CFT PROBES OF BULK GEOMETRY
& CAUSAL HOLOGRAPHIC INFORMATION

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Based on:

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NORDITA Oct. 1, 2012

OUTLINE

- Motivation & Background
- Features of Extremal Surfaces
- Probing Horizons
- Causal Holographic Information
- Summary & Future directions

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS × K *"on boundary"*

Key aspects:

* Gravitational theory maps to non-gravitational one!
* *Holographic*: gauge theory lives in fewer dimensions.
* Strong/weak coupling duality.

Invaluable tool to:

~ Use gravity on AdS to learn about strongly coupled field theory

~ Use the gauge theory to define & study quantum gravity in AdS

Pre-requisite: Understand the AdS/CFT 'dictionary'...

Motivation

To understand the AdS/CFT dictionary;
esp. how does spacetime (gravity) emerge?

- Most QG questions rest on bulk locality (& its breakdown)...
- Given a specific bulk location, what quantities in the CFT should we examine in order to learn about the physics at that location?
- How deep into the bulk can various CFT probes see?
- esp.: can convenient CFT probes see into a black hole?
- Given full knowledge of physics ($\rho_{\mathcal{A}}$) in a certain boundary region $\mathcal{A}$,
  - in what region of the bulk does it determine the bulk geometry?
  - in what region of the bulk is it sensitive to the bulk geometry?

Probes of bulk geometry

The bulk metric can be extracted using various CFT probes (which are described by geometrical quantities in the bulk):

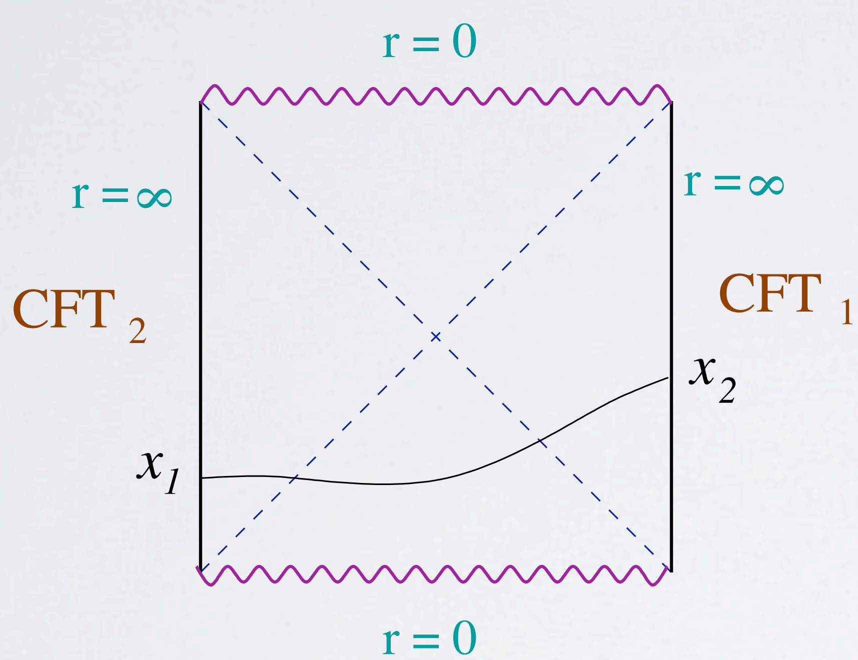
Examples:

CFT probe	bulk quantity
∗ expectation values of local gauge-invariant operators	asymptotic fall-off of corresponding conjugate field
∗ correlation functions of local gauge-invariant operators	in WKB approx., proper length of corresponding geodesic
∗ Wilson loop exp. vals.	area of string worldsheet
∗ entanglement entropy	vol of extremal co-dim.2 surface

Correlators see inside horizons

Consider correlators of high-dimension operators in CFT

3-dim BH:



- ✳ 2 asymptotic regions ⟹ 2 CFTs
- ✳ correlators of high-dim operators:

$$\langle \Phi(x_1)\, \Phi(x_2) \rangle \sim e^{-m\,\mathcal{L}_{12}}$$

- ✳ where $\mathcal{L}_{12}$ = regularised proper distance along spacelike geodesic between x_1 and x_2.

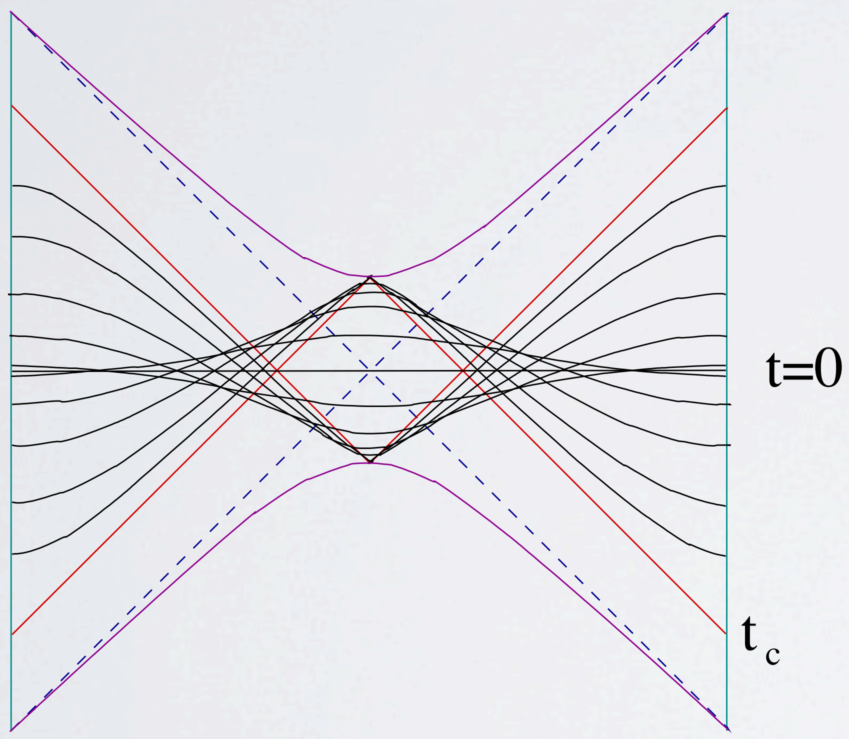
Correlators between operators in CFT$_1$ and CFT$_2$ give access to full spacetime.
[Louko, Marolf, Ross; Kraus, Ooguri, Shenker]
However, in 3-dim, these are insensitive to the BH singularity.

Probing black hole singularity

Signatures of bulk black hole singularity are encoded in analytically continued correlators of high-dimension operators in CFT

[Fidkowski, VH, Kleban, Shenker]

5-dim BH:



- spacelike geodesics repelled by singularity

$$\langle \Phi\, \Phi \rangle (t) \sim \frac{1}{(t - t_c)^{2m}}$$

- divergence in CFT correlation fn. as $t \to t_c$
- subtlety: requires analytic continuation

Hence CFT correlators are sensitive to geometry deep in bulk, even when causally disconnected from the boundary.

Bulk-cone singularities

More direct CFT probes (not reliant on analytic continuation) = *bulk-cone singularities*:

[VH, Liu, Rangamani]

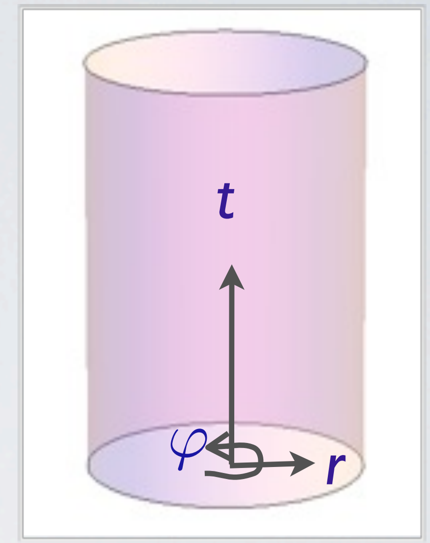
- Green's functions on curved bulk spacetime are singular at null-separated points
- Boundary correlation functions $\langle \Phi(x) \Phi(y) \rangle$ inherit these singularities
- Hence $\langle \Phi(x) \Phi(y) \rangle \to \infty$ when x and y are null-separated

  (either along boundary or through the bulk)

- The set of bulk-cone singularities in the CFT directly give the endpoints of bulk null geodesics.
- One can use this information to learn about the bulk geometry

Bulk-cone singularities

- Consider asymp.(global) AdS bulk, and study projection of null geodesics in $(r\,,t)$ and $(r\,,\varphi)$
- geodesic endpoints clearly distinguish between different bulk geometries:

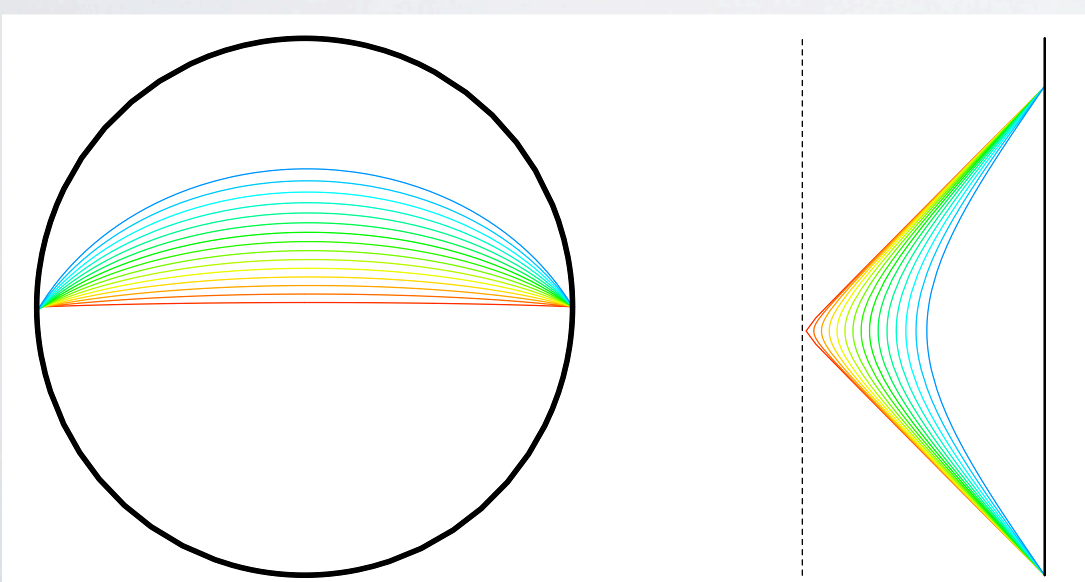


Null geodesics in AdS:

cf. Null geodesics in AdS 'star' geometry:

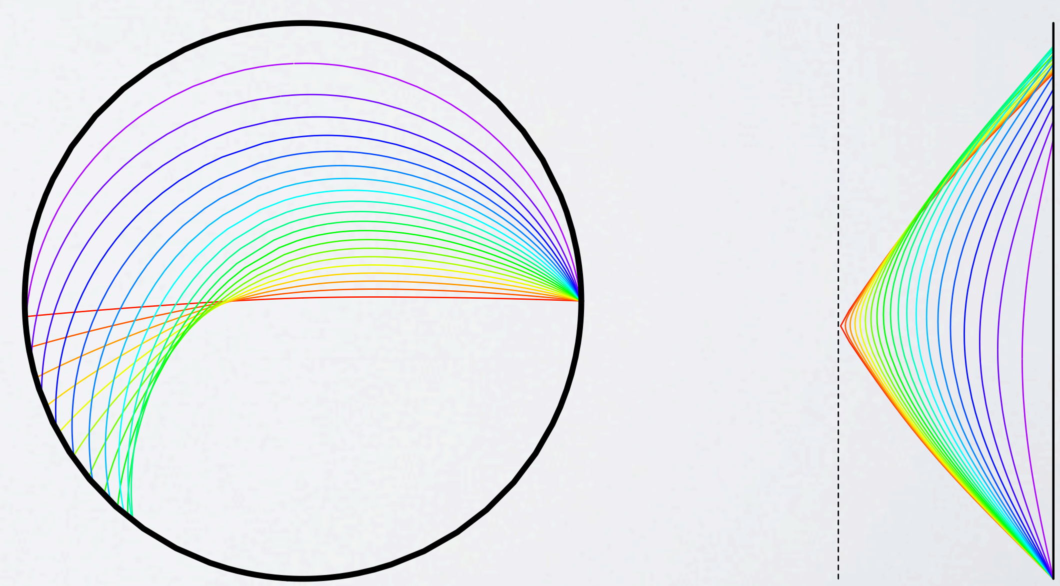
const. t

(r , t) plane



const. t

(r , t) plane



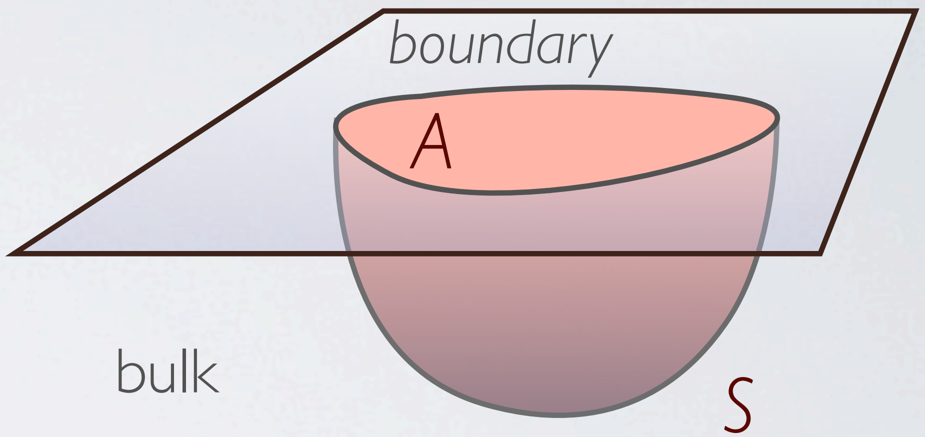
Holographic entanglement entropy

Proposal [Ryu & Takayanagi] for static configurations (at fixed t):

* Entanglement entropy of region $\mathcal{A}$ is

$$\mathcal{S}_{\mathcal{A}} = -\mathrm{Tr}\, \rho_{\mathcal{A}} \,\log \rho_{\mathcal{A}}$$

* In the bulk this is captured by area of minimal co-dimension 2 bulk surface S anchored on $\partial\mathcal{A}$.



