Gravitational turbulent instability of AdS

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energie atomique • energies alternatives

Based on: 1109.1825 and **1208.5772** [also: **1105.4167**]

Collaborators:

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[Related work: Bizon, Rostworowski, 1104.3702 ; Holzegel, Smulevici 1110.6794]

The Holographic Way, NORDITA, Stockholm, Sweden **Oct. 2012**

Motivation :

- **The AdS / CFT correspondence relates a (d-1)-dim QFT with a d-dim theory of (quantum) gravity:**

  - Any gravitational phenomena should have an equivalent CFT description, and vice-versa.

  - General gravity is now a tool to study field theory open questions:
  holographic description of condensed matter systems;
  transport properties / hydrodynamic description strongly coupled field theories;
  AdS/QCD (RHIC); **quantum turbulence . . .**

  - Also works the other way around in its strong version:
  weak coupling CFT as a definition for non-perturbative String Theory.

- **Here, we want to study *far from equilibrium dynamics* in gravity, and try to understand its field theory interpretation.**

  Two options:

  1. Full time evolution … hard!

  2. Poor's man approach:

  break down of perturbation theory → onset of interesting dynamics.

Outline :

1. **Anti-de Sitter (AdS) properties. Standard lore & Heuristics**
2. **Outline of Perturbative construction**
3. **Linear Perturbations**
4. **General Structure of non-linear construction:**
   - **4a. *Geons***
   - **4b. Colliding Geons → AdS is non-linearly unstable**
5. **String Theory Embedding & Field theory implications**
6. **Gravitational hairy black holes with a single $U(1)$.**
7. **Conclusions & Open questions**

Anti-de Sitter spacetime :

Anti-de Sitter (AdS) space is a maximally symmetric solution of

$$S = \frac{1}{16\pi G}\int \mathrm{d}^d x\sqrt{-g}\left[R + \frac{(d-1)(d-2)}{L^2}\right]$$

which in *global* coordinates can be written as: ($\Lambda = -1/L^2$)

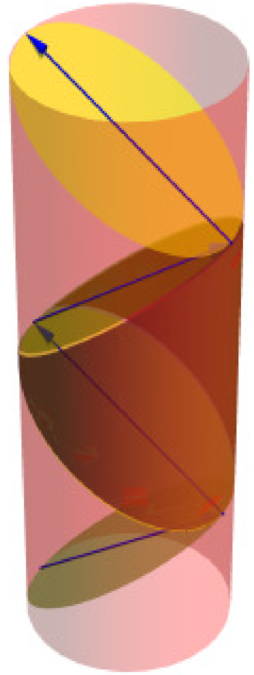
$$\mathrm{d}s^2 \equiv \bar{g}_{ab}\mathrm{d}x^a \mathrm{d}x^b = -\left(\frac{r^2}{L^2}+1\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\frac{r^2}{L^2}+1} + r^2 \mathrm{d}\Omega_{d-2}^2$$

Note that the Poincaré coordinates

$$\mathrm{d}s^2 = R^2(-\mathrm{d}\tau^2 + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}) + \frac{L^2\mathrm{d}R^2}{R^2}$$

do <u>not</u> cover the entire spacetime. We will <u>not</u> use them.

Anti-de Sitter spacetime :

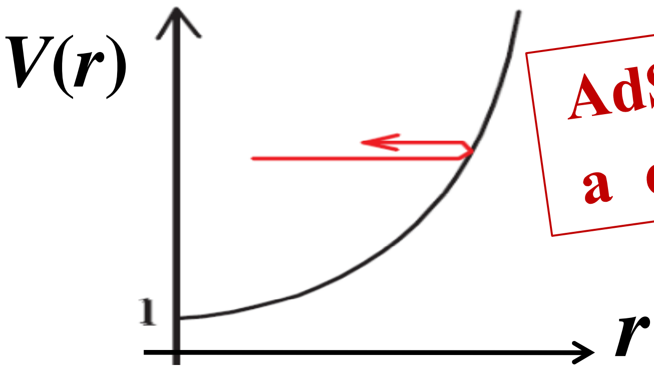


- **The turbulent instability will be described in <u>global</u> AdS**
- **Conformally, global AdS is described by interior of cylinder**

