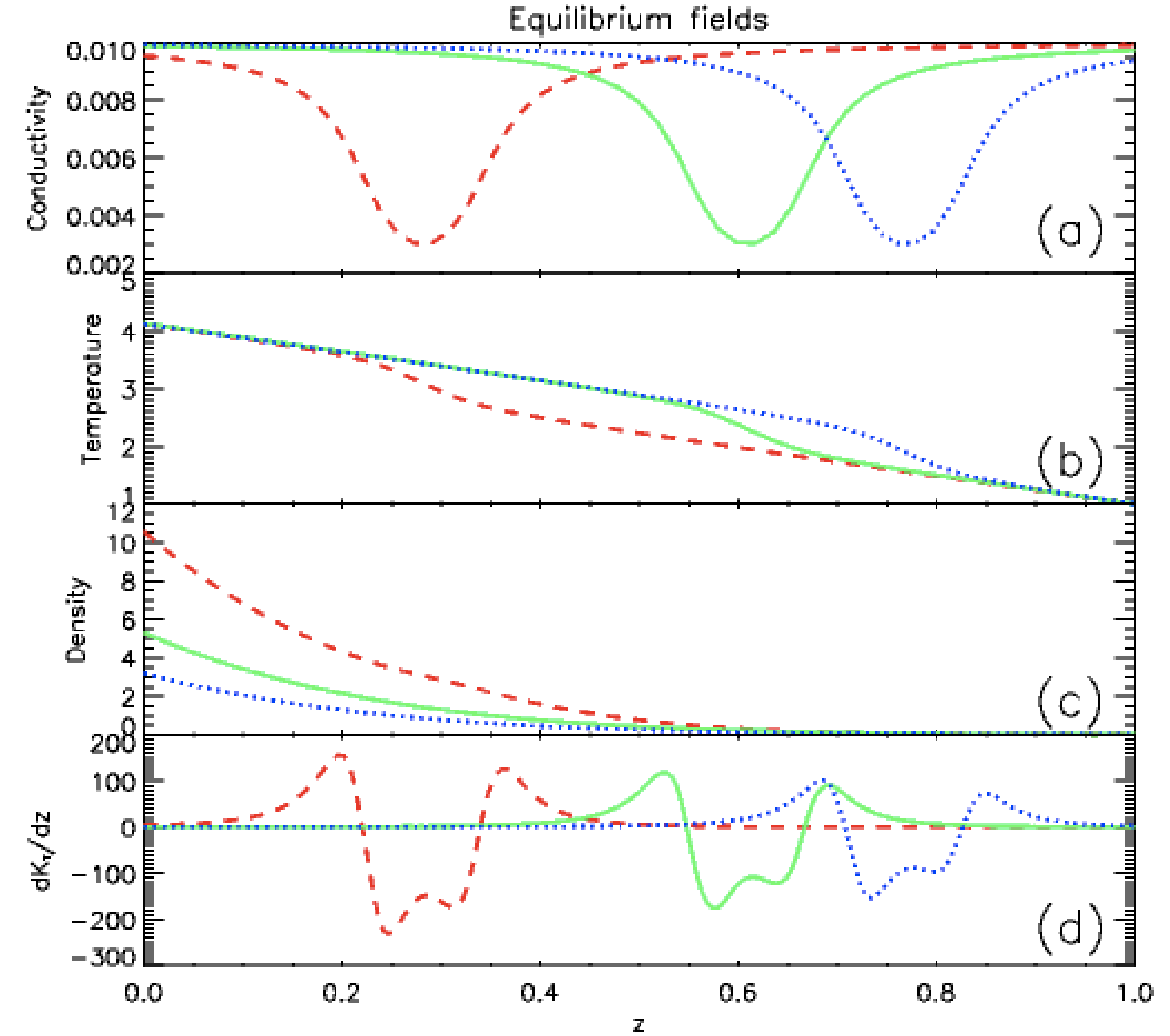


Fig. 3. a) three different conductivity's hollow: $T_{\mathrm{bump}} = 3.1$ (dashed red line), $T_{\mathrm{bump}} = 2.3$ (solid green line) and $T_{\mathrm{bump}} = 1.9$ (dotted blue line), with $\mathcal{A} = 70\%$, $e = 0.4$ and $\sigma = 7$; **b)** corresponding equilibrium temperatures; **c)** equilibrium densities; **d)** and equilibrium parameters $d\mathcal{K}_T/dz$.