In time-dependent situations, prescription must be covariantised:

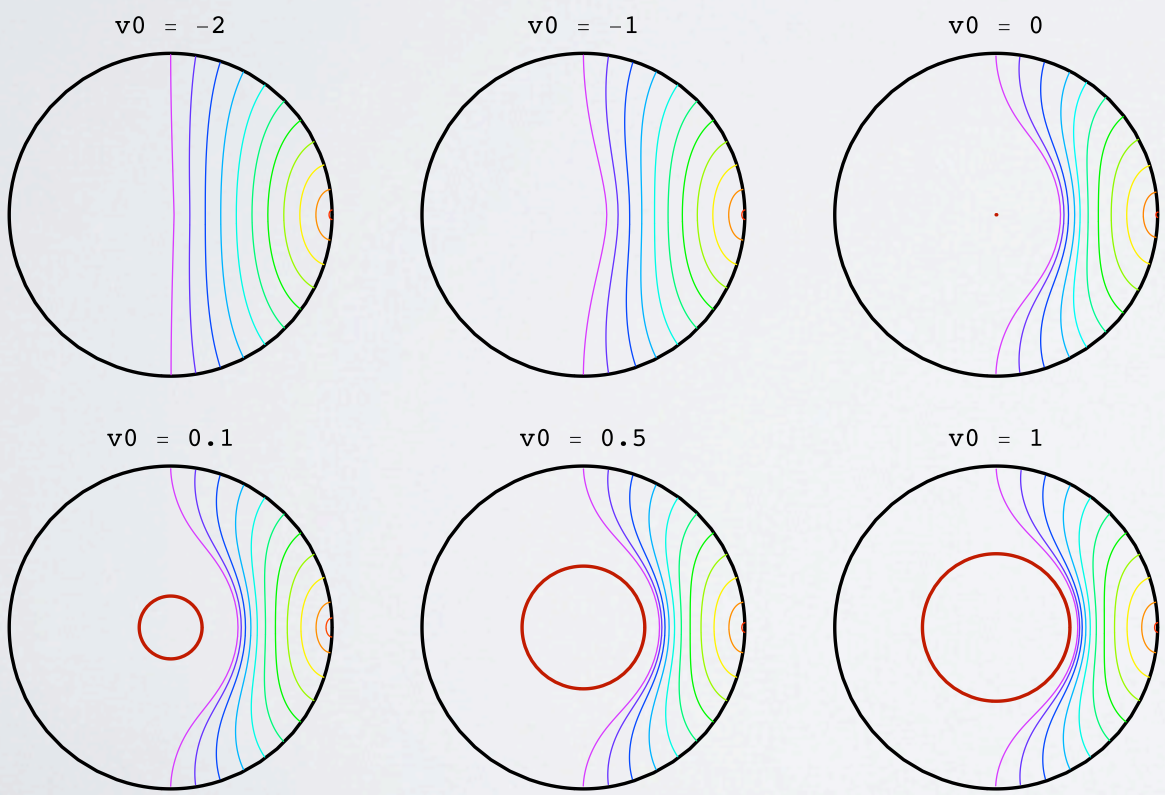
* minimal surface → extremal surface
* equivalently, S is the surface with zero null expansions; cf. light sheet construction [Bousso]

[VH, Rangamani, Takayanagi]

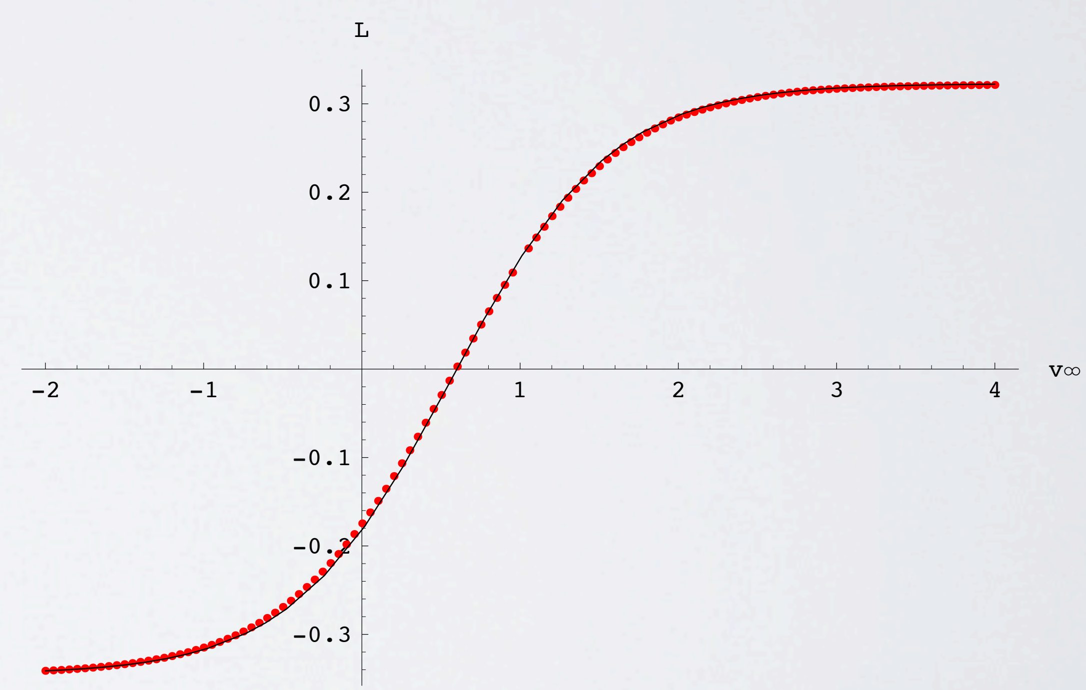
Holographic entanglement entropy

Entanglement entropy growth during thermalisation:
Bulk geometry = collapsing black hole (in 3-d):

behaviour of extremal surfaces
at times v_0 during collapse



corresponding entanglement entropy:



[VH, Rangamani, Takayanagi]

Recent developments

- What is the bulk region dual of a CFT restricted to a (globally hyperbolic subset of the AdS boundary?)

  [Bousso, Leichenauer, & Rosenhaus, *Light-sheets and AdS/CFT*]: Causal wedge

- Consider a density matrix $\rho_{\mathcal{A}}$ for a given spatial region $\mathcal{A}$ in the CFT. How much of the bulk geometry does $\rho_{\mathcal{A}}$ encode?

  [Czech, Karczmarek, Nogueira, Van Raamsdonk, *The Gravity Dual of a Density Matrix*]:

  - $\rho_{\mathcal{A}}$ is more naturally associated with domain of dependence, $D[\mathcal{A}]$
  - corresponding bulk dual must contain the causal wedge of $D[\mathcal{A}]$
  - but in general extends beyond the causal wedge

- [VH, Rangamani, *Causal Holographic Information*]:
  - Most natural (causally constructed) region associated w/ $\mathcal{A}$ = causal wedge
  - Causal holographic information surface $\Xi_{\mathcal{A}}$ characterizes bulk info in $\mathcal{A}$

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Probe geodesics

- For simplicity, focus on static, spher. sym., asymp.(global)AdS

$$ds^2 = -f(r)\, dt^2 + h(r)\, dr^2 + r^2\, d\Omega^2$$

- Probe geodesics = bulk geodesics with both endpoints anchored on the (same) AdS bdy
- Can only be spacelike or null (timelike geods don't reach bdy)
- Consider how deep into the bulk these can probe (= r_{min}) and the regularized proper length (= L_{reg}), for a given angular ($\Delta\varphi$) and temporal (Δt) separation of the endpoints:
- What part of the bulk is accessible to probe geodesics?
- Which are optimal geodesics for probing the bulk?

Probe geodesics

- geodesics w/ energy E and ang.mom. L have radial potential

$$\dot{r}^2 + V_{\rm eff}(r) = 0\ , \qquad V_{\rm eff}(r) = \frac{1}{h(r)}\left[-\kappa - \frac{E^2}{f(r)} + \frac{L^2}{r^2}\right]$$

- for turning point $r_{\rm min}$, the endpoints are separated by

$$\Delta t \equiv 2\int_{r_{\rm min}}^{\infty} \frac{E}{f(r)}\, g(r)\, dr \qquad \text{and} \qquad \Delta\varphi \equiv 2\int_{r_{\rm min}}^{\infty} \frac{L}{r^2}\, g(r)\, dr\ ,$$

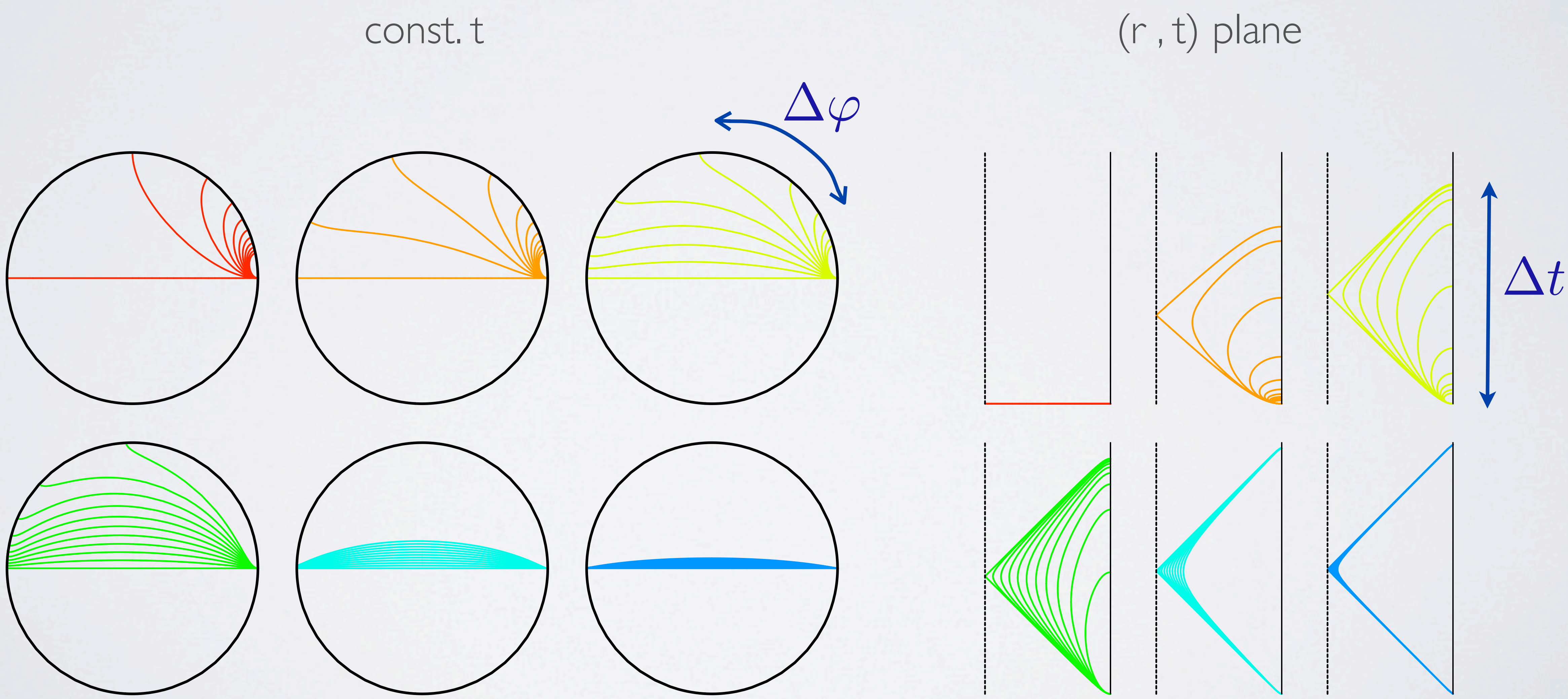
where

$$g(r) \equiv \sqrt{\frac{h(r)}{\kappa + \frac{E^2}{f(r)} - \frac{L^2}{r^2}}} = \frac{1}{\sqrt{-V_{\rm eff}(r)}} = \frac{1}{|\dot{r}(r)|}$$

- with proper length $\mathcal{L}_R = 2\int_{r_{\rm min}}^{R} g(r)\, dr$

Probe geodesics in AdS

- Distinct *E* in distinct panes, denoted by color-coding
- Distinct *L* are distinct curves in each pane



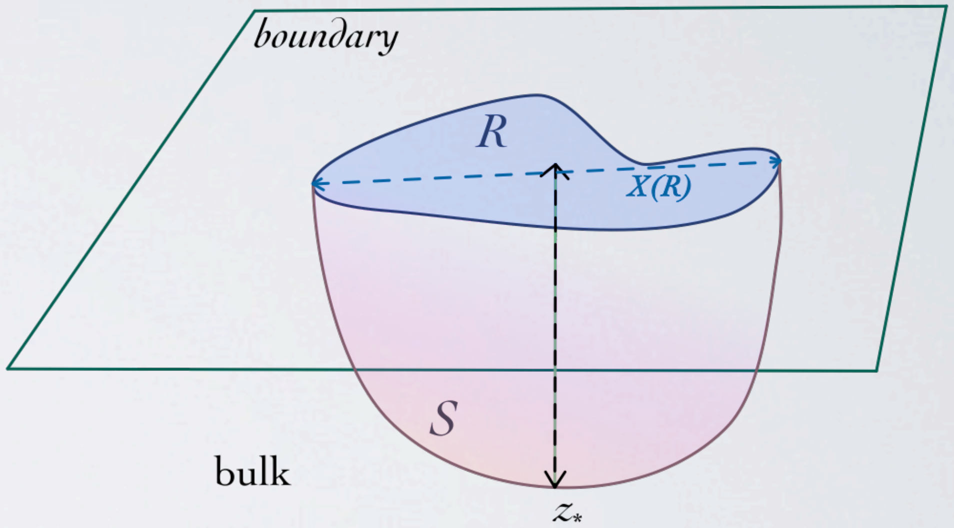
Results for probe geodesics

- In causally trivial spacetime, all of the bulk is accessible to spacelike as well as null geodesics
- In presence of a black hole, probe geodesics cannot penetrate the horizon (cf. part II)
- Spacelike geodesics probe deeper than null ones for fixed parameters E & L
- The `optimal' geodesics for probing bulk are the $E=0$ spacelike ones, as these minimize r_{min} at fixed $\Delta\varphi$
- (However, they have larger L_{reg})

Extremal surfaces

Simplified context: Focus on extremal surfaces S anchored on bdy n-dim region R in static planar asymp. (Poincare) AdS_{d+1}

Parameters we can dial:



- bulk geometry (specified by 2 fns of 1 variable):

$$ds^2 = \frac{1}{z^2}\,\left[-g(z)\,dt^2 + k(z)\,dx_i\,dx^i + dz^2\right]$$

- shape (1 fn of d-1 variables) & extent X (1 real number) of bdy region R
- dimensionality n (=1,2,...,d-1) of the surface S

Key feature of S:

- Bulk depth reached z_*
  - (Note that z_* is geometrically well-defined.)