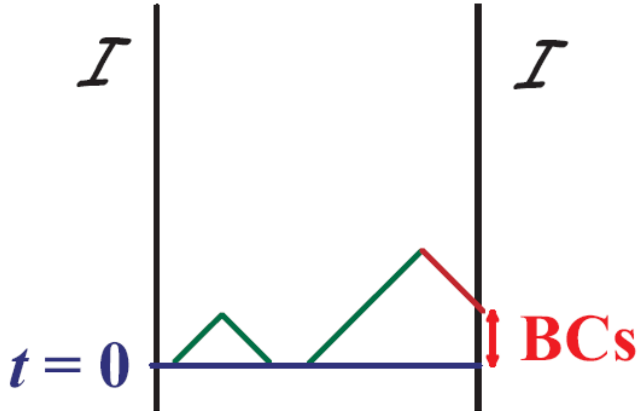
  The dual field theory lives on $R_t \times S^{d\text{-}2}$.

- **With *E, J* preserving boundary conditions,**

  waves bounce off at infinity and return in finite time.



AdS behaves as a confining box



- Poincaré coordinates cover only the brown-shaded region;

  Poincaré horizon destroys confining box property;

  Therefore, the instability *should* <u>not</u> be present

A difference between Minkowski, dS & AdS :

- **At the linear level,**
  AdS spacetime is as stable as the Minkowski or de-Sitter (dS) spacetimes.

- **For the Minkowski & dS spacetimes, it has been shown that**
  small, but finite, perturbations remain small [Christodoulou-Klainerman `93]
  So, Minkowski & dS are also non-linearly stable [Friedrich `86]

- **But this has not been shown for AdS !**

- **Claim: AdS is linearly stable but non-linearly unstable**
  Generic small (but finite) perturbations of AdS become large
  and *eventually* form black holes.

- **The energy cascades from low to high frequency modes**
  in a manner reminiscent of the onset of turbulence.

… oops :

- **Doesn't this claim contradict the fact that AdS is supersymmetric ?**

- **Doesn't this contradict the fact that there is a positivity energy theorem for AdS ?**

The short answer is NO :

- **Positivity energy theorem:** if matter satisfies the dominant energy condition, then $E \geq 0$ for all non-singular, asymptotically AdS initial data, being zero for AdS only.

  - This ensures that **AdS cannot decay** into state with lower E.

  - **It does <u>not</u> ensure that a small amount of energy added to AdS will not generically form a small BH.**

  - That is usually ruled out by arguing that waves disperse.

    … this does *not* happen in AdS because " it's a box ".

Why is AdS unstable? (Heuristics)

- **Dafermos & Holzegel:** linearized perturbations of AdS do not decay
  … suggests that non-linear corrections will grow in time
- **M. Anderson:** AdS acts like a confining finite box.

  Any generic finite excitation added to this box might be expected to
  explore *all* configurations consistent with the conserved charges of AdS
  ... including small black holes (argument predicts ergodic time).
- Special (fine tuned) solutions might *not* lead to formation of BHs:
  - We will see that for each linearized gravitational mode
    there will be an associated non-linear solution: a *geon* .
  - These solutions are special since they are exactly periodic in time
    and invariant under a single continuous symmetry (single KVF).
  - (AdS) Geons are analogous to gravitational plane waves (flat background)
- We then expect colliding geons to behave like colliding Grav. plane waves :

  …well, colliding exact plane waves produce singularities (BHs) [Penrose '71]

Perturbative construction of geons (1)

- **Expand the metric around global AdS as** $g = \bar{g} + \sum_i \epsilon^i h^{(i)}$
- **At each order *i* in perturbation theory, the Einstein equations yield:**

$$\tilde{\Delta}_L h^{(i)}_{ab} = T^{(i)}_{ab}$$

where $T^{(i)}$ depends on $\{h^{(j \leq i-1)}\}$ and their derivatives and

$$2\tilde{\Delta}_L h^{(i)}_{ab} \equiv -\bar{\nabla}^2 h^{(i)}_{ab} - 2\bar{R}_a{}^c{}_b{}^d h^{(i)}_{cd} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a}\bar{\nabla}^c h^{(i)}_{b)c}$$

- Technically, work with **Kodama-Ishibashi '03 gauge invariant formalism.**

That is, work with **gauge –invariant scalars** that obey **master equation:**

$$\Box_2 \Phi^{\alpha,(i)}_{\ell m}(t,r) + V^{(i)}_\ell(r)\Phi^{\alpha,(i)}_{\ell m}(t,r) = \tilde{T}^{\alpha,(i)}_{\ell m}(t,r)$$

ℓ, m : polar & azimuthal quantum # of the KI **spherical harmonics $Y_{\ell m}(\theta,\phi)$**

There are many **{ *ℓ, m* } building blocks excited** at higher order !