Computation of the eigenmodes

$$
\left\{
\begin{aligned}
\lambda R &= -\frac{du_z}{dz} - \frac{d\ln\rho_0}{dz}u_z \\
\lambda u_z &= -T_0\frac{dR}{dz} - T_0\frac{d\theta}{dz} + \tilde{g}\theta \\
\lambda\theta &= \gamma\chi_0\left[\frac{d^2\theta}{dz^2} + 2\left(\frac{d\ln K_0}{dz} + \frac{d\ln T_0}{dz}\right)\frac{d\theta}{dz}\right. \\
&\qquad \left. + \frac{d\ln T_0}{dz}\frac{d\mathcal{K}_T}{dz}\theta\right] - (\gamma-1)\frac{du_z}{dz} - \frac{d\ln T_0}{dz}u_z,
\end{aligned}
\right. \tag{34}
$$

$$A\psi = \lambda B\psi,$$

We compute the whole spectrum (QZ solver) or a given eigenvalue/eigenvector (iterative Arnoldi-Chebyshev solver)

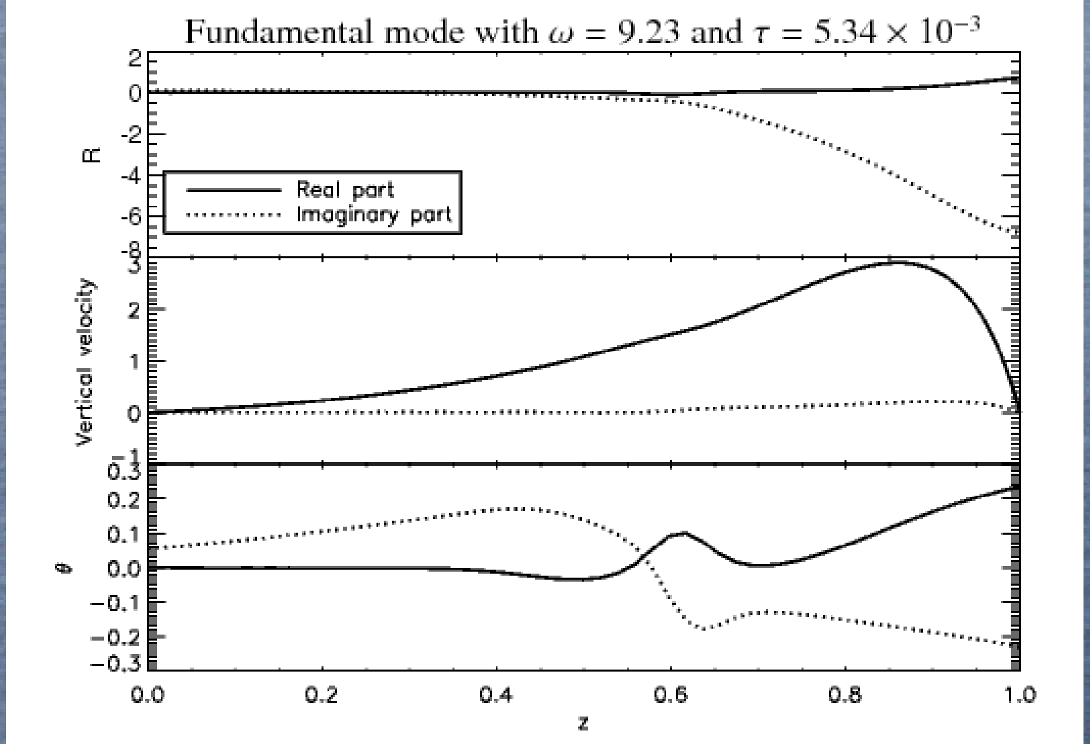
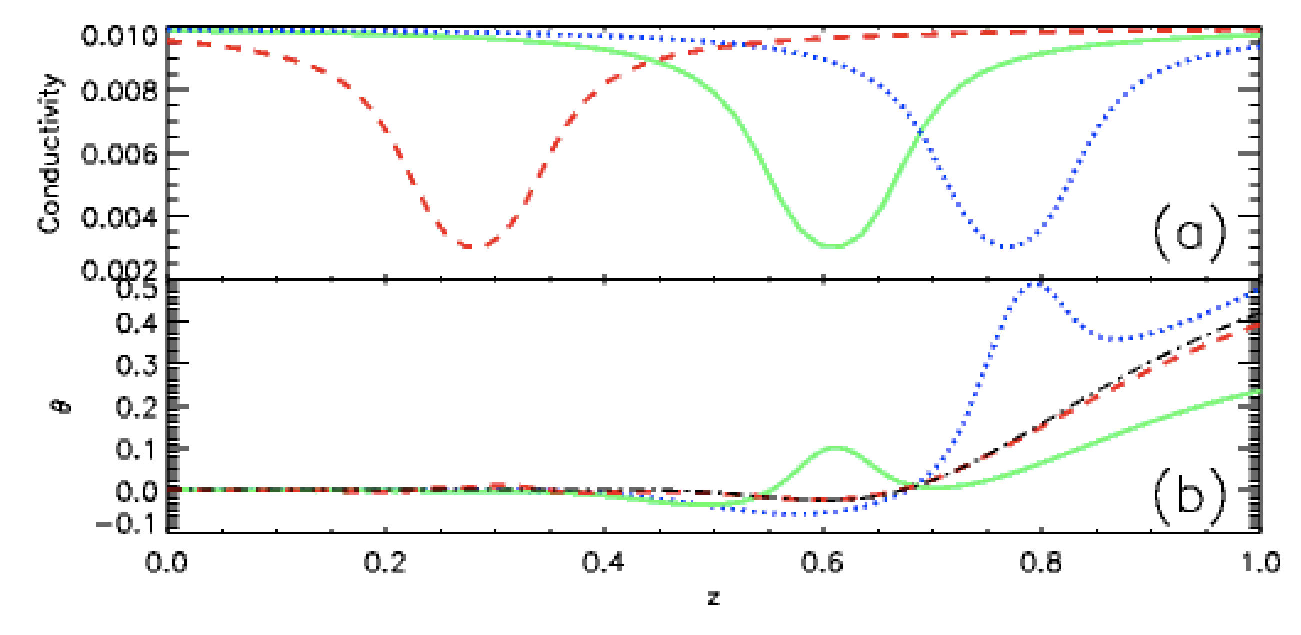


Fig. 4. Example of unstable eigenfunctions (R, u_z, θ) solutions of the system (34), for $T_{\text{bump}} = 2.3$, $e = 0.4$ and $\sigma = 7$ (the equilibrium setup is the one displayed by a solid green line in Fig. 3).