Results for extremal surfaces

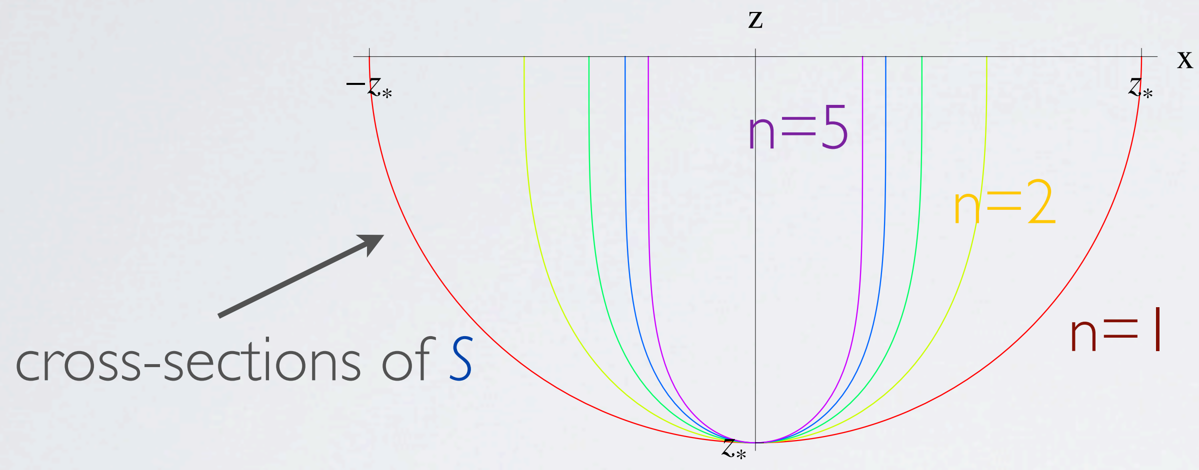
Preview:

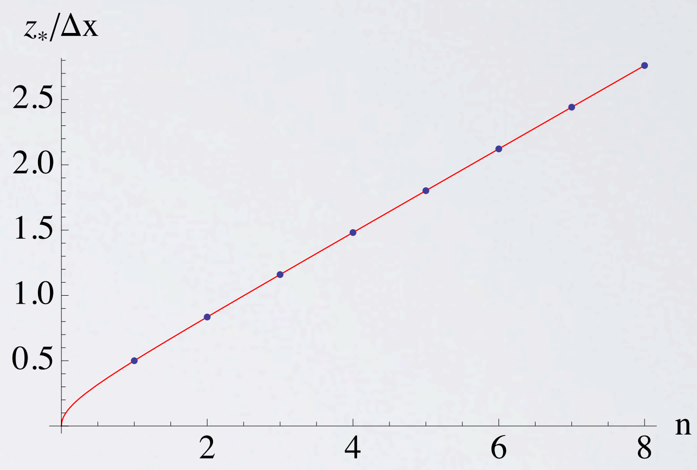
- Higher-dimensional surfaces probe deeper
  i.e. z_* increases with n for fixed extent $X(R)$
- Surfaces anchored on R = ball reach deepest
  compared to differently-shaped R with same extent or area
- Surfaces in pure AdS reach deeper
  for fixed R, compared asymp.AdS geometry

Results for extremal surfaces

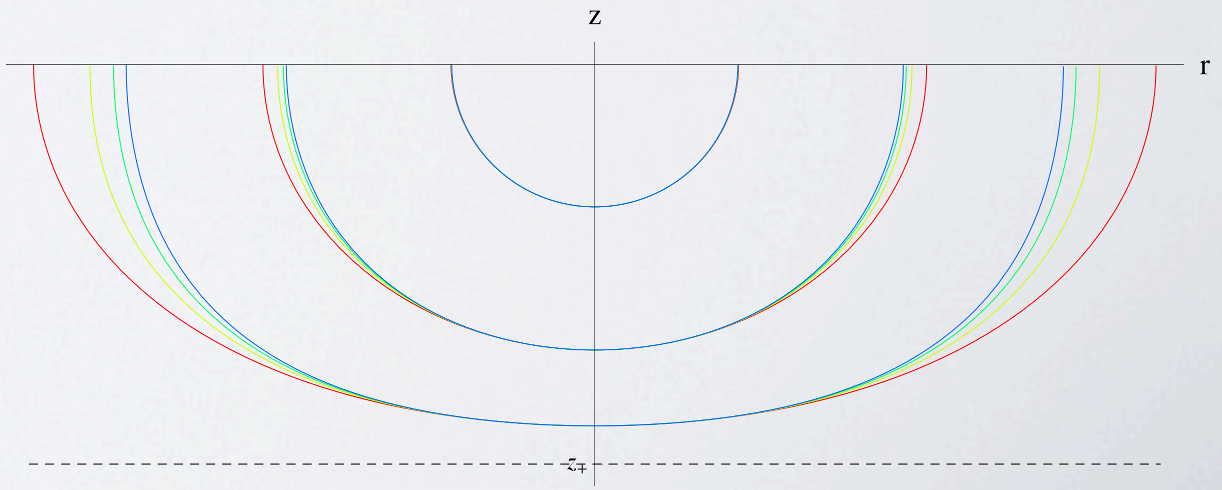
Higher-dimensional surfaces probe deeper:

- Consider *R* = n-dim'l strip on bdy of pure AdS



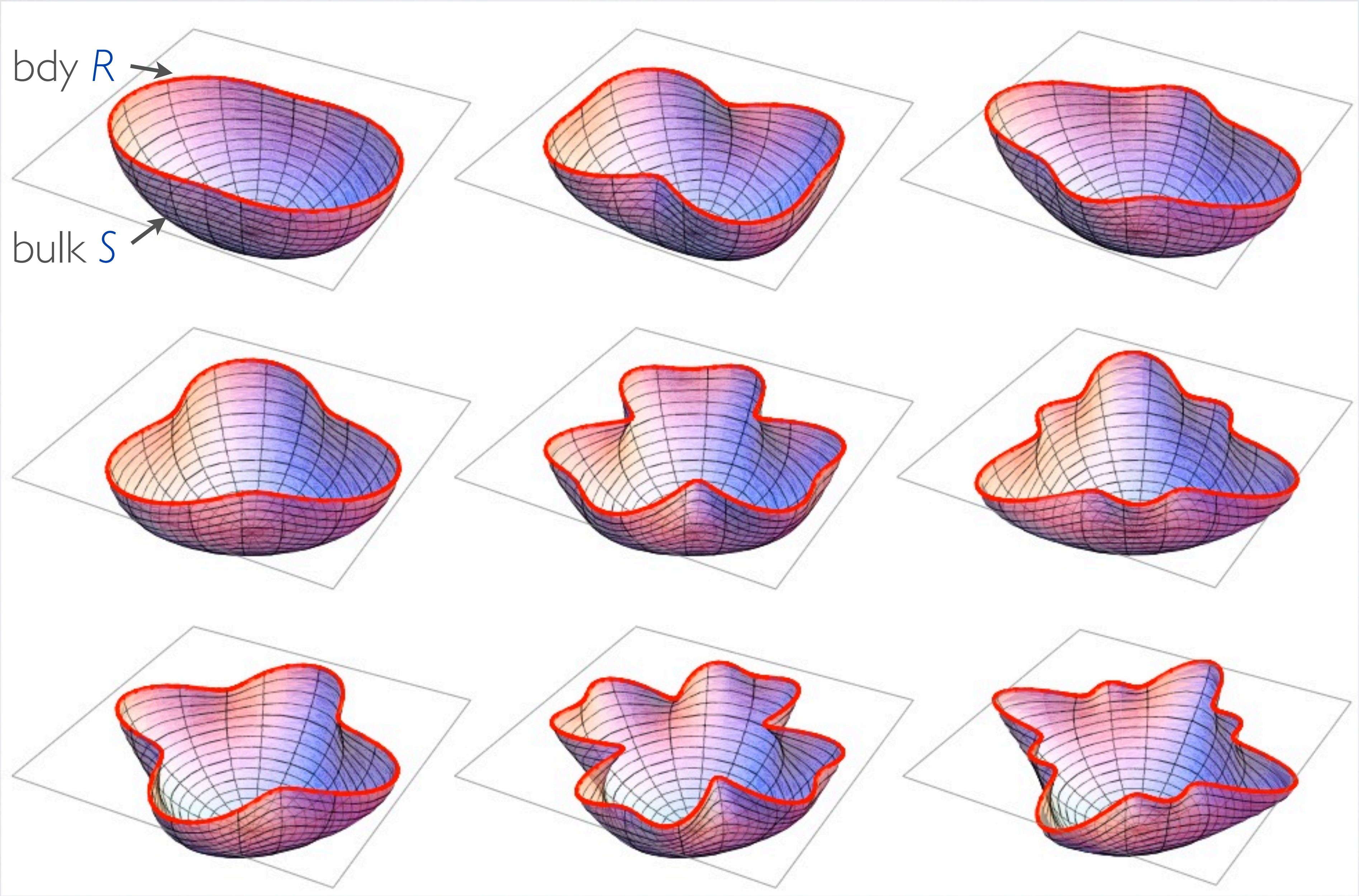


- We should compare z_* for fixed extent *X(R)*:
- For *R* = n-ball, $z_* = X/2$ ∀n (i.e. all S hemispherical) in pure AdS
- For *R* = n-ball in deformed AdS,
  - at fixed z_*, *X* decreases with *n*
  - at fixed *X*, z_* increases with *n*



Results for extremal surfaces

- Consider R with fixed dimensionality n and `area' $A(R)$
- What shape of R maximizes z_*, i.e. when does S reach deepest?



Results for extremal surfaces

Surfaces anchored on R=ball reach deepest:

- Linearize around the hemisphere $\rho(\theta,\phi) = \rho_0$ in pure AdS:

$$ds^2 = \frac{1}{\rho^2\,\cos^2\theta}\left[-dt^2 + d\rho^2 + \rho^2\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) + \sum_{j=3}^{d-1} d\tilde{y}_j^2\right]$$

- To 2nd order,

$$\rho(\theta,\phi) = \rho_0 + \epsilon\,\underline{\rho_1(\theta,\phi)} + \epsilon^2\,\underline{\rho_2(\theta,\phi)} + \mathcal{O}(\epsilon^3)$$

$$\rho_1(\theta,\phi) = \tan^\ell(\theta/2)\,(1+\ell\cos\theta)\,\cos\ell\phi\ .$$

$$\rho_2(\theta,\phi) = \frac{1}{4\,\rho_0}\,\tan^{2\ell}(\theta/2)\left\{(1+\ell\cos\theta)^2 + \left[\mu\,(1+2\,\ell\cos\theta) + \ell^2\,\cos^2\theta\right]\,\cos 2\ell\phi\right\}$$

- At fixed $\rho(\theta=0,\phi) = \rho_0$, the area $A(R) = \pi\,\rho_0^2\left(1 + \frac{\epsilon^2}{\rho_0^2} + \mathcal{O}(\epsilon^4)\right)$ increases as R is perturbed from ball
- Hence *R*=round ball ⇒ *S* has greatest reach for fixed *A(R)*
- Confirmed numerically at non-linear level as well.

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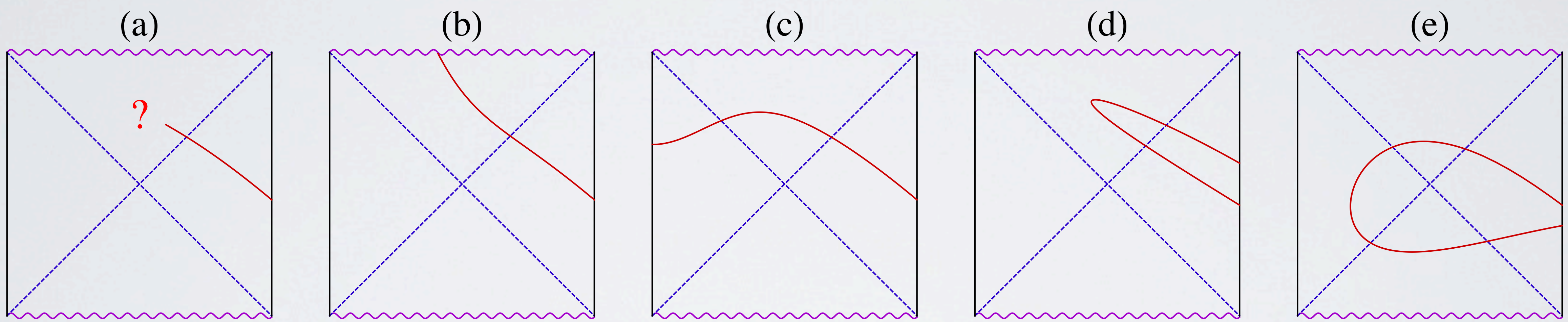
Probe geods can't penetrate horizons in static bulk

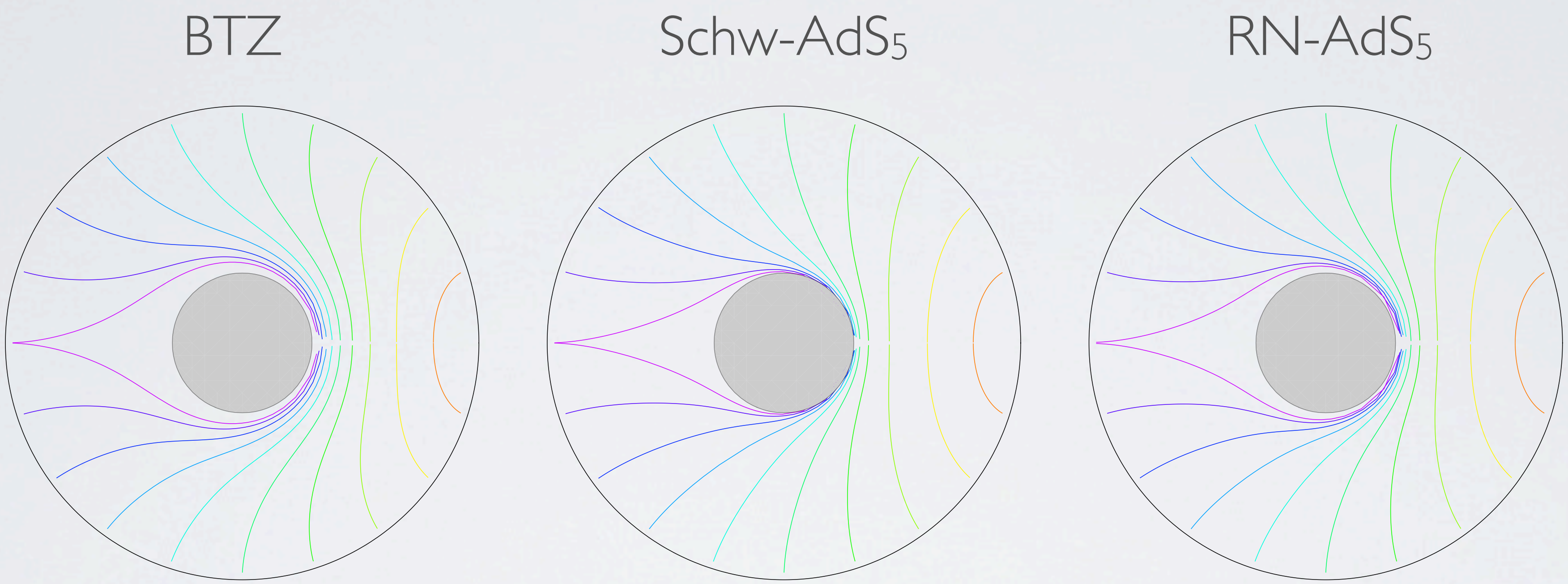
Consider a geod. crossing the horizon; what can happen?