- **Metric perturbation 2-tensor recovered through a linear differential map:**

$$h_{ab} = h_{ab}(\Phi) \qquad \text{(in a given gauge)}$$

Perturbative construction of geons (2)

Boundary conditions:

- Regularity at the origin ($r = 0$) requires (at least) the decay:

$$\Phi_{\ell m}^{\alpha,(i)} \sim \mathcal{O}(r^\ell)$$

- Close to the AdS conformal boundary (as $r \to \infty$)

$$\Phi_{\ell m}^{\alpha,(i)}(t,r) \sim R_{\ell m}(t) + \frac{S_{\ell m}(t)}{r} + \ldots .$$

Surprisingly, if we want to keep the boundary metric fixed (ie, if we want the perturbations to preserve *global* AdS asymptotics), we need to choose:

$$S_{\ell m}(t) = 0$$

This is also the choice that gives finite energy perturbations for the standard definition of "gravitational energy"

Linear Perturbations ($i = 1$)

- **At the linear level, $T = 0$, we can decompose our perturbations in t as**

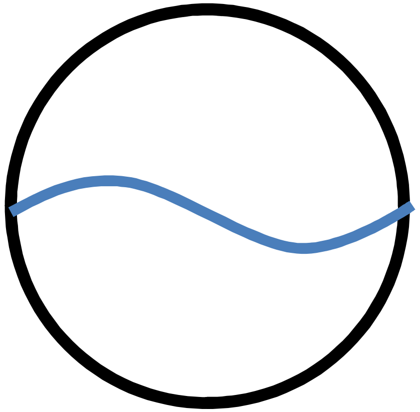
$$\Phi^{\alpha,(i)}_{\ell m}(t,r) = \Phi^{\alpha,(i),c}_{\ell m}(r)\cos(\omega_\ell t) + \Phi^{\alpha,(i),s}_{\ell m}(r)\sin(\omega_\ell t)$$

= 0 (initial data choice)

- **Because AdS acts like a confining box,**

only certain frequencies are allowed to propagate (p is radial overtone):

$$\omega_\ell L = 1 + \ell + 2p$$



These are the so-called normal modes of (global) AdS.

- **$\mathrm{Im}\,\omega = 0 \;\rightarrow\;$ AdS is linearly stable**

General Structure of higher order ($i > 1$)

1. **Start with a given perturbation** $\Phi_{\ell m}^{\alpha,(i),\kappa}(r)$ **, and determine the corresponding** $h_{\ell m}^{(i)}(t,r,\theta,\phi)$ **through the KI linear differential map**
   [Kodama-Ishibashi '03]

$$h_{ab} = h_{ab}(\Phi) \qquad \text{(in a given gauge)}$$

2. **Compute** $T_{ab}^{(i+1)}$**, in RHS of Einstein eq** $\tilde{\Delta}_L h_{ab}^{(i+1)} = T_{ab}^{(i+1)}$

   and decompose it as a sum of the fundamental building blocks $\mathcal{T}_{ab}^{\ell m}$

3. **Compute the source term** $\tilde{T}_{\ell m}^{\alpha,(i+1)}(t,r)$ **in the RHS of KI master eq**

$$\Box_2 \Phi_{\ell m}^{\alpha(i+1)}(t,r) + V_\ell^{(i)}(r) \Phi_{\ell m}^{\alpha(i+1)}(t,r) = \tilde{T}_{\ell m}^{(i+1)}(t,r)$$

   and determine $\Phi_{\ell m}^{\alpha,(i+1)}(t,r)$

General Structure of higher order ($i > 1$)

4. **If** $\tilde{T}^{\alpha,(i+1)}_{\ell m}(t,r)$ **has an harmonic time dependence** $\cos(\omega\, t)$ **, then** $\Phi^{\alpha,(i+1)}_{\ell m}(t,r)$ **will exhibit the same dependence,**

   <u>EXCEPT</u> when ω **agrees with one of the normal frequencies of AdS:**

$$\Phi^{\alpha,(i+1)}_{\ell m}(t,r) = \Phi^{\alpha,(i+1),c}_{\ell m}(r)\cos(\omega\, t) + \Phi^{\alpha,(i+1),s}_{\ell m}(r)\, t\, \sin(\omega\, t)$$

The latter mode is said to be *<u>RESONANT</u>* .