Parametric survey and the corresponding instability strips

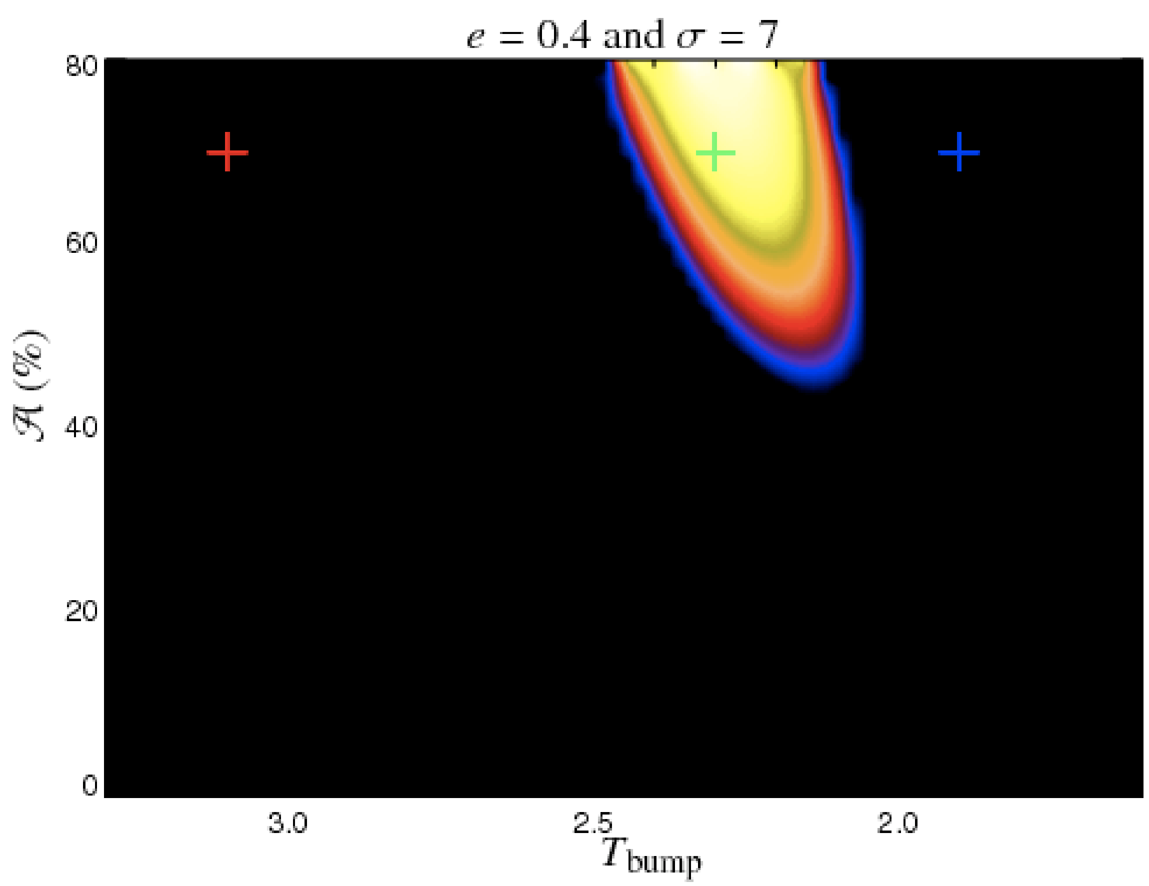


Fig. 7. Instability strip given by isocontour in growth rate τ of the fundamental mode for different values of $T_{\rm bump}$ and $K_{\rm min}$ expressed in % of $K_{\rm max}$. The three crosses correspond to the three particular computations done in Fig. 3 and 6 for $T_{\rm bump} = 1.9$, $T_{\rm bump} = 2.3$ and $T_{\rm bump} = 3.1$.