- (b) & (c) are allowed, but don't correspond to probe geods
- (d) is disallowed by energy conservation:
  - Ingoing coords: $ds^2 = -f(r)\,dv^2 + 2\sqrt{f(r)\,h(r)}\,dv\,dr + r^2\,d\Omega^2$
  - conserved energy along geods: $E = f(r)\,\dot{v} - \sqrt{f(r)\,h(r)}\,\dot{r}$
- (e) is disallowed by assumption of reaching horizon from bdy

Probe geods can't penetrate horizons in static bulk

Eg:



- spacelike geodesics can penetrate arb. close to horizon
- however as $r_{\min} \to r_+$, $\Delta\varphi \to \infty$ and $\mathcal{L}_{\rm reg} \to \infty$
- limiting $\Delta\varphi = 2\pi$,

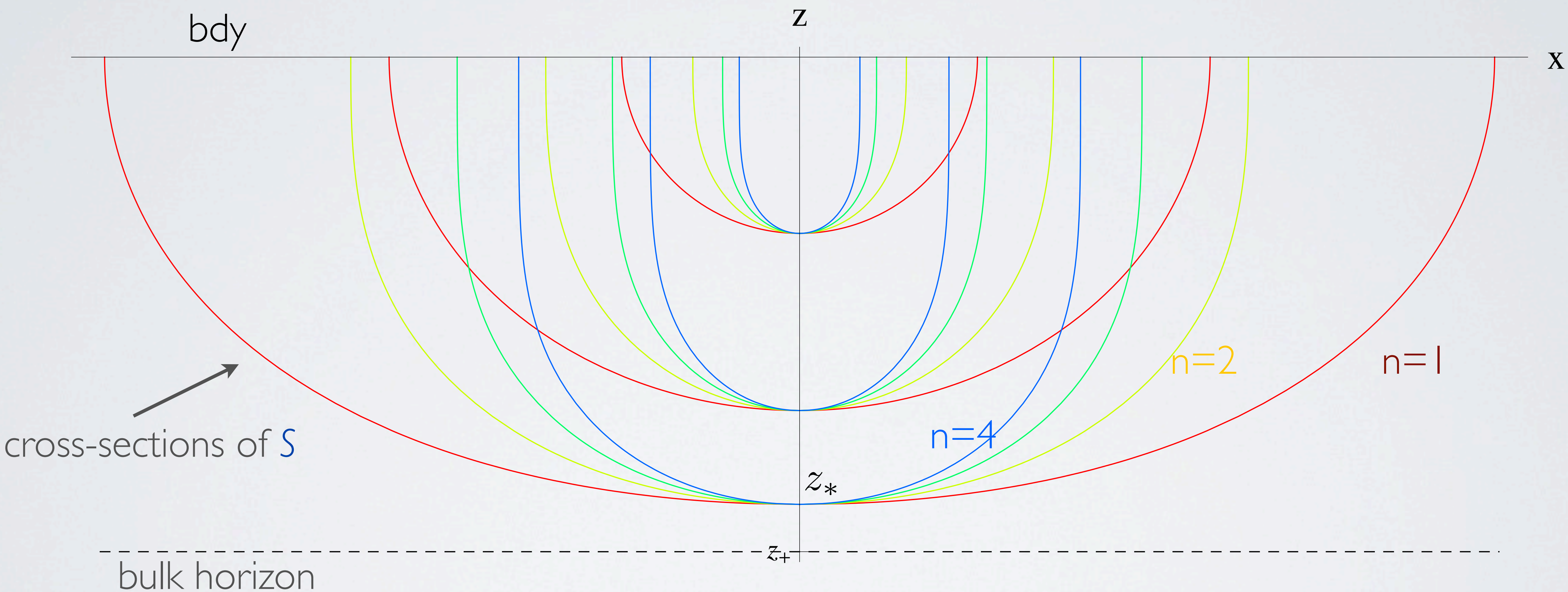
  for $r_+ = 0.5$

$$\frac{r_{\min} - r_+}{r_+} = \begin{cases} 9.03 \times 10^{-2} & \text{for BTZ} \\ 1.85 \times 10^{-3} & \text{for SAdS}_5 \\ 6.54 \times 10^{-2} & \text{for RNAdS}_5 \end{cases}$$

- null geodesics can only reach a finite distance from horizon

Extremal surfaces can't penetrate horizons in static bulk

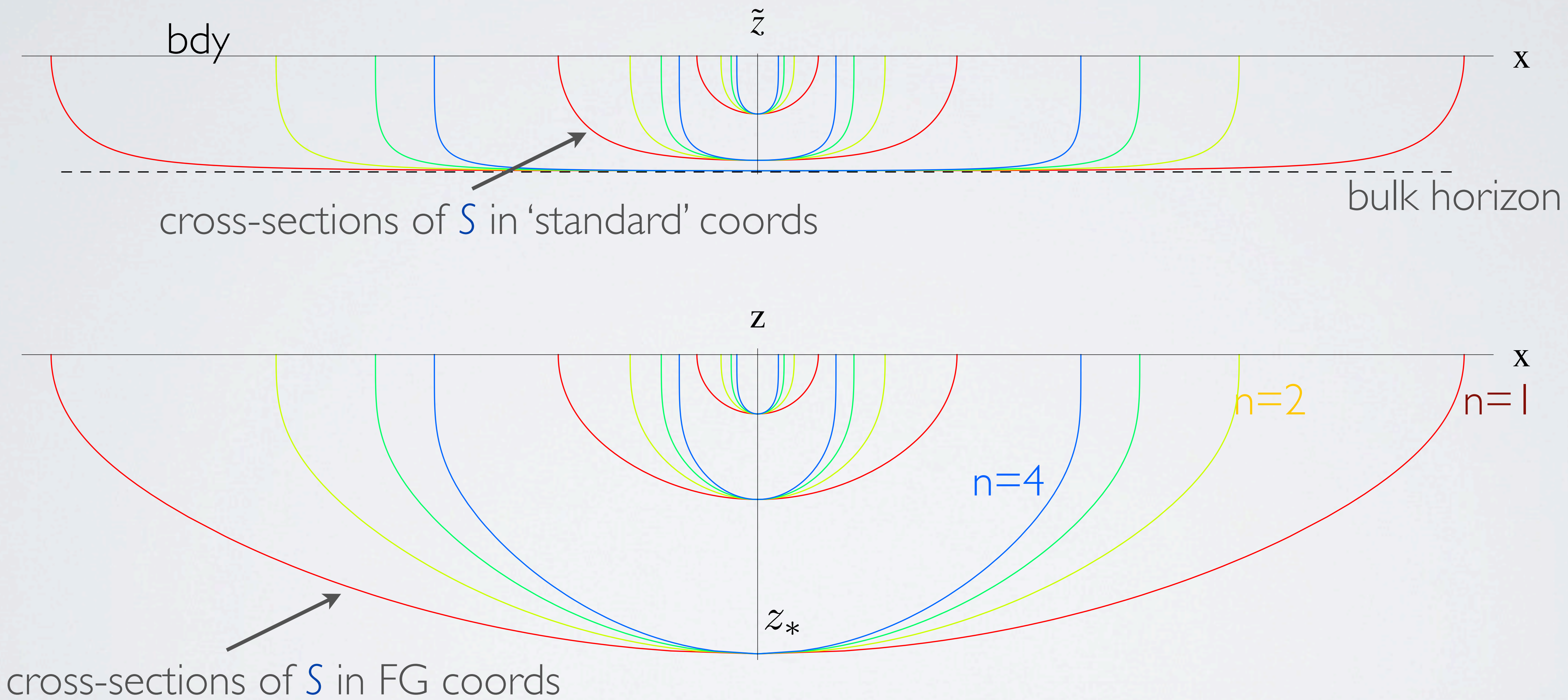
- Consider extent *X(R)* for *n*-strip in Schw-AdS5 at fixed z_*



- We see extremal surfaces are repelled by the horizon

Extremal surfaces can't penetrate horizons in static bulk

- Cf. extent *X(R)* for *n*-strip in extremal RN-AdS5 at fixed z_*



- Again, extremal surfaces are repelled by the horizon

Extremal surfaces can't penetrate horizons in static bulk

Completely general proof, for any n, R, & bulk geom:

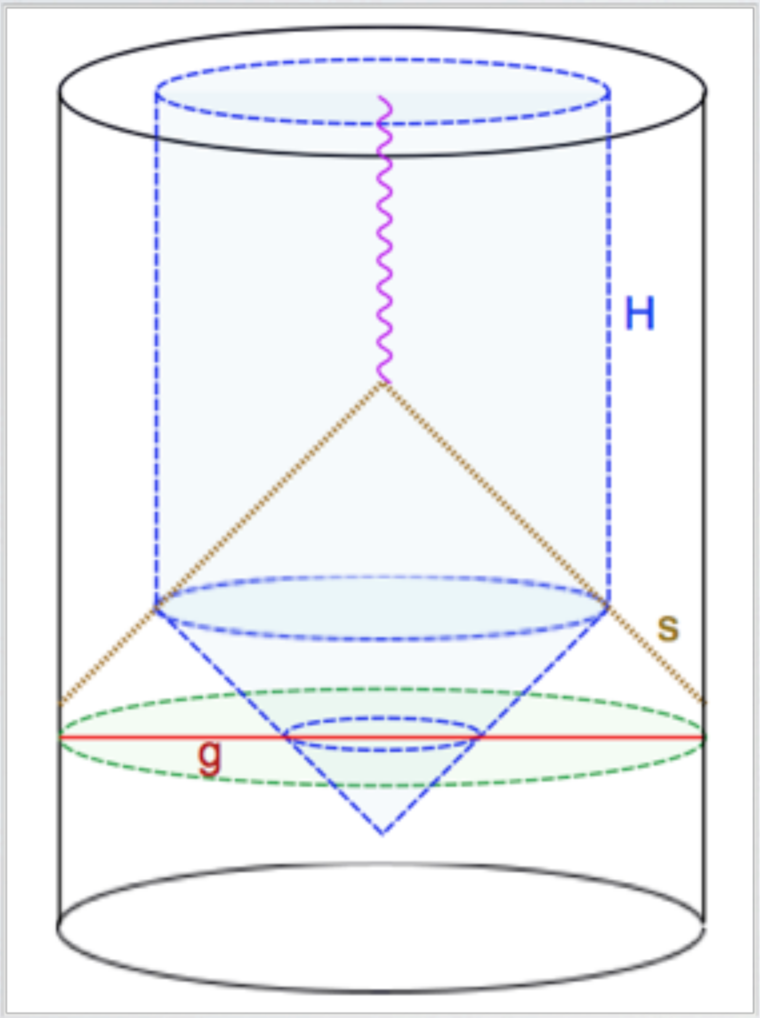
- Consider extremal surface S param. by $z(x^1, \ldots, x^n)$

  in bulk spacetime $ds^2 = \frac{1}{z^2}\left[-f(z)\, dt^2 + dx_i\, dx^i + h(z)\, dz^2\right]$
- At horizon, $f \to 0$ and $h \to \pm\infty$
- Lagrangian: $\mathcal{L}(z\,,\, z_{,1}\,,\, \ldots\,,\, z_{,n}\,;\, x^1, \ldots, x^n) = \sqrt{G} = \frac{\sqrt{1 + h(z)\left(z_{,1}^2 + \ldots + z_{,n}^2\right)}}{z^n}$
- EOM: $\sum_i z_{,ii}\left(1 + h(z) \sum_j z_{,j}^2\right) - h(z) \sum_{i,j} z_{,ij}\, z_{,i}\, z_{,j} + \sum_i z_{,i}^2 \left(\frac{n}{z} + \frac{h'(z)}{2\, h(z)}\right) + \frac{n}{z\, h(z)} = 0$
- Around the turning point, $z = z_*$: $\sum_i z_{,ii} + \frac{n}{z\, h(z)} = 0$
- In order for z to be maximum, $z_{,ii}(z_*) < 0$, which forces $h(z_*) > 0$
- Hence turning point z_* must be outside the horizon.

Extremal surfaces can penetrate horizons in time-evolving bulk

Gedanken-experiment to demonstrate that causality does not pose a fundamental obstacle to extracting information via CFT:

[VH]



- uses the teleological nature of event horizon & non-local nature of AdS/CFT:
  - Measure bulk event by spacelike CFT probe (precursor), e.g. geodesic **g**
  - Afterwards, collapse a shell **s**,
  - such that the resulting event horizon ***H*** encompasses the measured event.
- Then **g** is a probe geodesic which penetrates the event horizon
  - seen explicitly for geods in Vaidya-AdS

[VH, Maxfield]

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    - Construction of causal wedge and CHI
    - Properties of CHI for stationary configurations
    - Behavior of CHI in dynamical settings
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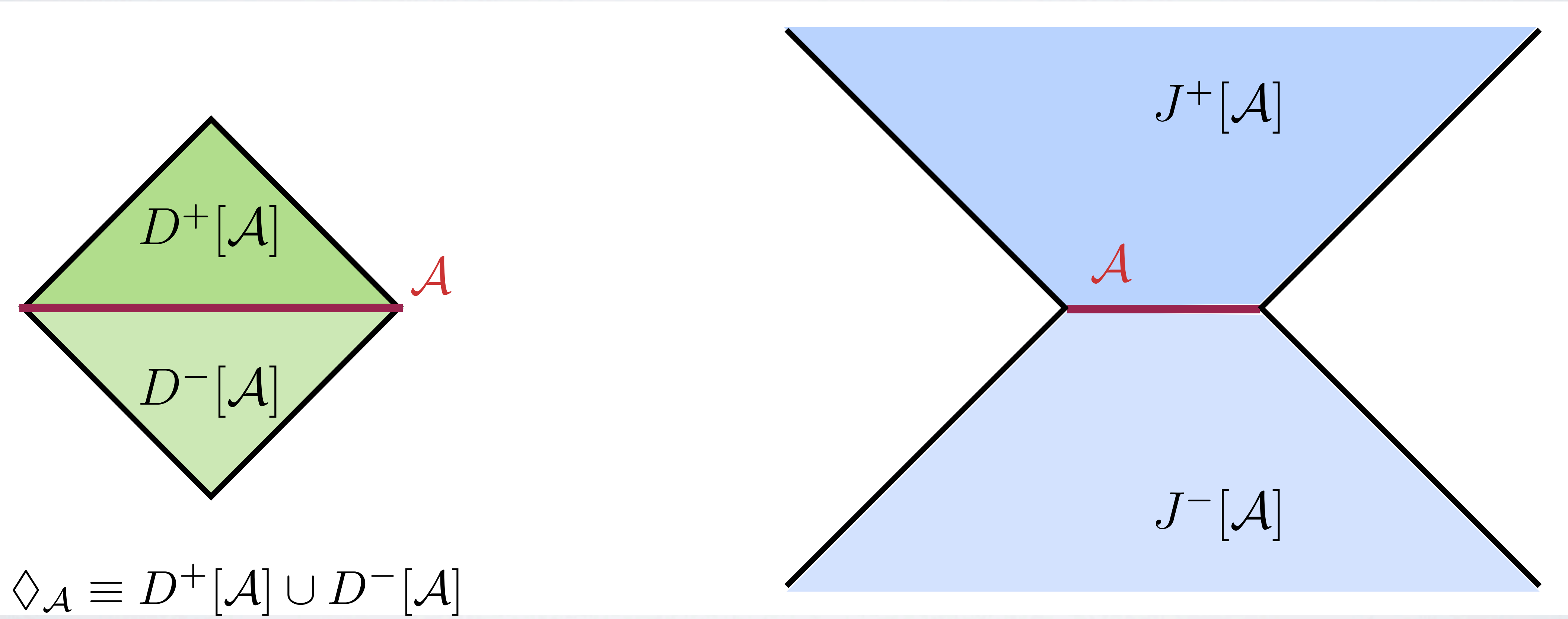
Bulk dual to a bdy region $\mathcal{A}$?

What is the most natural bulk region
associated to a given region $\mathcal{A}$ on the bdy?