5. **If for a given perturbation one can construct** $\Phi^{\alpha,(i)}_{\ell m}$ **to any order** i **, without ever introducing a term growing linearly in time, the solution is said to be stable ; otherwise it is unstable .**

Construction 1 : single Geon

[ℓ, m : quantum # $Y_{\ell m}(\theta,\phi)$]

1. **Start with a single mode** $\ell = m = 2$ $(\omega_2 L = 3)$ **initial data [a normal mode] .**

2. **At** 2^{nd} **order** there are **no resonant modes: solution is regular everywhere**

3. **At** 3^{rd} **order,** there is **a resonant mode, but** one can **set the amplitude of** of the **growing mode to zero by changing the** ω **slightly:**

$$\omega L = 3 - \frac{14703}{17920}\epsilon^2$$

- **The structure of the equations indicate that there is only one resonant term at each odd order, and that the amplitude of the growing mode can be set to zero by correcting the frequency**

- **One can compute the asymptotic charges to fourth order, and they obey to the first law of thermodynamics:**

$$E_g = \frac{3J_g}{2L}\left(1 - \frac{4901\,J_g}{7560\pi L^2}\right), \quad \omega_2 = \frac{3}{L}\left(1 - \frac{4901\,J_g}{3780\pi L^2}\right)$$

This also defines our expansion parameter ϵ **:** $J_g = \frac{27}{128}\pi L^2 \epsilon^2$

Construction 1: single Geon (2)

- **We adjust our initial data such that the time dependence of our linear mode can always be recast as $\cos(\omega t - m\phi)$ which is invariant under:**

$$K = \frac{\partial}{\partial t} + \frac{\omega}{m}\frac{\partial}{\partial \phi}$$

Single **Killing vector field (KVF) of geon !**

∂_t , ∂_ϕ of original AdS are not KVFs !

- **At non-linear level, we have the same type of symmetry … but ω changes . So, it's stationary (periodic) but not axisymmetric neither time-independent !**

Construction 2: linear combination of Geons

1. **Start with linear combination of $\ell = m = 2$** $(\omega_2 L = 3)$ ***and*** **$\ell = m = 4$** $(\omega_4 L = 5)$

2. **Like in the single mode initial data, at second order there are no resonant modes and the solution can be rendered regular everywhere**

3. **At third order, there are three resonant modes:**

   - **The amplitude of the growing modes in two of the resonant modes can be removed by adjusting the frequency of the initial data** ($\omega_2 L = 3 + \ldots$ and $\omega_4 L = 5 + \ldots$) **like we did for single mode initial data**

   - **The amplitude of the growing mode with the largest frequency cannot be set to zero** ($\omega L = 7, \ \ell = m = 6$) **!**

AdS is non-linearly unstable !

Construction 2: linear combination of Geons (2)

- The frequency $\omega L = 7$ of the growing mode is higher than any of the frequencies we started with: $\omega_2 L = 3$ and $\omega_4 L = 5$ **!**

- The energy (amplitude) is thus transferred to modes of higher frequency

- Expect this to continue: When the $\omega L = 7,\ \ell = m = 6$ mode grows, it will source even higher frequency modes with growing amplitude

Conjecture:

The endpoint of this gravitational turbulent instability is a rotating AdS black hole

- Timescale for BH formation given by breakdown of perturbation theory:

$$\epsilon^3 t \sim \epsilon \quad \rightarrow \quad t_{\rm BH} \sim 1/\epsilon^2$$

Further support for the conjecture:
time evolution of similar spherical scalar field instability in AdS