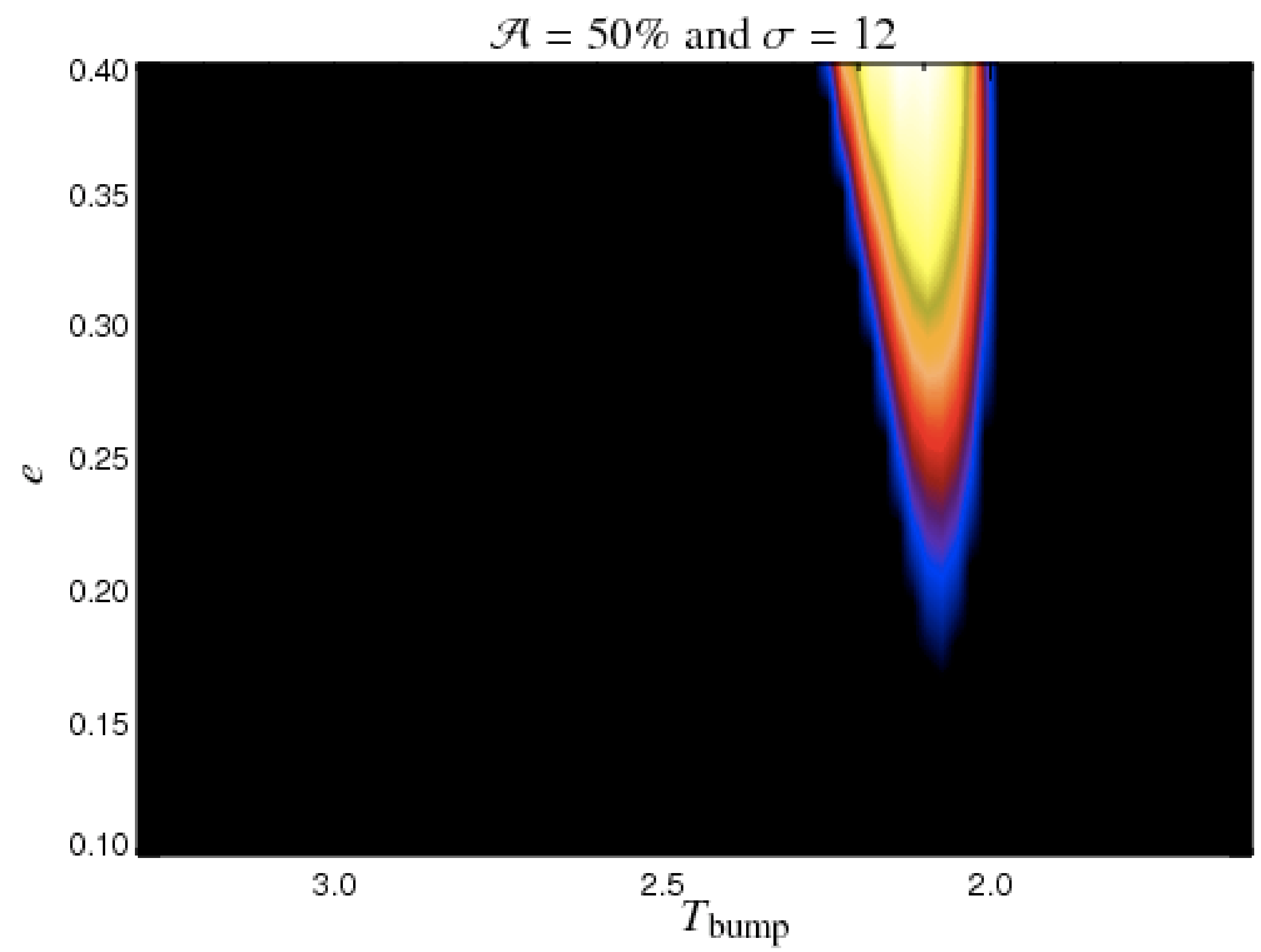


Fig. 9. Instability strip given by isocontour in growth rate τ of the fundamental mode for different values of $T_{\rm bump}$ and e.

Why we got instability strips?