- 'natural': try to be minimalistic, use only bulk causality
- Take $\mathcal{A}$ to be d-1 dimensional spatial region on bdy of asymp. AdS$_{d+1}$ bulk spacetime.
- The unique minimal construction gives a bulk causal wedge associated with $\mathcal{A}$, and a corresponding d-1 dimensional bulk surface $\Xi_{\mathcal{A}}$
- Using geometrical information, we can associate a number $\chi_{\mathcal{A}}$ to $\mathcal{A}$, corresponding to area of $\Xi_{\mathcal{A}}$

Causal construction

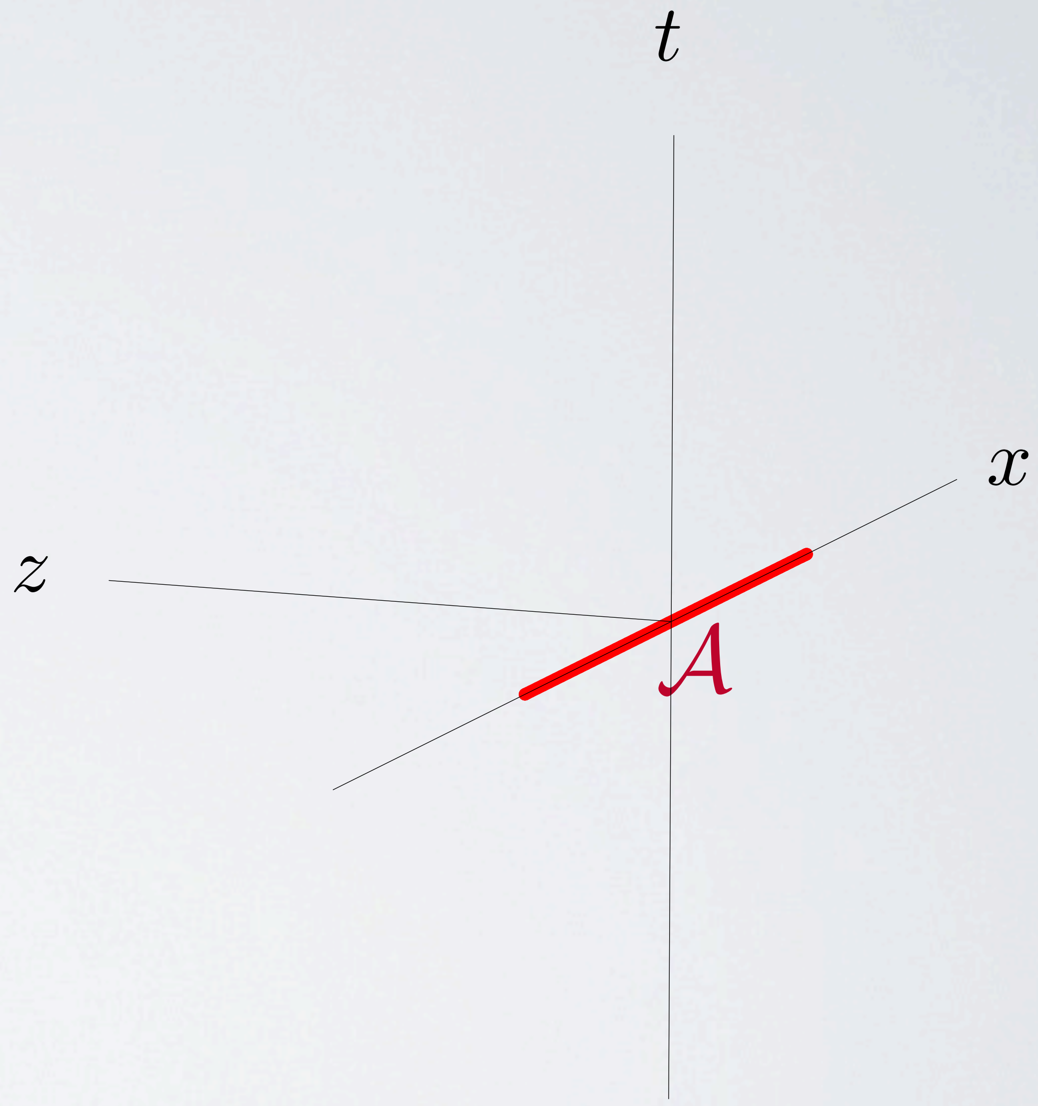
- domain of dependence $D^{\pm}[\mathcal{A}]$ = region which must influence or be influenced by events in $\mathcal{A}$
- domain of influence $I^{\pm}[\mathcal{A}]$ = region which can influence or be influenced by events in $\mathcal{A}$



- Given $\rho_{\mathcal{A}}$, we can determine observables in the entire $\Diamond_{\mathcal{A}}$

Causal construction

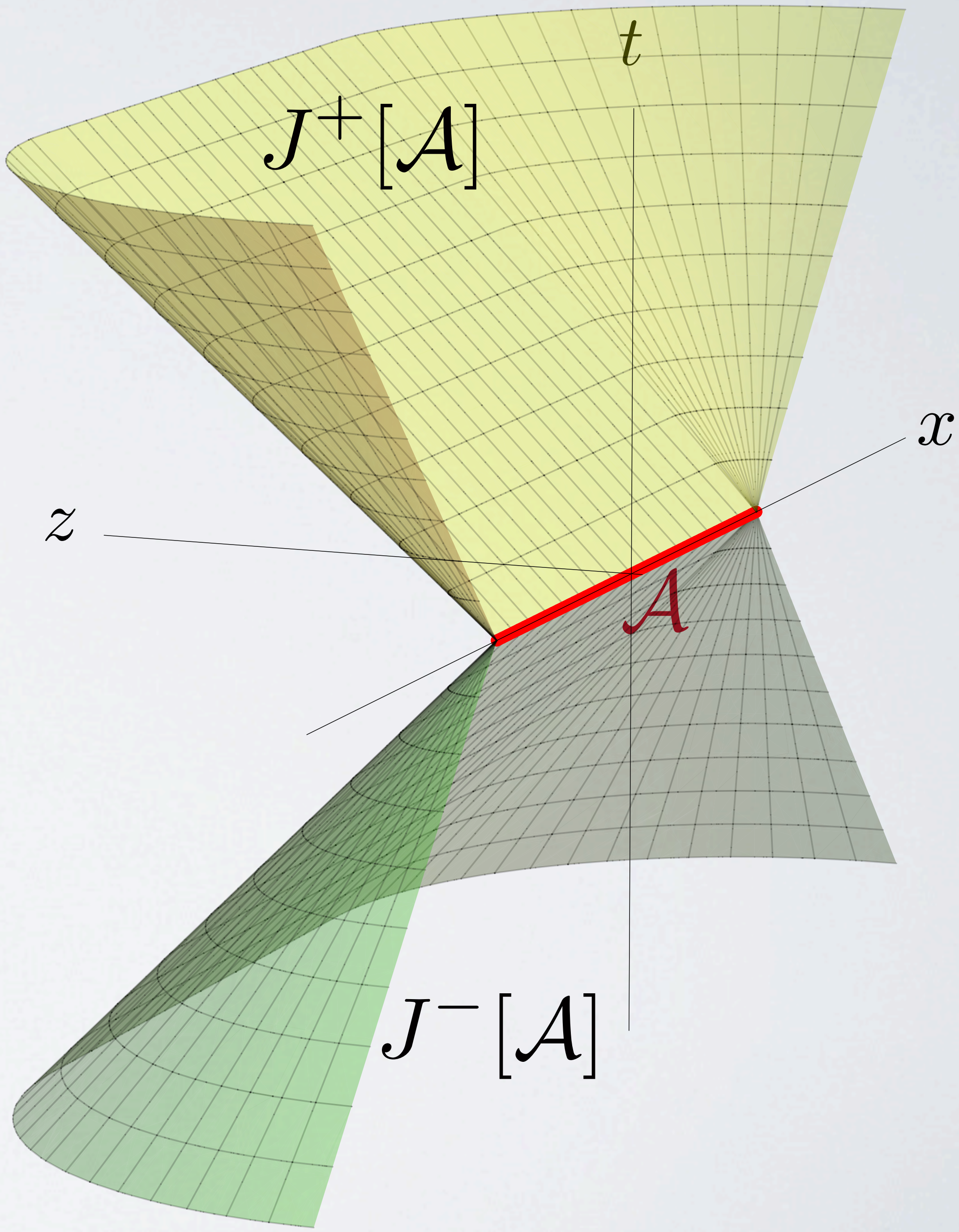
- Consider a bdy region $\mathcal{A}$



Causal construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection

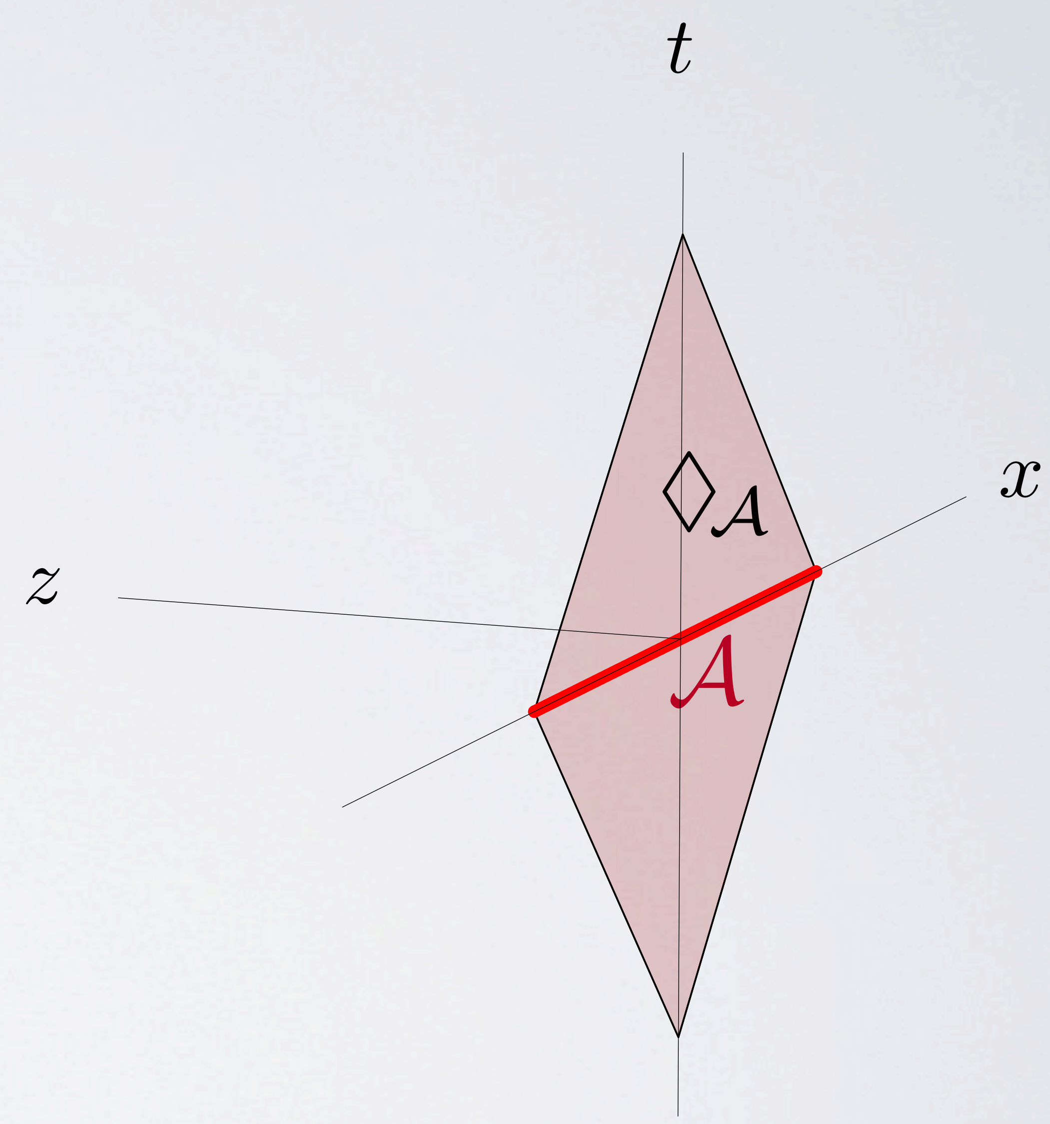
(and bulk domain of dependence of $\mathcal{A}$ is just the region $\mathcal{A}$ itself).



Causal construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\Diamond_{\mathcal{A}}$

(observables in the entire region $\Diamond_{\mathcal{A}}$ can be determined solely from the initial conditions specified on $\mathcal{A}$)



Causal construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\lozenge_{\mathcal{A}}$
- Its bulk domains of influence extend arb. deep, but their intersection doesn't

t

$J^{+}[\lozenge_{\mathcal{A}}]$

x

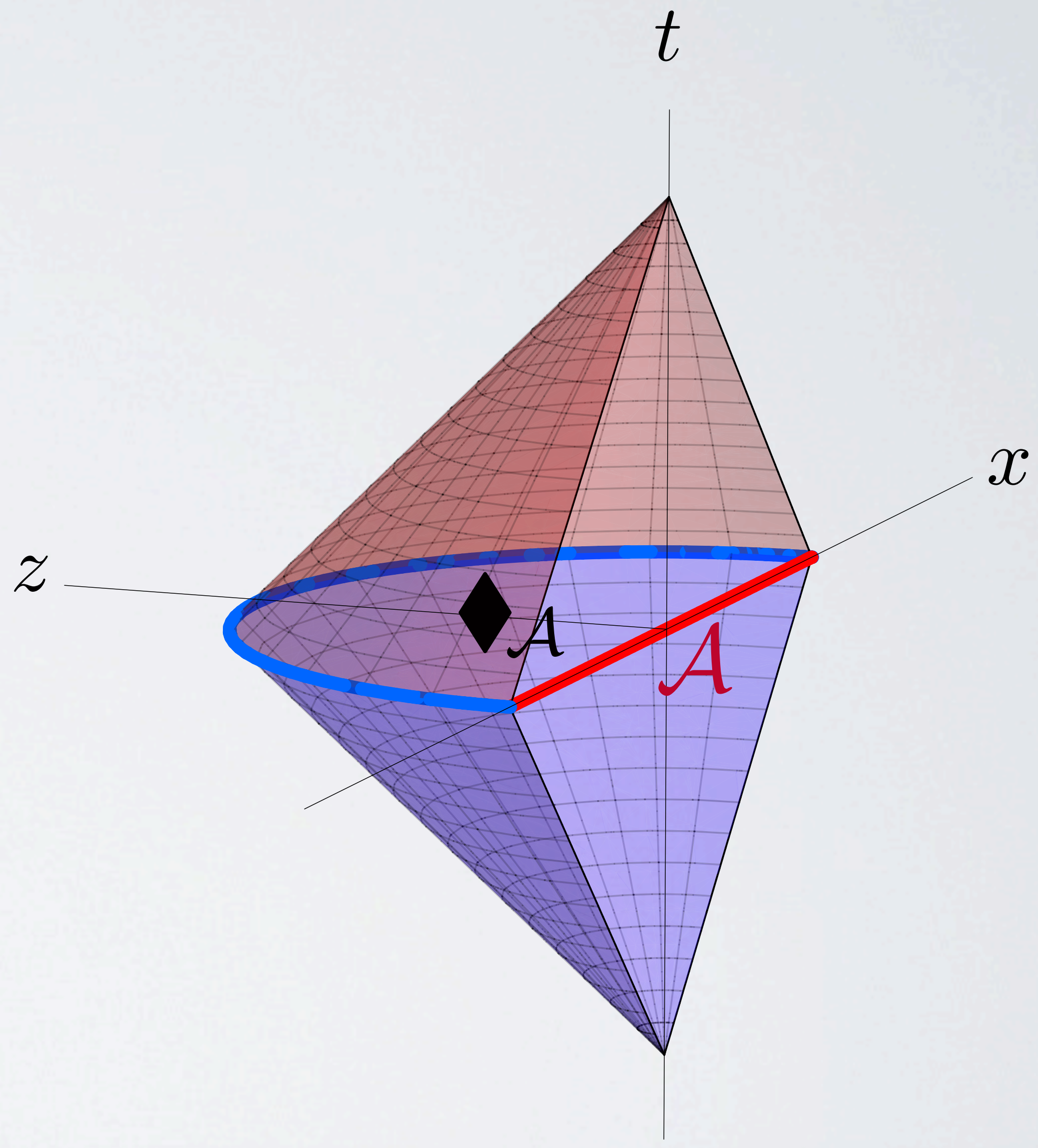
z

$\mathcal{A}$

$J^{-}[\lozenge_{\mathcal{A}}]$

Causal construction

- Consider a bdy region $\mathcal{A}$
- Its bulk domains of influence extend arb. deep into the bulk and have trivial intersection
- Consider a bdy domain of dependence of $\mathcal{A}$, denoted $\lozenge_{\mathcal{A}}$
- Its bulk domains of influence extend arb. deep, but their intersection doesn't
- This defines for us the bulk causal wedge of $\mathcal{A}$, denoted $\blacklozenge_{\mathcal{A}}$



Causal construction

- Bulk causal wedge $\blacklozenge_{\mathcal{A}}$

$$\blacklozenge_{\mathcal{A}} \equiv J^-[\lozenge_{\mathcal{A}}] \cap J^+[\lozenge_{\mathcal{A}}]$$