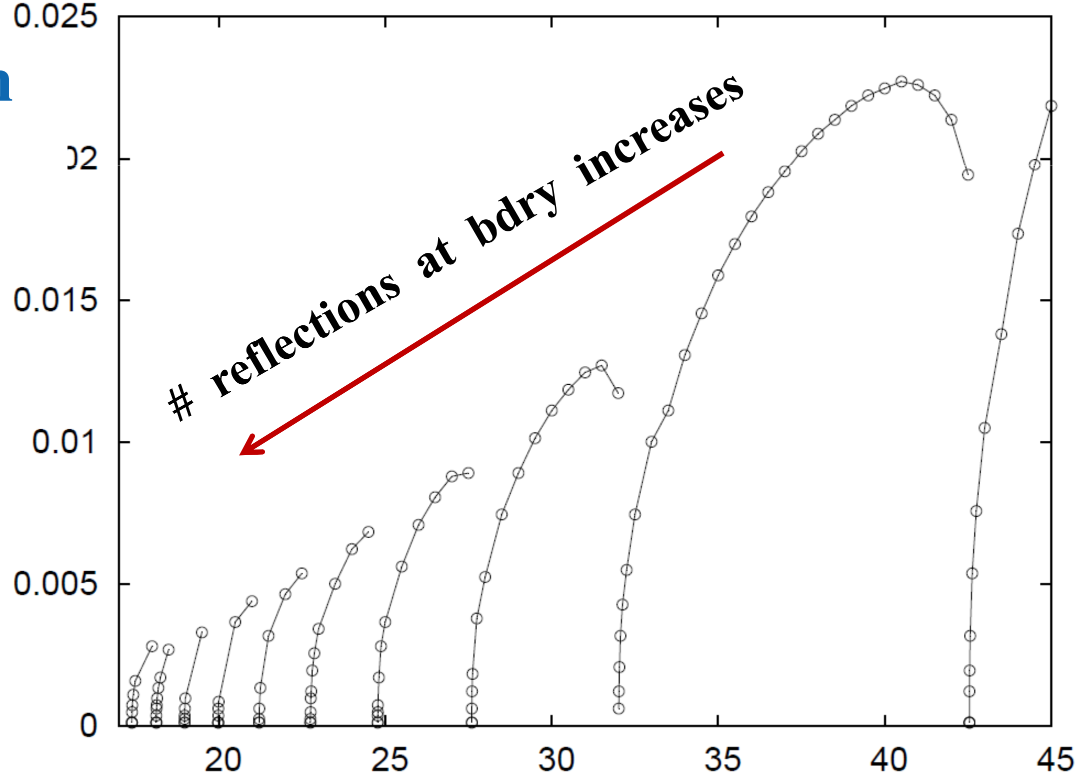
- **Time evolution of Spherical scalar field shell in AdS: collapse to BH**

  [Bizon-Rostworowski '11, Garfinkle '11]

- **No matter how small the initial amplitude is ,**

  the curvature at the origin grows and a small BH forms.

Horizon radius



Amplitude α of initial perturbation

- At $r_H \sim 0$ a naked singularity forms (but very fine-tuned initial data).

- Same critical behavior as Choptuik (BHs so small that don't see AdS radius)

- In the flat case Choptuik told us:

  Initial scalar field profile $\Phi \sim \alpha$ f(r).

  .Small α → waves scatter and die-off at ∞
  .Critical α^* → naked singularity forms.
  Near it: $M_{BH} \sim (\alpha - \alpha^*)\gamma$ with $\gamma \sim 0.37$
  .Large α → large BH forms

What are the necessary ingredients for instability?

- **Resonances occur because normal modes of AdS are commensurable**

$$\omega_i - \omega_j = \omega_k$$

- **Geons** are **non-linearly** **stable** to this non-linear mechanism :

  normal modes of Geons are continuous deformations of AdS normal modes…

  Asymptotic resonances at large ℓ *not* "strong enough" to trigger instability.

  $\Sigma\, \omega_i = O(\ell^{-|a|})$ **OD, Horowitz, Marolf, Santos, 1208.5772 .**

- **More intricate!**

  It seems that even systems with "just" asymptotic resonances are unstable:

  "Minkowski in a cavity", Maliborski , 1208.2934

  "AdS enclosed in a box ", Buchel, Lehner, Liebling, 1210.0890

Description within String theory

- **Consider II B string theory on $AdS_5 \times S^5$, with AdS length scale L**
- **There are two energy scales:**
  the Planck energy E_P and the string energy E_S, with $E_S < E_P$ ($E_P = N^2/L$)
- **Possibilities:**
  - **If the initial energy is larger than $E > E_P$, one forms a 5D AdS BH**
  - **If the initial energy is $E_{corr} < E < E_P$, one forms a 10D black hole**

    Here, E_{corr} is the energy of a BH of the string scale size

    [Susskind, Horowitz-Polchinski]
  - **If the initial energy is $E_s < E < E_{corr}$, one forms an excited string**
  - **If $E < E_S$, cascade stops at freq. $\omega = E$: gets a gas of particles in AdS**
- **Thus, at the quantum level there is no continuous cascade or instability!**
  The instability is probably *not* present at finite N (weak coupling QFT)
  $\Longrightarrow$ no source of a problem for the dual field theory
- **But, what is dual description of the instability at large N** (strong coupling) **?**