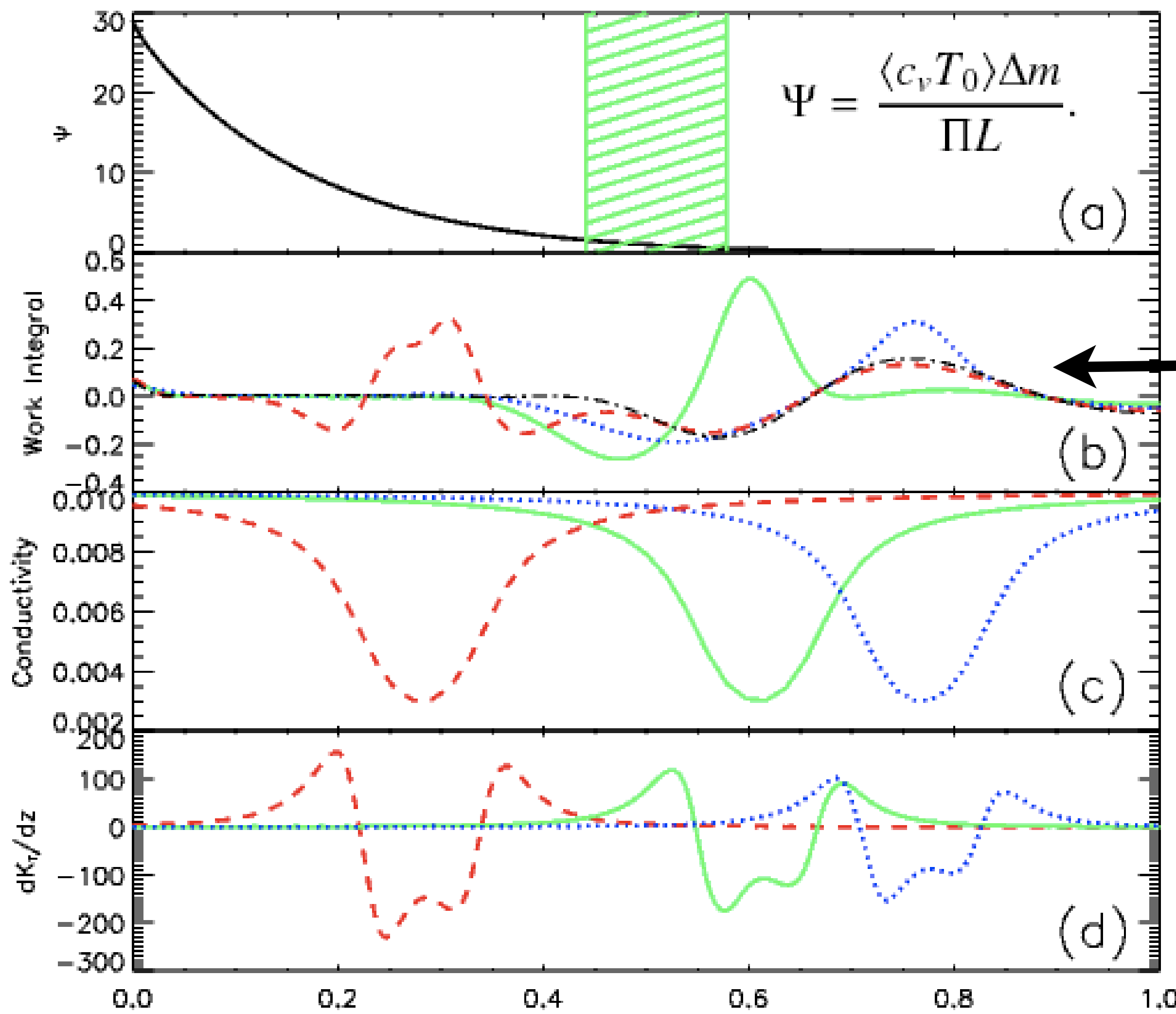


Fig. 10. a) coefficient Ψ as introduced in Eq. (19) plotted over the entire box. The green hatched zone superimposed represents the location where we have $\Psi \sim 1$. **b**) the real part of numerator of the work integral as given in Eq. (39) plotted for three different equilibrium models ($T_{\rm bump} = 1.9$ in dotted blue line, $T_{\rm bump} = 2.3$ in solid green line and $T_{\rm bump} = 3.1$ in dashed red line) corresponding to those studied in Fig. 4 and in Fig. 6. The case with a constant radiative conductivity $K_0(T_0) = K_{\max}$ is superimposed as a dot-dashed black line. **c**) corresponding radiative conductivity profiles. **d**) corresponding equilibrium field $d\mathcal{K}_T/dz$.

work integral

$$\lambda = \frac{\int_0^1 \left(p' \frac{du_z^*}{dz} - \tilde{g}\rho' u_z^* \right) dz}{\int_0^1 |u_z|^2 \rho_0 dz}. \qquad (39)$$

The physical criterion for instability:
$\Psi \approx 1$

Let's go back to the Pencil Code...

