Quantization of the Black Hole Horizons

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1 A Quantization Condition?

- Many black holes satisfy the quantization condition:

$$\frac{1}{(8\pi G_4)^2} A_+ A_- = \text{integer} \ .$$

  Here A_+ is the usual area of the event horizon and A_- is the area of the *inner* horizon.

- *This is very surprising!*

- For example, the inner horizon is known to be an unstable Cauchy horizon; so how can its area be significant?

- The purpose of this talk: review and elaborate on aspects of the story.

2 Example: 4D Kerr

- Primary evidence for the quantization condition: inspection of explicit examples.
- This talk: focus on asymptotically flat spacetime in $D = 4$.
- The simplest example: the Kerr black hole.
- The "entropies" of Kerr black holes computed from outer and inner areas:

$$S_\pm = \frac{A_\pm}{4G_4} = 2\pi \left(G_4 M^2 \pm \sqrt{G_4^2 M^4 - J^2} \right) \ .$$

- The product

$$\frac{1}{4\pi^2} S_+ S_- = J^2 = \text{integer}$$

3 Adding Charges

- In the context of $N = 4$ or $N = 8$ SUGRA the black holes can be generalized to carry arbitrary charges.
- The outer and inner areas then give the product:

$$\frac{1}{(2\pi)^2} S_+ S_- = J^2 + J_4 \ ,$$

where the quartic invariant (for the $N = 4$ theory, for definiteness):

$$J_4 = \vec{Q}^2 \vec{P}^2 - (\vec{Q} \cdot \vec{P})^2 \ .$$

- The right hand side of the product rule is an integer because J_4 is integral for correctly quantized charges.
- In fact: the right hand side is a *positive* integer since $J^2 > -J_4$ is the condition to avoid CTCs.

4 The Non-BPS Branch

- There is another branch of solutions with areas such that

$$\frac{1}{(2\pi)^2} S_+ S_- = -J_4 - J^2 \ .$$

- Again, the right hand side is not only an integer, it is a positive integer since $J_4 < -J^2$ is the condition to avoid CTCs.

- The two black hole branches can be continuously deformed to each other, passing through the locus $J^2 = -J_4$.

- These transition solutions correspond to singular limits where $A_- = 0$ and the inner horizon has become singular.

- Example: the Schwarchild soution.

5 Extremal Limits

- In the extremal limit the two horizons approach each other: $S_- \to S_+$.
- In this case
$$S_+ = 2\pi\sqrt{|J_4 + J^2|}\ .$$
- So: the integer in the quantization is the one that appears "under the square root" in the regime where extreme entropy is accounted for by the Cardy formula.
- The surprise is that this integer remains part of the story arbitrarily far from extremality.

6 Always an Integer?

- In the special case of a Kerr-Newman black hole:
$$\frac{1}{(2\pi)^2} S_+ S_- = \frac{1}{4} Q^4 + J^2 \ .$$
- This is *not* generally an integer:
$$Q^2 = \alpha n_e^2 \qquad \text{where} \qquad \alpha^{-1} \simeq 137 \ .$$
- Construction of Kerr-Newman as a solution in $N \geq 2$ SUGRA involves an embedding such as
$$Q_{\rm KN} = \frac{1}{2} Q_{\rm N=2} = \frac{1}{2} P_{\rm N=2} \ .$$
- The Dirac quantization condition on the $N = 2$ theory effectively takes
$$Q_{N=2} P_{N=2} = 2\pi \times \text{integer} \ \Rightarrow \ \frac{1}{2}\alpha = \frac{e^2}{8\pi} = \text{integer} \ .$$

7 Generalized Attractor Mechanism

- The Right Hand Side of the quantization condition

$$\frac{1}{(2\pi)^2} S_+ S_- = \text{integer} \ ,$$

  is independent of the *black hole mass.*

- It is also independent of *scalar VEVs*, i.e. on the *position in moduli space.*

- In particular, the relation can be continued from weak to strong coupling.

- This aspect can be thought of as a *generalized attractor mechanism.*

8 A First Law for the Inner Horizon

- "Thermodynamics" of the inner horizon must be considered in Lorentzian signature.
- For this Wald's formalism works well (see later).
- The conservation of the Noether charge for the Killing generator of the inner horizon gives a "first law"

$$T_- dS_- = dM - \Omega_- dJ - \Phi_- dQ - \Psi_- dP \ .$$

- In order to circumvent stability issues the differentials must be taken "in the space of solutions"; so there are no physical perturbations involved.
- The temperature $T_- < 0$ in the convention here (normal outward).

9 Consistency Relation

- Differentiation of the quantization rule with respect to mass gives:

$$T_+ S_+ = -T_- S_- \ .$$

  This is a stange relation that is highly nonobvious in concrete examples.

- Recall $T_- < 0$.

- Derivation of this consistency relation is tantamount to derivation of the quantization rule.