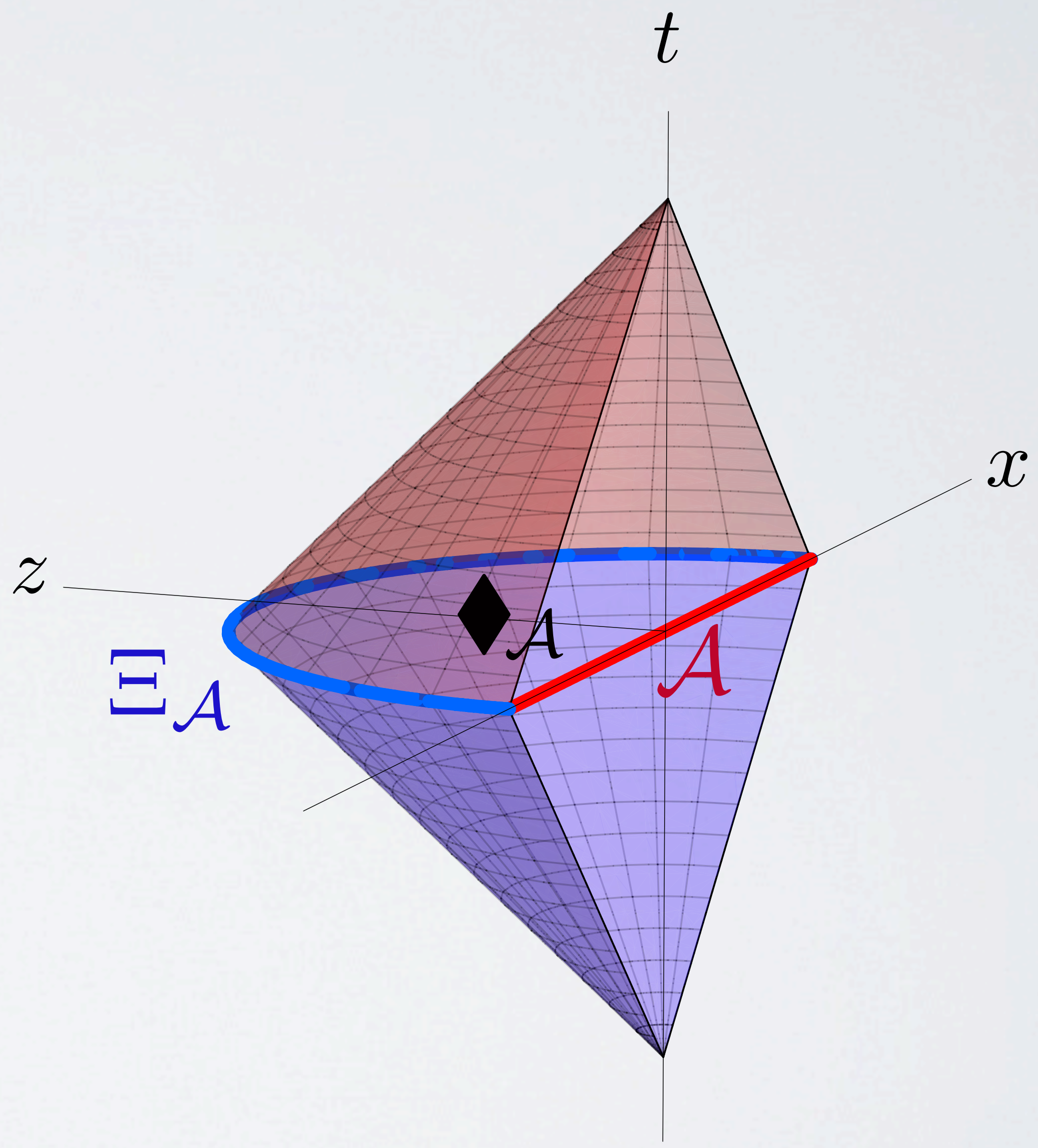
= { bulk causal curves which begin and end on $\lozenge_{\mathcal{A}}$ }

- Causal information surface $\Xi_{\mathcal{A}}$

$$\Xi_{\mathcal{A}} \equiv \partial_+(\blacklozenge_{\mathcal{A}}) \cap \partial_-(\blacklozenge_{\mathcal{A}})$$

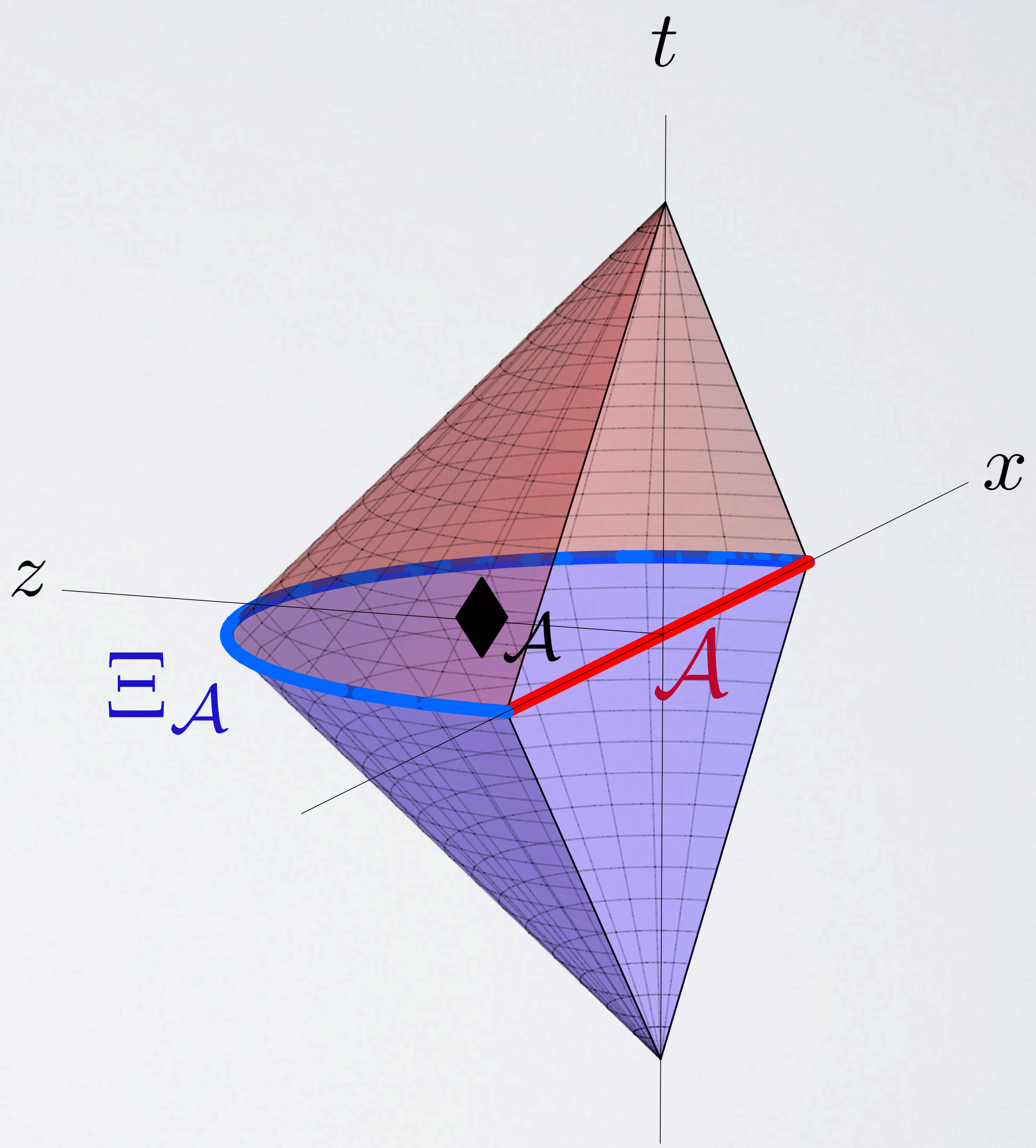
- Causal holographic information $\chi_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\, G_N}$$



Main question:

What is the CFT interpretation of $\Xi_{\mathcal{A}}$ and $\chi_{\mathcal{A}}$?

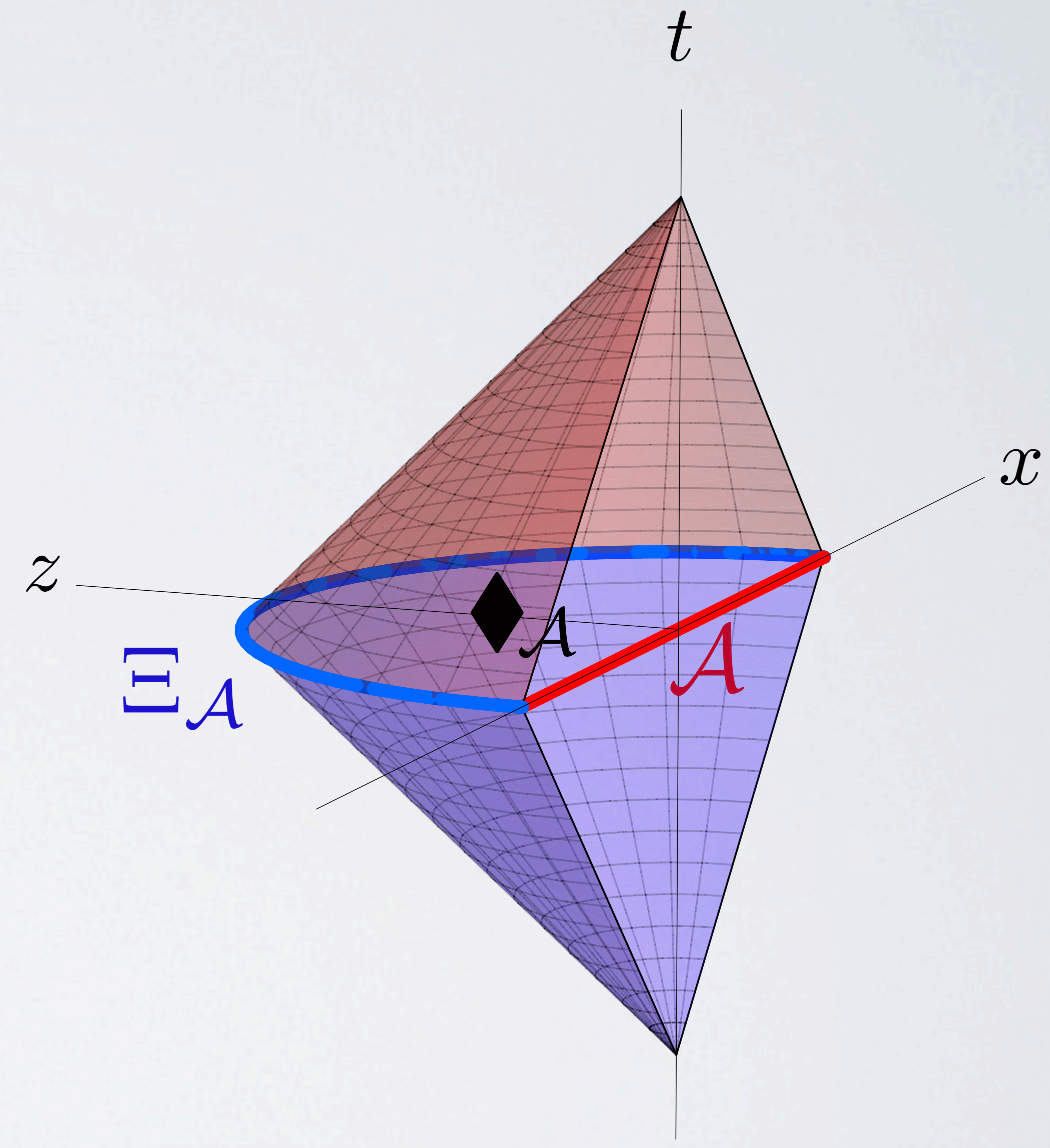


Gather hints

by considering geometrical properties and behavior of $\Xi_{\mathcal{A}}$ ...

General properties of $\Xi_{\mathcal{A}}$:

- Causal information surface $\Xi_{\mathcal{A}}$ is a *d-1* dimensional spacelike bulk surface which:
  - is anchored on $\partial\mathcal{A}$
  - lies within (on boundary of) $\blacklozenge_{\mathcal{A}}$
  - reaches deepest into the bulk from among surfaces in $\blacklozenge_{\mathcal{A}}$
  - is a minimal-area surface among surfaces on $\partial(\blacklozenge_{\mathcal{A}})$ anchored on $\partial\mathcal{A}$
- However, $\Xi_{\mathcal{A}}$ is in general not an extremal surface $\mathfrak{E}_{\mathcal{A}}$ in the bulk.

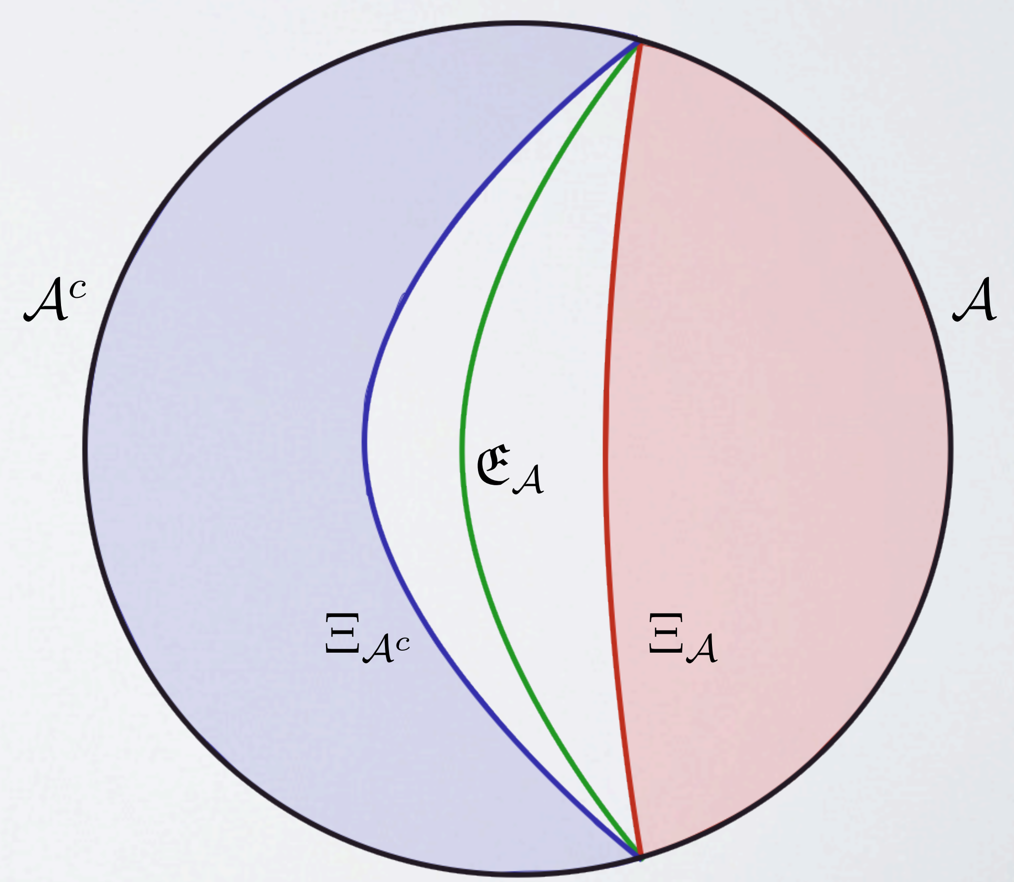


General properties of $\Xi_{\mathcal{A}}$:

- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $\mathcal{A}$
  - Justification 1: explicit calculation
    e.g. A=infinite strip in d>2 dim:

$$z^*_{\Xi} = \frac{w}{2} \; , \qquad z^*_{\mathfrak{E}} = \frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2(d-1)}\right)} \, \frac{w}{2}$$

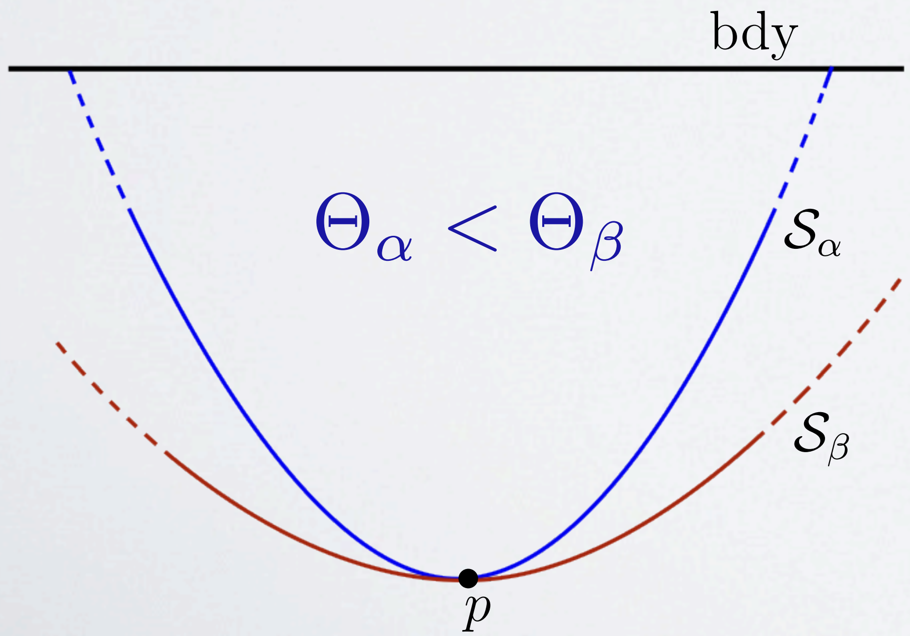
  - Justification 2: general argument:
    e.g. A=disk on bdy of global AdS and bdy state = pure: $S_{\mathcal{A}} = S_{\mathcal{A}^c}$
  - causal wedge differs for $\mathcal{A}$ and $\mathcal{A}^c$; and reach furthest in pure AdS, wherein $\Xi = \mathfrak{E}$, so in general Ξ recedes towards the bdy...

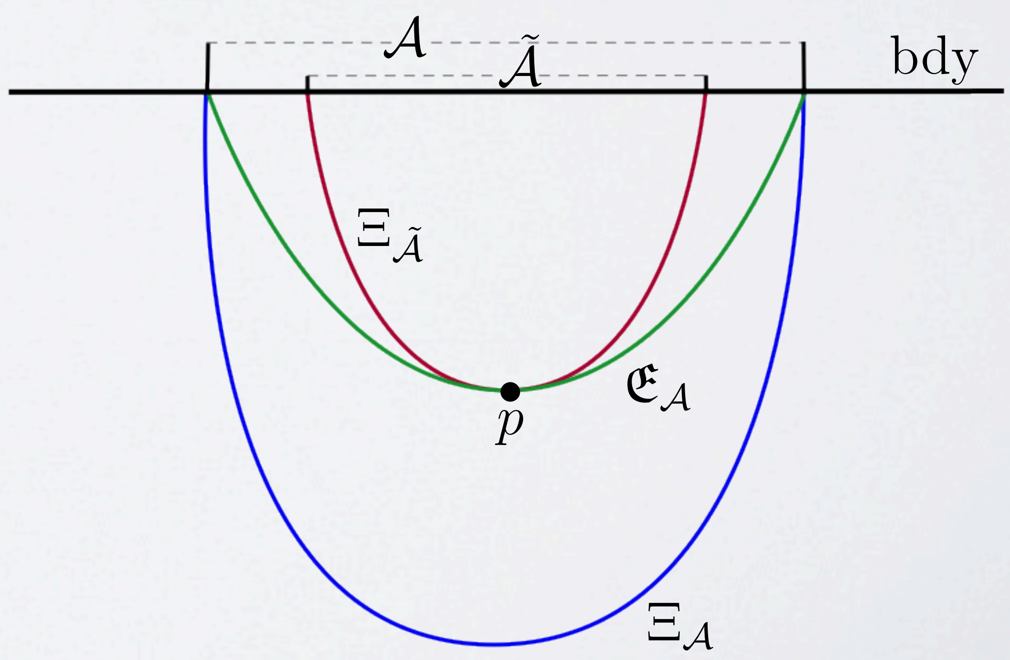


General properties of $\Xi_{\mathcal{A}}$:

- In general $\Xi_{\mathcal{A}}$ does not penetrate as far into the bulk as the bulk extremal surface $\mathfrak{E}_{\mathcal{A}}$ associated with $\mathcal{A}$

- Justification 3: general argument based on expansion of null generators: By construction, $\Theta_{\Xi} \geq 0$ while $\Theta_{\mathfrak{E}} = 0$
- Proof by contradiction: suppose $\mathfrak{E}_{\mathcal{A}}$ lay closer to bdy than $\Xi_{\mathcal{A}}$. Then tangent to $\mathfrak{E}_{\mathcal{A}}$, there is a surface $\Xi_{\tilde{\mathcal{A}}}$ for some smaller region $\tilde{\mathcal{A}}$. But for such configuration, $\Theta_{\Xi_{\tilde{\mathcal{A}}}} < 0$, which is a contradiction.

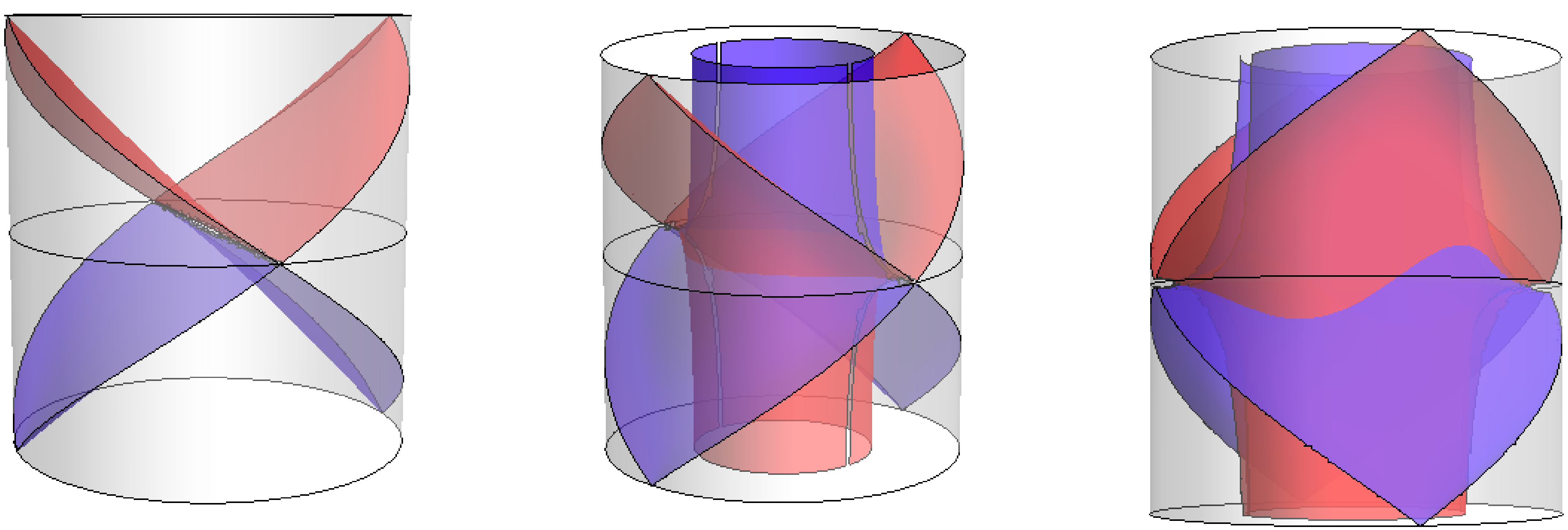




Cases when $\Xi_{\mathcal{A}}$ and $\mathfrak{E}_{\mathcal{A}}$ coincide:

- However, in all cases where one is able to compute entanglement entropy in QFT from first principles, independently of coupling, the surfaces $\mathfrak{E}_{\mathcal{A}}$ and $\Xi_{\mathcal{A}}$ agree!
- = When EE can be related to thermal entropy... cf. [Myers et.al.]

bdy:	CFT vacuum:	thermal density matrix:	grand canonical density matrix:
bulk:	pure AdS:	static BTZ:	rotating BTZ:



(a) (b) (c)

Cases when $\Xi_{\mathcal{A}}$ and $\mathfrak{E}_{\mathcal{A}}$ coincide:

bdy: CFT vacuum: | thermal density matrix: | grand canonical density matrix:

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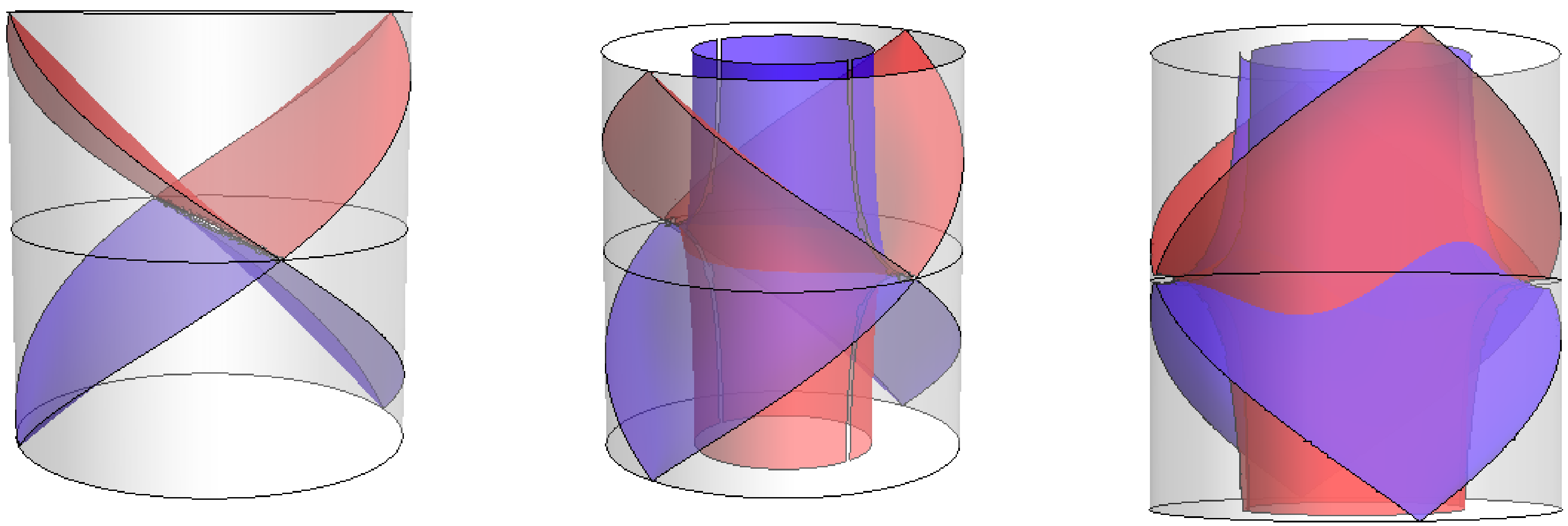


(a) (b) (c)

$$\text{(a).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{3} \log\left(\frac{2\varphi_0}{\varepsilon}\right)$$

$$\text{(b).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{3} \log\left[\frac{\beta}{\pi\,\varepsilon} \sinh\left(\frac{2\pi\,\varphi_0}{\beta}\right)\right]$$

$$\text{(c).} \qquad S_{\mathcal{A}} = \chi_{\mathcal{A}} = \frac{c_{\text{eff}}}{6} \log\left[\frac{\beta_+\,\beta_-}{\pi^2\,\varepsilon^2} \sinh\left(\frac{2\pi\,\varphi_0}{\beta_+}\right) \sinh\left(\frac{2\pi\,\varphi_0}{\beta_-}\right)\right]$$

General properties of $\chi_{\mathcal{A}}$:

- The Causal Holographic Information $\chi_{\mathcal{A}}$
  - in special* cases, coincides with Entanglement entropy $\mathcal{S}_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\mathrm{Area}(\Xi_{\mathcal{A}})}{4\,G_N} \quad = \quad \mathcal{S}_{\mathcal{A}} \equiv -\mathrm{Tr}\left(\rho_{\mathcal{A}}\, \log \rho_{\mathcal{A}}\right) = \frac{\mathrm{Area}(\mathfrak{E}_{\mathcal{A}})}{4\,G_N}$$

  - but in general diverges more strongly than entanglement entropy
    e.g. for d=4, $\mathcal{A}$ = strip of width w , w/ IR regulator L & UV regulator ε ,

$$S_{\mathcal{A}} = c_{\mathrm{eff}}\, L^2 \left(\frac{1}{\varepsilon^2} - \frac{0.32}{w^2} \right) \;, \qquad \chi_{\mathcal{A}} = c_{\mathrm{eff}}\, L^2 \left(\frac{1}{\varepsilon^2} - \frac{2}{w^2} + \frac{4}{w^2}\, \log\left(\frac{w}{\varepsilon}\right) \right)$$

  - hence provides a bound on entanglement entropy $S_{\mathcal{A}} \leq \chi_{\mathcal{A}}$
  - unlike entanglement entropy, always varies smoothly with size of $\mathcal{A}$

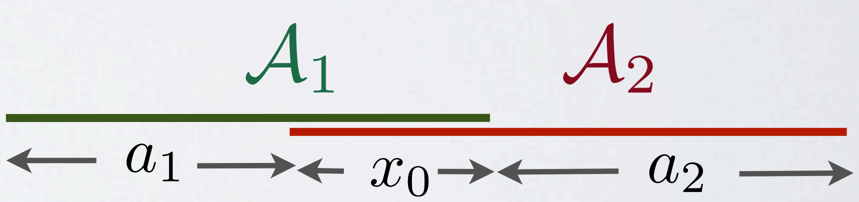
General properties of $\chi_{\mathcal{A}}$:

- The Causal Holographic Information $\chi_{\mathcal{A}}$
  - unlike entanglement entropy, does NOT satisfy strong subadditivity

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$
$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \setminus \mathcal{A}_2} + S_{\mathcal{A}_2 \setminus \mathcal{A}_1}$$

Geometric proof in static bulk and support in time-dep bulk [Headrick et.al.]
But counter-examples for $\chi_{\mathcal{A}}$:

  - explicit counter-example: 2 strips in d=4:



SS requires

$$F(a_1 + x_0) + F(a_2 + x_0) - F(a_1 + a_2 + x_0) - F(x_0) > 0 \,, \qquad F(x) = \frac{1}{x^2} \log\left(\frac{x}{\tilde{\varepsilon}}\right)$$

but this can be violated - e.g. by $x_0 = a_1 = a_2$

Toy model for dynamics:

Vaidya-AdS spacetime, describing a null shell in AdS:

$$ds^2 = -f(r,v)\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$

where $\quad f(r,v) = r^2 + 1 - \vartheta(v)\,m(r)$

with $\quad m(r) = \begin{cases} r_+^2 + 1 \quad , & \text{in AdS}_3 \\ \frac{r_+^2}{r^2}\,(r_+^2+1) \quad , & \text{in AdS}_5 \end{cases}$