What we are doing **What we are supposed to do! ;-)**

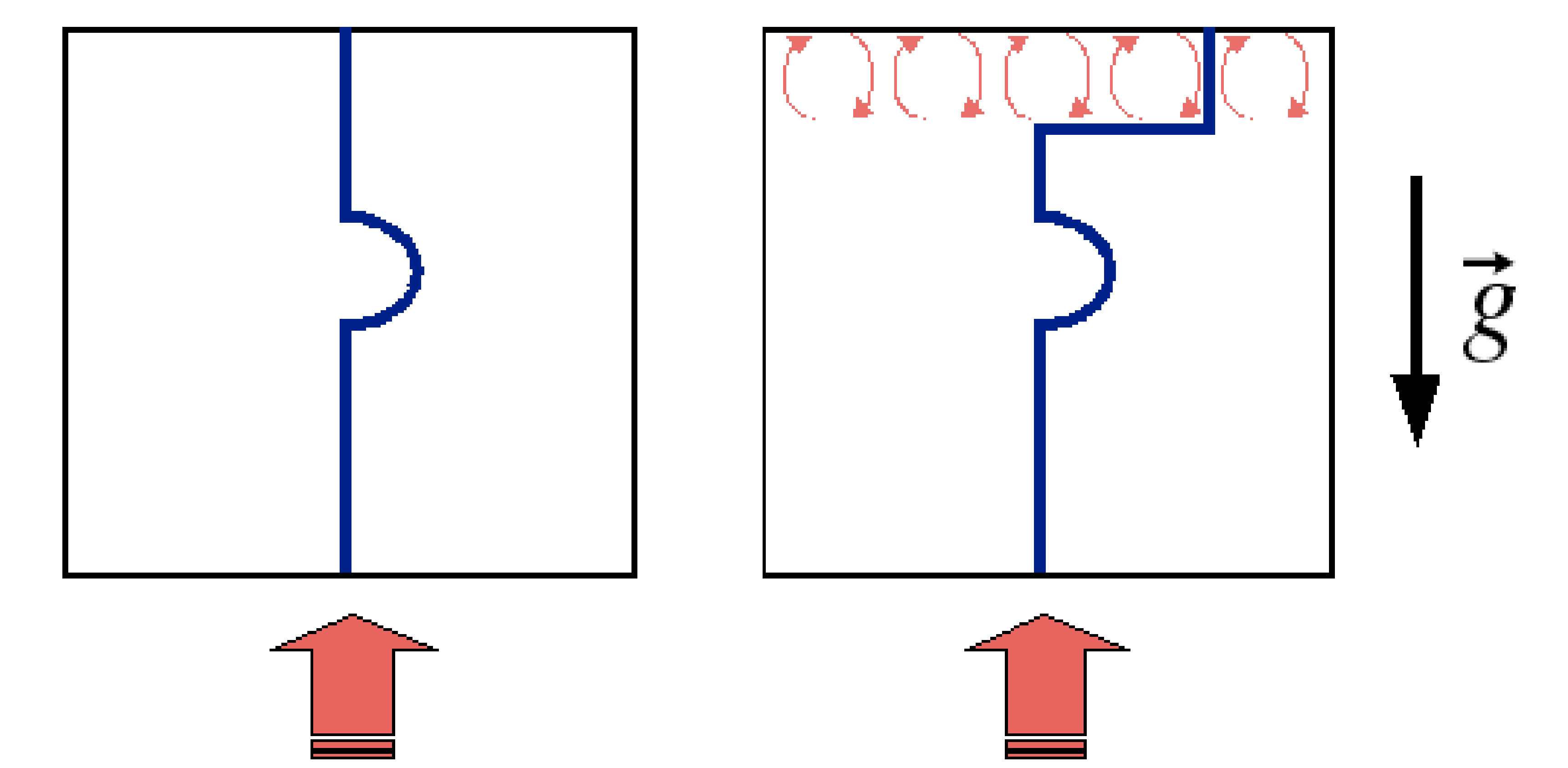


Fig. 11. *Left*: entirely radiative model which we have discussed in this article. *Right*: the (blue) curve emphasises the conductivity profile whereas small (red) rolls denote a convective zone close to the surface. The large (red) arrow expresses the radiative flux coming from the bottom of our layer.