10 Interpretation of the Consistency Relation

- Suppose the general entropy can be divided into "chiral halves", as in 2D CFT:

$$S_+ = S_L + S_R \equiv \frac{1}{2}(S_+ + S_-) + \frac{1}{2}(S_+ - S_-) \ .$$

- The $T_+ S_+ = -T_- S_-$ relation gives

$$\frac{S_R}{T_R} = \frac{S_L}{T_L} \ .$$

- Interpretation: the central charge is the same in R and L sectors — so there is no diffeomorphism anomaly.

11 Another Consistency Relation?

- There is also the relation:

  $$\frac{\Omega_-}{T_-} = -\frac{\Omega_+}{T_+} \ .$$

  This relation is equally bizarre.

- In the context of a CFT with two halves this relation amounts to

  $$\left(\frac{\partial S_L}{\partial J}\right)_M = 0 \ .$$

- Interpretation: only R-movers in the 2D CFT have the ability to carry angular momentum.

12 Interpretation: Quantization Cond.

Model for the CFT (motivated by perturbative strings):

- Each chiral half has a quantized oscillator level $N_{L,R}$.
- Each has a zero mode contribution that depends on mass and moduli.
- Translational invariance imposes level matching:

$$N_L - N_R = J^2 + J_4 \ .$$

Extremal limits:

- BPS states: $N_R = 0$ and $J = 0$ *preserves SUSY* .
- Kerr/CFT states: $N_R = 0$ and $J \neq 0$ *breaks SUSY spontaneously.*
- Non-BPS states: $N_L = 0$.

13 Multistring Theory?

- DVV proposed a chiral multistring theory with level matching condition:

  $$\sum_{k,l} k^{(l)} N_k^{(l)} = J_4 \ .$$

  Here k is the momentum quantum number and l is *the number of strings*.

- A version of the theory is realized by the standard precision counting formulae (Igusa cusp form).

- The structure needed for non-extremal black holes would have both right and left movers.

- The general level matching condition can approximated for large charges as

  $$\sum k^{(l)} N_{L,k}^{(l)} - \sum k^{(l)} N_{R,k}^{(l)} = J^2 + J_4 \ .$$

14 A GR Challenge

- The central observation is simply:

$$A_+ A_- = \text{ independent of mass} \Leftrightarrow T_+ S_+ + T_- S_- = 0 \ .$$

- It should be possible to prove this in classical GR!
- The following is an attempt in this direction (which unfortunately is not complete).

15 Noether-Wald Charge

- The variation of the Lagrangian n-form:

$$\delta \mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta} \ .$$

  where ϕ denotes all fields, both metric (g) and matter fields (ψ); and $\mathbf{E} = 0$ is the equations of motion.

- The current $(n-1)$-form corresponding to a diffeomorphism ξ is

$$\mathbf{J} = \mathbf{\Theta}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot \mathbf{L} \ .$$

- Its divergence vanishes upon imposing the equations of motion so on-shell there is a Noether charge $(n-2)$-form $\mathbf{Q}$ for each diffeomorphism ξ:

$$\mathbf{J} = d\mathbf{Q} \ .$$

16 Killing Horizons

- Wald's entropy is essentially the Noether charge of a Killing vector, computed at the Killing horizon. In this context

$$\mathbf{Q} = \frac{1}{2}\epsilon_{abcd}\nabla^a\chi^b dx^c \wedge dx^d$$

- The outer and inner horizons are Killing horizons of the Killing vectors

$$\chi_\pm = \partial_t + \Omega_\pm \partial_\phi \ .$$

- The covariant derivatives of these Killing simplify at their respective horizons:

$$\nabla^a \chi^b_\pm \big|_{\pm\text{hor}} = \pm\kappa_\pm \epsilon^{ab} \ ,$$

  where the bimetric satisfies $g^{ab} = \epsilon^{ac} g_{cd} \epsilon^{cb}$.

- The *sign* of the bimetric ϵ^{ab} is such that it defines an outgoing normal at the outer horizon.

17 Killing Horizons

- For the Killing vectors, the *total* charges (after integration over the respective horizon at the bifurcation point) :

$$\mathbf{Q}[\xi_\pm] = \pm\kappa_\pm A_\pm = \pm 8\pi G T_\pm S_\pm$$

- Charge conservation gives the desired identity: $T_+ S_+ + T_- S_- = 0$.
- Well, almost.
- The "cross-terms" (the charges for $\chi_\pm$ evaluated at the $\mp$ horisons) cancel in explicit computations but it is not clear that there is a general argument.

18 Summary

A discussion of an apparent quantization condition:

$$\frac{1}{(8\pi G_4)^2} A_+ A_- = \text{integer} \ .$$

Comments:

- The right hand side does not depend on black hole mass.
- So it appears that there is some kind of index that can be continued from extremality to non-extremality.
- The right hand side also does not depend on scalars.
- So it appears that the index can be continued from weak to strong coupling (before of after going to the extreme limit).

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