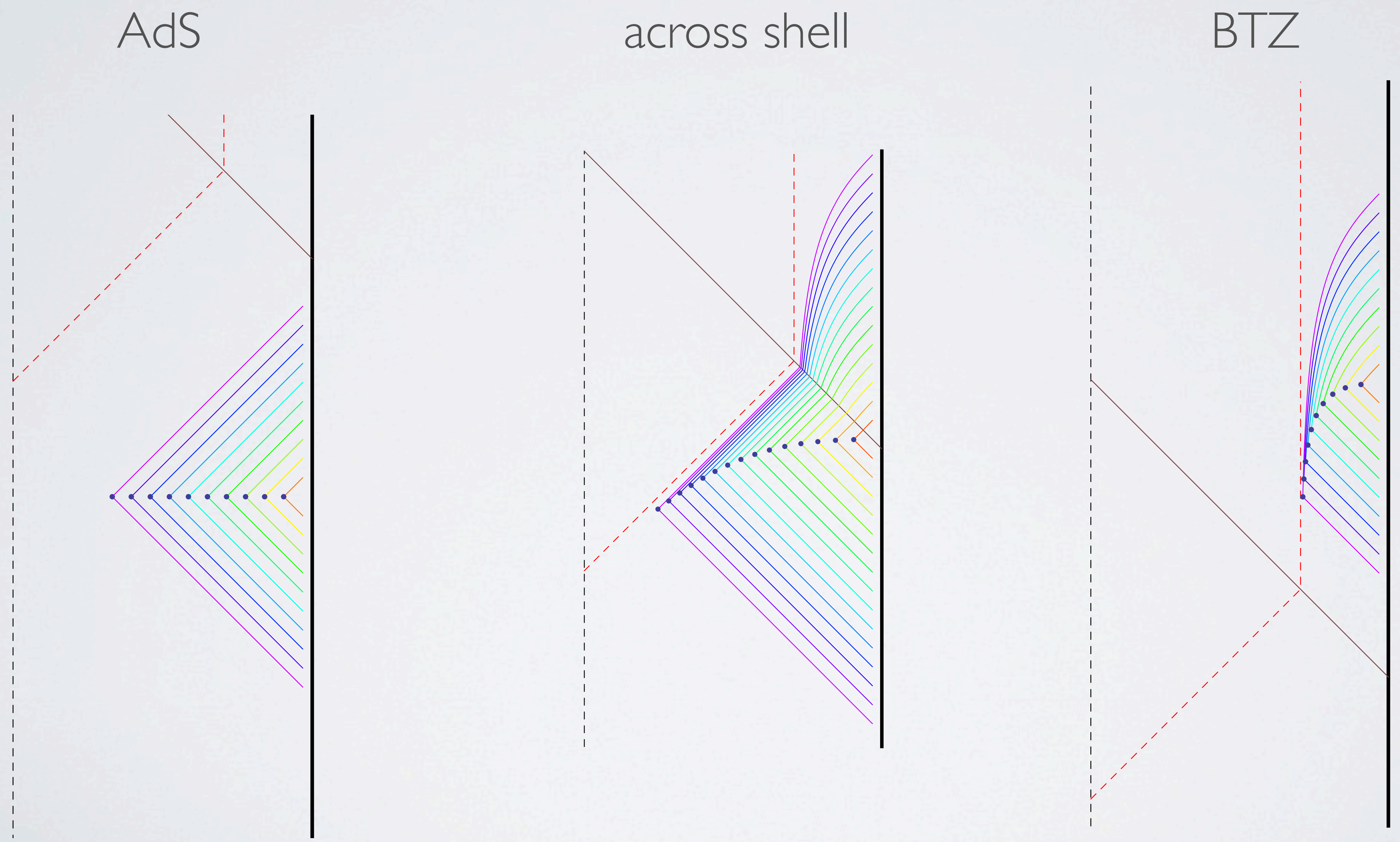
and $\quad \vartheta(v) = \begin{cases} 0 \quad , & \text{for } v < 0 \\ 1 \quad , & \text{for } v \geq 0 \end{cases}$ $\longrightarrow$ pure AdS / $\longrightarrow$ Schw-AdS (or BTZ)

we can think of this as $\delta \to 0$ limit of smooth shell with thickness δ :

$$\vartheta(v) = \frac{1}{2}\left(\tanh\frac{v}{\delta} + 1\right)$$

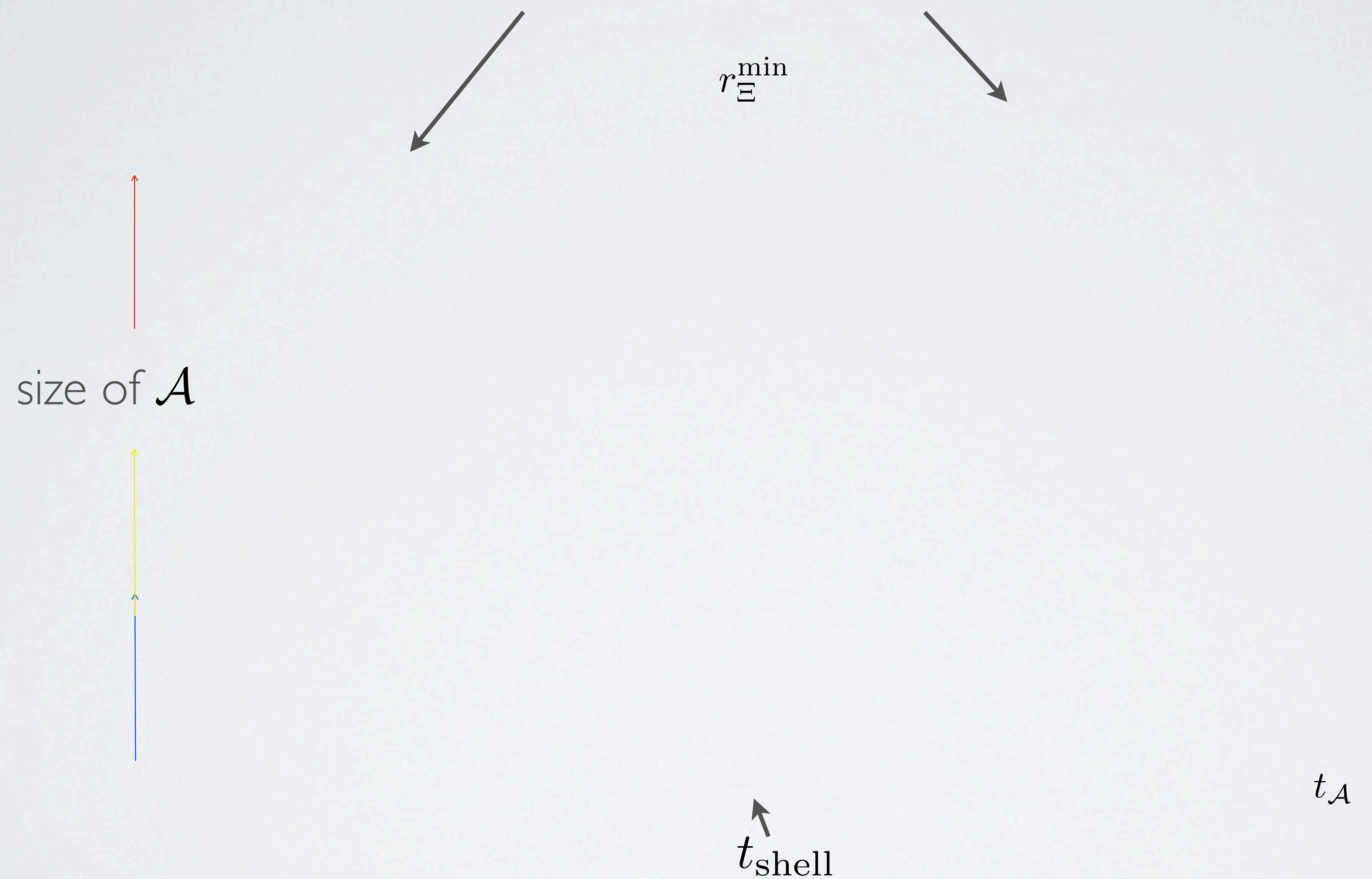
Causal wedge profile in Vaidya:

For fixed size of $\mathcal{A}$, causal wedge profile changes in time:



Quasi-teleological nature of $\chi_{\mathcal{A}}$:

For fixed size of $\mathcal{A}$, deepest reach of $\Xi_{\mathcal{A}}$ monotonically increases from AdS value to BTZ value:



Similarly for $\chi_{\mathcal{A}}$: Note that it starts increasing before $t_{\mathcal{A}} = t_{\text{shell}}$

Cf. deepest reach of $\Xi_{\mathcal{A}}$ vs. $\mathfrak{E}_{\mathcal{A}}$:

Unlike $\Xi_{\mathcal{A}}$, extremal surface $\mathfrak{E}_{\mathcal{A}}$ depends only on spatial info; starts increasing only at $t_{\mathcal{A}} = t_{\text{shell}}$:

$\Xi_{\mathcal{A}}$

$\mathfrak{E}_{\mathcal{A}}$

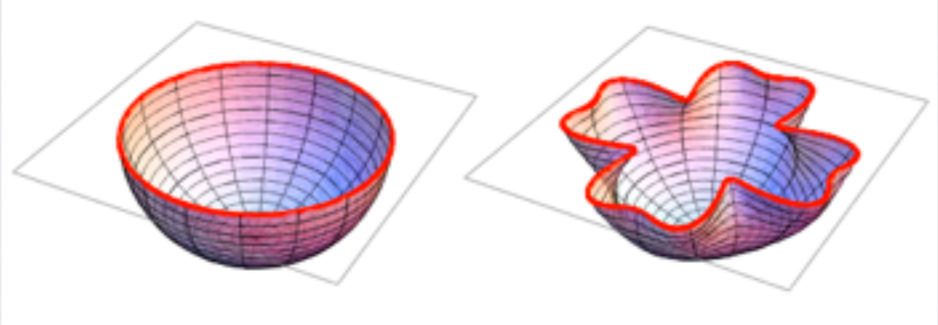
t_{shell}

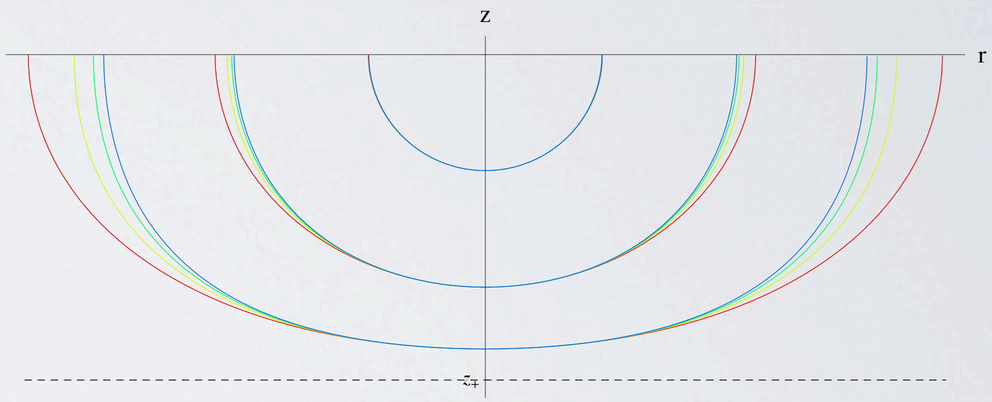
OUTLINE

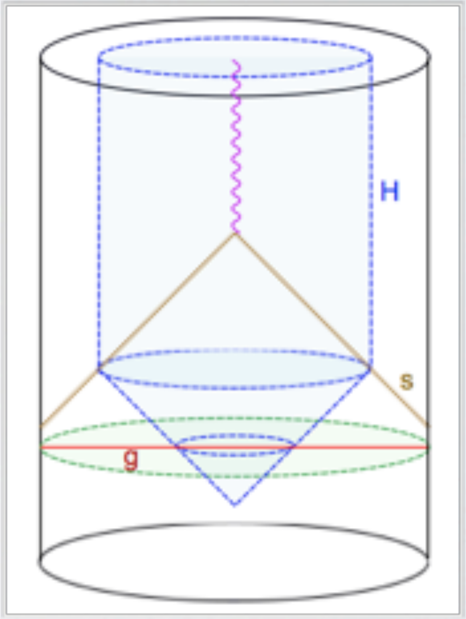
- Motivation & Background
- Features of Extremal Surfaces
- Probing Horizons
- Causal Holographic Information
- Summary & Future directions

Summary for extremal surfaces

- Spacelike geodesics reach deeper than null geodesics (at fixed spatial separation of endpoints).
- Higher-dimensional extremal surfaces reach deeper (at fixed extent of bounding region).
- Extremal surfaces anchored on sphere reach deepest (at fixed extent or volume of bounding region).
- Extremal surfaces of *any* dimension, anchored on *any* region, in *any* static planar black hole spacetime, cannot penetrate the horizon.
- Extremal surfaces can penetrate horizon of dynamically evolving black hole.







Summary for CHI

- The causal wedge $\blacklozenge_{\mathcal{A}}$
  - is the most natural (minimal nontrivial) bulk spacetime region related to $\mathcal{A}$
  - corresponds to bulk region most easily reconstructed from $\rho_{\mathcal{A}}$
  - cannot penetrate event horizon of a black hole
- The causal holographic information $\chi_{\mathcal{A}}$
  - coincides with entanglement entropy $S_{\mathcal{A}}$ in certain special cases (when DoFs in $\mathcal{A}$ are maximally entangled with those outside)
  - in general provides an upper bound on entanglement entropy
  - monotonically increases during thermalization
  - behaves quasi-teleologically, but only on light-crossing timescales
  - remains smooth as a function of time and the size of $\mathcal{A}$

Conjectured meaning of $\chi_{\mathcal{A}}$:

- We conjecture that $\chi_{\mathcal{A}}$ characterizes the amount of information contained in $\mathcal{A}$ which can be used to reconstruct the bulk geometry (entirely in $\blacklozenge_{\mathcal{A}}$ but possibly further)...
  - cons. set of local bulk `observers' starting & ending on bdy inside $\lozenge_{\mathcal{A}}$
  - these have access to full $\blacklozenge_{\mathcal{A}}$, but the info contained can be reduced:
    - bulk evolution: suffices to consider just Cauchy slice for $\blacklozenge_{\mathcal{A}}$
    - holography: suffices to consider just screen: natural region associated to $\mathcal{A}$ $= \Xi_{\mathcal{A}}$
  - hence natural to identify $\chi_{\mathcal{A}}$ with amount of info contained in $\mathcal{A}$
- This has entropy-like behavior, however, it does *not* correspond to a Von Neumann entropy:
  - e.g. it violates strong subadditivity.
- However, it provides a bound on Entanglement entropy;
  - and coincides in special, maximally-entangled, cases.

Future directions

Most important questions still remain:

- What is the direct boundary interpretation/construction of the causal holographic surface $\Xi_{\mathcal{A}}$ and 'information' $\chi_{\mathcal{A}}$?
- What bulk region can we fully reconstruct?

  (& What is the most efficient reconstruction method?)
- e.g. suppose we know $\{\chi_{\mathcal{Q}}\}$ for all sub-regions $\mathcal{Q} \in \mathcal{A}$; does this provide sufficient info to recover bulk metric in $\blacklozenge_{\mathcal{A}}$?
- What is the bulk dual of the reduced density matrix $\rho_{\mathcal{A}}$?
- Given a bulk location, how do we extract the geometry there from the CFT?

  (& How deep / late into BHs can various probes see?)
- How does the CFT encode bulk locality and causality?