The big problem: the radiative diffusion

The most favorable setups lead to $\chi = \frac{K}{\rho c_p} \sim 1$ at the surface

Why? Because of the condition $\Psi = \frac{<c_v T_0> \Delta m}{\Pi L} \sim 1$

One needs to deal implicitly the radiative diffusion term

Our semi-implicit approach: The Rosenbrock scheme

$$\frac{dT}{dt} = \frac{1}{\rho c_p}\,\mathrm{div}(K(T)\vec{\nabla}T) \Rightarrow \frac{T^{n+1}-T^n}{dt} = \frac{\Lambda_x(T^n)+\Lambda_x(T^{n+1})}{2} + \frac{\Lambda_y(T^n)+\Lambda_y(T^{n+1})}{2}$$

$$\Lambda_x(T) = \frac{1}{\rho_{i,j}c_p 2dx^2}\left[(K_{i+1,j}+K_{i,j})(T_{i+1,j}-T_{i,j}) - (K_{i,j}+K_{i-1,j})(T_{i,j}-T_{i-1,j})\right]$$

where $K_{i,j}$ means $K(T_{i,j})$

$$\Lambda(T^{n+1}) \simeq \Lambda(T^n) + \left(\frac{\partial\Lambda}{\partial T}\right)_n (T^{n+1}-T^n)$$

jacobian matrix J

$$\left(I - \frac{dt}{2}J_x - \frac{dt}{2}J_y\right)T^{n+1} = \left(I - \frac{dt}{2}J_x - \frac{dt}{2}J_y\right)T^n + dt\left[\Lambda_x(T^n)+\Lambda_y(T^n)\right]$$

$$\boxed{\left(I - \frac{dt}{2}J_x - \frac{dt}{2}J_y\right)}\left(T^{n+1} - T^n\right) = dt\left[\Lambda_x(T^n) + \Lambda_y(T^n)\right]$$

matrix A of size N²x N² quite heavy to invert
using SOR or conjuguate gradient methods

The operator-splitting approach:

$$A(T^{n+1} - T^n) = \left(I - \frac{dt}{2}J_x\right)\left(I - \frac{dt}{2}J_y\right)\left(T^{n+1} - T^n\right) - \boxed{\frac{dt^2}{4}J_xJ_y(T^{n+1} - T^n)}$$

enclosed in the explicit
RK3 errors

$$\Rightarrow \left(I - \frac{dt}{2}J_x\right)\left(I - \frac{dt}{2}J_y\right)\left(T^{n+1} - T^n\right) = dt\left[\Lambda_x(T^n) + \Lambda_y(T^n)\right]$$

Rosenbrock splitting:

$$\begin{cases} \left(I - \frac{dt}{2} J_x\right) \alpha = \Lambda_x(T^n) + \Lambda_y(T^n) \\ \left(I - \frac{dt}{2} J_y\right) \beta = \alpha \\ T^{n+1} = T^n + dt\beta \end{cases}$$

Now the final question is: where should we put this implicit stuff in the code?

Our try: the explicit physics become a source term after the rk_2n call in run.f90

$$\text{source} = \frac{T^{n+1}_{\text{explicit}} - T^{n}_{\text{explicit}}}{dt}$$

⇒ two ways to do that...

First way: half of this source term is added in each direction

$$\begin{cases} \left(I - \frac{dt}{2} J_x\right) \alpha = \Lambda_x(T^n) + \Lambda_y(T^n) + \dfrac{\text{source}}{2} \\ \\ \left(I - \frac{dt}{2} J_y\right) \beta = \alpha + \dfrac{\text{source}}{2} \\ \\ T^{n+1} = T^n + dt\beta \end{cases}$$

Second way: the source term is added at the end

$$\begin{cases} \left(I - \frac{dt}{2} J_x\right) \alpha = \Lambda_x(T^n) + \Lambda_y(T^n) \\ \\ \left(I - \frac{dt}{2} J_y\right) \beta = \alpha \\ \\ T^{n+1} = T^n + dt\left(\beta + \text{source}\right) \end{cases}$$

And the winner is...