Field theory implications

- **Fact that one evolves to state of max entropy (BH forms, 2^{nd} law → S↗) can be viewed as thermalization (evolution towards equilibrium) ; not in the canonical ensemble (T is not fixed!), but in the microcanonical ensemble since E, J is fixed by our BCs**

- **All field theories with a gravity dual will show this cascade of energy like the onset of turbulence**

- Interesting observation:
  - In 2+1 dimensions, classical turbulence has an inverse energy cascade due to an extra conserved quantity - the enstrophy. This is responsible for hurricanes and other weather phenomena
  - Our gravitational system is dual to a strongly coupled quantum theory
  - Our results indicate that in 2+1 strongly coupled QFT there is a standard energy cascade.

Field theory implications (2)

- **More intriguing, from the CFT perspective, is the existence of Geons**

- **At the linear level, these are spin-2 excitations**

- **A nonlinear geon is like a bose condensate of these excitations**

These high energy states do NOT thermalize ! … no BH forms, no decay in *t* …

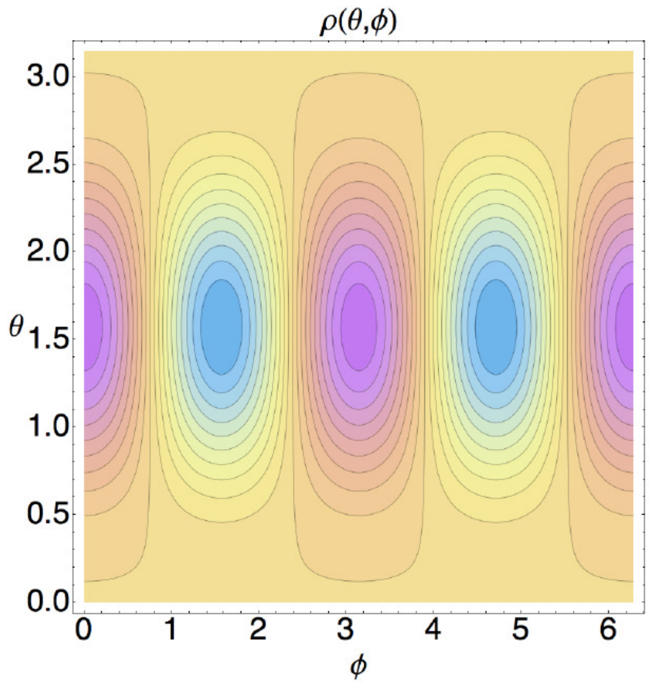
[Large *N* (strongly coupled) field theories act classically, and classical field theories almost never thermalize]

- **The boundary stress-tensor contains regions of negative and positive energy density around the equator.**

  It is invariant under :

$$K = \frac{\partial}{\partial t} + \frac{\omega}{m}\frac{\partial}{\partial \phi}$$

which is timelike near the poles but spacelike near the equator



Gravitational hairy BHs with a single $U(1)$

- **One can add a small black hole inside a geon: only constraint is that the Killing field of the Geon must be null on the horizon:**

$$\Omega_H = \frac{\omega}{m}$$

- There are **many Geons (*m's*); thus whole new class of BHs w/ single $U(1)$: they are stationary but not axisymmetric neither time independent !**

  - **This seems to contradict rigidity theorems**
    [Hawking,'72; Hollands, Ishibashi, Wald, '06; Isenberg, Moncrief, '06]

    which show that stationary black holes must be axisymmetric…
    (RT assumes $\exists$ stationary KV ∂_t that is not normal to H^+ … $\Longrightarrow \exists\ \partial_\phi$)

  - **Well,** these theorems are **not applicable to these BHs,** since our **(stationary) single KVF generates the horizon, ie it is normal to horizon**

- **Aside note: *Scalar hairy* BHs** with **single** $U(1)$ ***explicitly* constructed** in
  [OD, Horowitz, Santos 1105.4167]

The Kerr-AdS BH is *NOT* the unique stationary black hole in AdS

Conclusions & Open questions

Conclusions:

- AdS spacetime is non-linearly unstable:
  generic small perturbations become large and (probably) form BHs
- For each linearized gravity mode, there is an exact, nonsingular geon
- Dual field theory shows generic turbulent cascade to maximum entropy state but there are special states (geons) that do not thermalize

Open questions:

- Construct non-linearly (numerically) the geons
- What are fundamental ingredients for the non-linear instability ?
- Time evolution needed
- What can we learn about turbulence ?

Thank you!